Integrating factors for first order differential equations

1. First order linear differential equations

Consider the homogeneous first order linear differential equation,

$$y' + P(x)y = 0\,, \tag{1}$$

where $y' \equiv dy/dx$. This is a separable equation, since it can be rewritten in the form

$$\frac{dy}{y} = -P(x)dx\,.$$

The solution then is immediately obtained by integration,

$$\int \frac{dy}{y} = -\int P(x)dx\,.$$

Carrying out the integration yields

$$\ln|y| = -\int^x P(x')dx' + A\,, \tag{2}$$

where the symbol $-\int^x P(x')dx'$ refers to the indefinite integral of $P(x)$ but without including the integration constant A , which is separately exhibited in eq. (2). Since y is a function of x, we shall henceforth denote $y = y(x)$. Exponentiating eq. (2) yields,

$$|y(x)| = C\exp\left\{-\int^x P(x')dx'\right\}\,,$$

where $C \equiv e^A$. Noting that $|y| = y\,\mathrm{sgn}\,y$, where

$$\mathrm{sgn}\,y = \begin{cases} +1\,, & \text{for } y > 0, \\ -1\,, & \text{for } y < 0, \end{cases} \tag{3}$$

one can absorb the sign of y in the definition of the constant C (calling the resulting constant C') and write,

$$y(x) = C'\exp\left\{-\int^x P(x')dx'\right\}\,, \tag{4}$$

which is the most general solution to eq. (1). For later reference, we shall introduce

$$u(x) \equiv \exp\left\{\int^x P(x')dx'\right\}\,, \tag{5}$$

which implies that eq. (4) is given by

$$y(x) = \frac{C'}{u(x)}\,.$$

The inhomogeneous first order linear differential equation,

$$y' + P(x)y = Q(x)\,, \tag{6}$$

is not separable as written. However, it is remarkable that one can turn eq. (1) into a separable differential equation simply by multiplying it by $u(x)$. The key observation is that the derivative of eq. (5) yields,

$$\frac{d}{dx}u(x) = P(x)u(x)\,.$$

Hence,

$$\frac{d}{dx}\big[u(x)y(x)\big] = u(x)\big[y' + P(x)y\big] = u(x)Q(x)\,, \tag{7}$$

after employing eq. (6) at the final step. Introducing a new variable, $v(x) \equiv u(x)y(x)$, we see that we have succeeded in converting the non-separable eq. (6) into a separable equation,

$$\frac{dv}{dx} = u(x)Q(x)\,.$$

The function $u(x)$ is called an *integrating factor*, since multiplication by $u(x)$ turned a non-separable differential equation into a separable differential equation. The solution for $v(x)$ [and hence $y(x)$] is now straightforward, and we end up with

$$y(x) = \frac{1}{u(x)}\left\{\int^x Q(x')u(x')dx + C\right\}\,, \tag{8}$$

where C is the integration constant that parameterizes a one-parameter family of solutions to eq. (1).

As a simple example, consider the differential equation.

$$y' + \frac{y}{x} = x^2\,. \tag{9}$$

Comparing with eq. (1), we identify $P(x) = 1/x$. In light of eqs. (5) and (7), one can identify the integrating factor for eq. (9) as $u(x) = \exp\big[\ln|x|\big] = |x| = x\,\text{sgn}(x)$, where the sgn function was defined in eq. (3). But since the integrating factor will multiply both the left and the right hand sides of eq. (9), we can simply drop the overall sign factor (which cancels out) and take $u(x) = x$. Hence, multiplying eq. (9) by x yields,

$$xy' + y = x^3\,. \tag{10}$$

One can now immediately recognize that

$$\frac{d}{dx}(xy) = xy' + y\,,$$

using the well known result for the derivative of a product. Hence, eq. (10) yields,

$$\frac{d}{dx}(xy) = x^3\,.$$

Integrating both sides of this equation yields

$$xy = \tfrac{1}{4}x^4 + C\,.$$

Finally dividing by x, we end up with[1]

$$y(x) = \tfrac{1}{4}x^3 + \frac{C}{x}\,. \tag{11}$$

In summary, it is straightforward to determine the integrating factor needed to convert a non-separable first order linear differential equation into a separable equation. We have shown above that the required integrating factor is given by eq. (5).

