Physics 110. Electricity and Magnetism. Professor Dine

Spring, 2008. Handout: Solving for the Scalar and Vector Potentials

1 Properties of the Dirac Delta Function

Your text has a good discussion of the Dirac delta-function. Here I wish to stress just a few other points.

The δ-function, $\delta(x)$, has the property that it diverges at $x = 0$ and is zero everywhere else, in such a way that, for any reasonably smooth function, $f(x)$:

$$\int dx f(x)\delta(x - x_o) = f(x_o). \tag{1}$$

This is a rather strange idea, so we can give two "representations" of the δ-function. The first is:

$$\delta(x) = \lim_{\Delta x \to 0} \frac{1}{\sqrt{\pi}\Delta x} e^{-\frac{x^2}{\Delta x^2}} \tag{2}$$

To understand this, think of Δx as extremely small, but fixed. Then the exponential makes the right hand side very small, except for a very narrow range of x near $x = 0$. Near $x = 0$, the function is huge. To check that this has the right integration properties, note that for f a smooth function, we can approximate $f(x)$ by $f(0)$ over the small interval Δx about the origin where the function is large ("has support", as mathematicians would say).

Exercise: Check that this representation has the right integration properties.

This representation makes the meaning of the δ-function clear, but another representation is more useful in practice, and will help us understand how to solve the equations for V and $\vec{A}$. Consider Fourier's theorem, for a function on an interval, $-L/2 < x < L/2$. In general,

$$f(x) = \sum a_n e^{\frac{2\pi i n x}{L}} \tag{3}$$

with

$$a_n = \frac{1}{L} \int_{-L/2}^{L/2} dx f(x) e^{\frac{-2\pi i n x}{L}}. \tag{4}$$

Take $f(x) = \delta(x)$. Then

$$a_n = \frac{1}{L}. \tag{5}$$

So

$$\delta(x) = \frac{1}{L} \sum e^{\frac{2\pi i n x}{L}}. \tag{6}$$

Now consider the limit of very large L. We can then replace the sum over n by an integral; calling $k = \frac{2\pi n}{L}$,

$$\delta(x) = \frac{1}{L} \int dn e^{\frac{2\pi i n x}{L}} \tag{7}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}.$$

So far, we have been talking about one-dimensional functions, but the generalization to three dimensions is not hard. We will write:

$$\delta(\vec{x}-\vec{x}_o)=\delta(x_1-x_{o\ 1})\delta(x_2-x_{o\ 2})\delta(x_3-x_{o\ 3}) \tag{8}$$

In terms of the δ-function, there is a natural description of the charge density and current associated with a collection of point charges, located at time t at points $\vec{x}_i(t)$, moving with velocities $\vec{v}_i(t)$:

$$\rho(\vec{x},t)=\sum_i q_i\delta(\vec{x}-\vec{x}_i(t)) \qquad \vec{j}(\vec{x},t)=\sum_i q_i\vec{v}_i(t)\delta(\vec{x}-\vec{x}_i(t)) \tag{9}$$

Exercise: Verify that the total charge, obtained by integrating ρ over all of space, is $\sum q_i$.

2 Solving the equations

We have seen that in the Lorentz gauge,

$$\partial_o V+\mu_o\epsilon_o\vec{\nabla}\cdot\vec{A}=0 \tag{10}$$

the equations for $\vec{A}$ and V are very symmetrical:

$$\nabla^2 V-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}V=-\frac{\rho}{\epsilon_o} \tag{11}$$

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}=-\mu_o\vec{J}$$

So V and the three components of $\vec{A}$ obey equations identical in form. So we we really have to figure out how to solve only one equation.

Your text, sensibly, guesses an answer and checks by plugging in the equations. We can proceed a bit more systematically. To do this, let's return first to the Poisson equation of electrostatics, and describe a method to solve this.

For a point charge, with $q=\epsilon_o$ located at $\vec{x}'$, the charge density is

$$\rho(\vec{x})=\epsilon_o\delta(\vec{x}-\vec{x}') \tag{12}$$

and the potential solves the equation:

$$-\nabla^2 V=\delta(\vec{x}-\vec{x}'). \tag{13}$$

We will give the solution of this equation a name, $G(\vec{x},\vec{x}')$,

$$G(\vec{x}-\vec{x}')=\frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|}. \tag{14}$$

Now knowing this function, the "Green's function", we can solve the problem of any charge distribution:

$$V(\vec{x})=\int d^3x' G(\vec{x}-\vec{x}')\rho(\vec{x}'). \tag{15}$$

Exercise: Check that V solves Poisson's equation, for general ρ, by simply plugging in the solution, and noting that $\nabla^2 G=\delta(\vec{x}-\vec{x}')$. Be careful to remember with respect to which variable you are differentiating.