2. First order exact differential equations

In this section we will partially relax the condition of linearity. We will still only allow the first power of dy/dx, but we will consider the coefficients of the differential equation to be arbitrary functions of x and y. The most equation of this type can be written in the following form,

$$M(x,y)dx + N(x,y)dy = 0\,. \tag{12}$$

Under the assumption that $N(x,y) \neq 0$ over the relevant range of the parameters x and y, one can rewrite eq. (12) in the form of a homogeneous differential equation,

$$y' + \frac{M(x,y)}{N(x,y)} = 0\,. \tag{13}$$

The differential equation given eq. (12) is called *exact* if it can be rewritten in the following form,

$$dF(x,y) = M(x,y)dx + N(x,y)dy = 0\,, \tag{14}$$

for some function $F(x,y)$. If such a function can be found, then one can immediately integrate eq. (14) to obtain,

$$F(x,y) = C\,,$$

for some constant C. In this case, $M(x,y)dx + N(x,y)dy$ is called an exact differential.

Here are some examples of exact differentials,

$$d(xy) = xdy + ydx\,, \tag{15}$$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}\,, \tag{16}$$

$$d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}\,, \tag{17}$$

$$d(x^2 + y^2) = 2(xdx + ydy) \tag{18}$$

$$d\left(\arctan\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}\,. \tag{19}$$

$$d\left(\ln\frac{y}{x}\right) = \frac{ydx - xdy}{xy}\,. \tag{20}$$

[1]Strictly speaking, the original differential equation given in eq. (9) is not defined at $x = 0$ due to the singular behavior of the term y/x as $x \to 0$.

How does one identify whether $M(x,y)dx+N(x,y)dy$ is an exact differential? Recall that the chain rule when applied to $F(x,y)$ yields

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy\,. \tag{21}$$

Comparing with eq. (14), one can identify,

$$M(x,y) = \frac{\partial F}{\partial x}\,, \qquad\qquad N(x,y) = \frac{\partial F}{\partial y}\,. \tag{22}$$

Under the assumption that $F(x,y)$ is a well behaved function, it then follows that

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}\,. \tag{23}$$

That is, in evaluating the second partial derivative, the end result does not depend on the order in which you perform the two partial derivative operations. Eq. (23) is called the *integrability condition* (the origin of this terminology is provided in Appendix A).

Using eq. (22), we see that

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}\,, \qquad\qquad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y}$$

Hence, eq. (23) yields,

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}\,. \tag{24}$$

Eq. (24) provides the condition that must be satisfied if $M(x,y)dx + N(x,y)dy$ is an exact differential. Indeed, it is a simple matter to check that the exact differentials listed in eqs. (15)–(20) all satisfy eq. (24).

A following simple example illustrates the procedure. Consider the differential equation,

$$-\frac{y\,dx}{x^2} + \frac{dy}{x} = 0\,. \tag{25}$$

Identifying $M(x,y) = -y/x^2$ and $Q(x,y) = 1/x$, it is straightforward to verify that eq. (24) is satisfied. Indeed, eq. (25) is equivalent to

$$d\left(\frac{y}{x}\right) = 0\,, \tag{26}$$

which immediately yields the solution $y/x = C$ or equivalently

$$y(x) = Cx\,.$$

In some cases, given $M(x,y)$ and $N(x,y)$ such that $dF(x,y) = M(x,y)dx+N(x,y)dy$, it may take a little effort to identify $F(x,y)$. Here is an example that shows how to identify the function $F(x,y)$. Consider the differential equation,

$$(3x^2 + 2xy)dx + (x^2 + y^2)dy = 0\,. \tag{27}$$

This is an exact differential, since

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = 2x\,.$$

Hence, there exists a function $F(x,y)$ such that $dF(x,y) = (3x^2+2xy)dx + (x^2+y^2)dy$. Using eq. (22),

$$\frac{\partial F}{\partial x} = M(x,y) = 3x^2 + 2xy\,, \tag{28}$$

which implies that

$$F(x,y) = x^3 + x^2y + f(y)\,, \tag{29}$$

where $f(y)$ is a function of y alone. To determine $f(y)$, we again employ eq. (22) to obtain

$$\frac{\partial F}{\partial y} = N(x,y) = x^2 + y^2 = x^2 + \frac{df}{dy}\,, \tag{30}$$

where the last step makes direct use of eq. (29) to compute $\partial F/\partial y$. Eq. (30) then implies that,

$$\frac{df}{dy} = y^2\,,$$

which yields $f(y) = \frac{1}{2}y^3 + C_0$, where C_0 is the integration constant. Inserting this result back into eq. (29) provides the final result,

$$F(x,y) = x^2 + x^2y + \tfrac{1}{3}y^3 + C_0\,. \tag{31}$$

The original differential equation given in eq. (27) is equivalent to $dF(x,y) = 0$. Hence, the solution to eq. (27) is $F(x,y) = C$ for some constant C. Having found the explicit form for $F(x,y)$, the solution to eq. (27) is then immediately given by,

$$x^2 + x^2y + \tfrac{1}{3}y^3 = C\,, \tag{32}$$

where the constant C_0 has be absorbed into the constant C. By plugging eq. (32) back into eq. (27), one can check that our solution is correct,

$$d(x^3 + x^2y + \tfrac{1}{3}y^3) = (3x^2+2xy)dx + (x^2+y^2)dy = 0\,. \quad \checkmark$$

Unfortunately, in the general case, $M(x,y)dx + N(x,y)dy$ is typically not an exact differential. However, if one could find a function $u(x,y)$ such that

$$dG(x,y) = u(x,y)\big[M(x,y)dx + N(x,y)dy\big]\,,$$

is an exact differential, then one would immediately be able to obtain the solution to eq. (12), namely $G(x,y) = C$ for some constant C. Once again, the function $u(x,y)$ is called an integrating factor, since it converts a non-separable differential equation into one that can be trivially solved by integration. In the next section, we will examine some techniques for finding the integrating factor $u(x,y)$.

3. Determining the integrating factor of a first order differential equation

We return to the first order differential equation,

$$M(x,y)dx + N(x,y)dy = 0\,. \tag{33}$$

In this section, we assume that $M(x,y)dx + N(x,y)dy$ is not an exact differential. We seek an integrating factor, $u(x,y)$, such that $u(x,y)\big[M(x,y)dx + N(x,y)dy\big]$ is an exact differential, in which case we can employ eq. (24) after making the substitutions, $M(x,y) \to u(x,y)M(x,y)$ and $N(x,y) \to u(x,y)N(x,y)$. Hence,[2]

$$\frac{\partial}{\partial x}\big[u(x,y)N(x,y)\big] = \frac{\partial}{\partial y}\big[u(x,y)M(x,y)\big]\,. \tag{34}$$

Performing the partial derivatives above using the product rule and rearranging terms, we end up with a partial differential equation for $u(x,y)$,

$$u(x,y)\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N(x,y)\frac{\partial u}{\partial x} - M(x,y)\frac{\partial u}{\partial y}\,. \tag{35}$$

So far, we have not made any real progress toward solving eq. (33), since finding a solution to eq. (35) is typically more difficult than our original problem. Nevertheless, in certain special cases, a solution of eq. (35) is easily obtained. We consider two interesting special cases below.