For Poisson's equation, this is rather heavy machinery, but for the general, time-dependent equations for V and $\vec{A}$, this is quite powerful. We will look for a solution of the equation:

$$[-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]G(\vec{x}-\vec{x}', t-t') = \delta(\vec{x}-\vec{x}')\delta(t-t'). \tag{16}$$

To solve this, we first fourier transform in time/frequency. For the left hand side we write:

$$G(\vec{x}-\vec{x}', t-t') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} g_\omega(\vec{x}-\vec{x}'). \tag{17}$$

On the right hand side, we use our Fourier representation of the δ-function:

$$\delta(t-t') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tag{18}$$

From this we get:

$$\int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \left[(-\nabla^2 - \frac{\omega^2}{c^2}) g_\omega(\vec{R}) - \delta(\vec{R})\right] = 0. \tag{19}$$

where $\vec{R} = \vec{x}-\vec{x}'$. Requiring that the integrand vanish (this follows from standard Fourier analysis) gives:

$$(-\nabla^2 - \frac{\omega^2}{c^2}) g_\omega(\vec{R}) - \delta(\vec{R}) = 0. \tag{20}$$

Now away from $\vec{R} = 0$, this equation is not hard to solve, since the δ-function is zero. There is no preferred direction, so g_ω should be a function of $|\vec{R}|$. Going to spherical coordinates, g satisfies:

$$-\frac{1}{R^2}\frac{d}{dR}(R^2 \frac{dg_\omega}{dr}) + k^2 g_\omega = 0 \tag{21}$$

($k = \frac{\omega}{c}$). Calling $g_\omega = \frac{\chi}{R}$ (remember this trick from 101B?), this becomes:

$$\frac{d^2\chi}{dR^2} + k^2\chi = 0. \tag{22}$$

This is easy to solve. As usual, we need to choose a boundary condition. We'll make a guess, and see that it gives a sensible result:

$$g_\omega(R) = \frac{e^{ikR}}{R} \tag{23}$$

Now what happens at $R = 0$? This is also easy. For small R,

$$g_\omega(R) \approx \frac{1}{R} \tag{24}$$

so the laplacian acting on g gives precisely the δ function. So we have "lucked out" and found the solution of the equation *with* the δ-function.

As a final stage, we need to Fourier transform back. Sounds scary, but really it's very easy.

$$G(\vec{x}-\vec{x}', t-t') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} g_\omega(\vec{x}-\vec{x}'). \tag{25}$$

$$= G(\vec{x}-\vec{x}', t-t') = \frac{1}{|\vec{x}-\vec{x}'|}\int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{i\omega|\vec{x}-\vec{x}'|/c}.$$

Now all we have to do is remember our fourier representation of the δ-function to see that:

$$G(\vec{x}-\vec{x}', t-t') = \frac{1}{|\vec{x}-\vec{x}'|}\delta(t-t'-\frac{1}{c}|\vec{x}-\vec{x}'|). \tag{26}$$

This is quite interesting. The δ-function vanishes for time such that light can propagate from $\vec{x}'$ at time t' to $\vec{x}$ at time t. The solution of this condition for t' we will call the "retarded time", t_R. So for V, for example, we now have the solution:

$$V(\vec{x}, t) = \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} \rho(\vec{x}', t_R). \tag{27}$$

There is a similar solution for $\vec{A}$.

As a final remark, what about the boundary condition? Suppose we had taken

$$g = e^{-ikR}? \tag{28}$$

Then we would have gotten a δ-function $\delta(t - t' + \frac{1}{c}|\vec{x} - \vec{x}'|)$. This would correspond to a solution where light started at $t, \vec{x}$ and ended at $t', \vec{x}'$! This violates causality, but this Green's function is mathematically useful. Actually, much of Feynman's fame (and the essence of Feynman diagrams) comes from his realization that it is often useful to take a linear combination of these two Green's functions!