1. Suppose that

$$\frac{1}{N(x,y)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)\,, \tag{36}$$

That is, the function that appears on the left hand side of eq. (36) is a function of x alone (i.e., there is no dependence on the variable y). In this case, we can assume that the integrating factor, $u(x,y) \equiv u(x)$, also has no dependence on the variable y. Plugging back into eq. (35), it follows that

$$u(x)f(x)N(x,y) = N(x,y)\frac{du}{dx}\,. \tag{37}$$

We can cancel out the factor $N(x,y)$ from both sides of eq. (37) as long as $N(x,y)$ does not vanish in the region of interest. Hence, the integrating factor $u(x)$ is the solution to the first order separable differential equation,

$$\frac{du}{dx} = u(x)f(x)\,. \tag{38}$$

This equation is of the form given by eq. (1), by identifying $P = -f$. Hence, we can use the solution already given in eq. (4) to obtain,

$$u(x) = \exp\left\{\int^x f(x')dx'\right\}\,, \tag{39}$$

up to an overall nonzero multiplicative constant C'. However, this constant is irrelevant, since we make use of the integrating factor by multiplying both sides of eq. (33) by $u(x)$, in which case any overall multiplicative constant drops out.

[2]The integrating factor, which must be a solution to eq. (34), is not unique. Nevertheless, just finding one possible solution for $u(x,y)$ is sufficient to convert eq. (33) into an exact differential equation.

2. Suppose that

$$\frac{1}{M(x,y)}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=g(y)\,, \tag{40}$$

That is, the function that appears on the left hand side of eq. (40) is a function of y alone (i.e.,there is no dependence on the variable x). In this case, we can assume that the integrating factor, $u(x,y)\equiv u(y)$, also has no dependence on the variable x. Plugging back into eq. (35), it follows that

$$u(y)g(y)M(x,y)=-M(x,y)\frac{du}{dy}\,. \tag{41}$$

We can cancel out the factor $M(x,y)$ from both sides of eq. (41) as long as $M(x,y)$ does not vanish in the region of interest. Hence, the integrating factor $u(y)$ is the solution to the first order separable differential equation,

$$\frac{du}{dy}=-u(y)g(y)\,. \tag{42}$$

This equation is again of the form given by eq. (1), by identifying $P=g$. Hence, we can use the solution already given in eq. (4) to obtain,

$$u(y)=\exp\left\{-\int^{y}g(y')dy'\right\}\,, \tag{43}$$

To illustrate the above method for determining the integrating factor, consider the differential equation,

$$xdy-ydx=0\,. \tag{44}$$

In this example, $M(x,y)=-y$ and $N(x,y)=x$, so that

$$\frac{\partial M}{\partial y}=-1\,,\qquad\qquad \frac{\partial N}{\partial x}=1\,. \tag{45}$$

Hence, $xdy-ydx$ is not an exact differential. Note that

$$\frac{1}{N(x,y)}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-\frac{2}{x}\,, \tag{46}$$

so that Case 1 above is applicable. Thus, we can immediately use eq. (39) to obtain

$$u(x)=\exp\left\{-2\int^{x}\frac{dx'}{x'}\right\}=\exp\bigl(-2\ln|x|\bigr\}=\exp\bigl(\ln|x|^{-2}\bigr)=\frac{1}{|x|^2}\,. \tag{47}$$

Since any overall nonzero multiplicative factor in defining $u(x)$ can be dropped, the overall sign of $u(x)$ is not significant. Hence, one can employ the integrating factor,

$$u(x)=\frac{1}{x^2}\,.$$

Indeed,

$$u(x)\bigl[xdy-ydx\bigr]=\frac{xdy-ydx}{x^2}=d\left(\frac{x}{y}\right)\,,$$

is an exact differential. Hence, eq. (44) is equivalent to $d(x/y)=0$, whose solution is given by $y=Cx$ for some constant C.

In the example just discussed, we also have

$$\frac{1}{M(x,y)}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{2}{y}\,, \tag{48}$$

so that Case 2 above is also applicable. One can then immediately use eq. (43) to obtain

$$u(y)=\exp\left\{-2\int^{y}\frac{dy'}{y'}\right\}=\exp\bigl(-2\ln|y|\bigr\}=\exp\bigl(\ln|y|^{-2}\bigr)=\frac{1}{|y|^2}\,. \tag{49}$$

Thus, another possible choice for the integrating factor is

$$u(y)=\frac{1}{y^2}\,.$$

Once again, we can check that

$$u(y)\bigl[xdy-ydx\bigr]=\frac{xdy-ydx}{y^2}=-d\left(\frac{y}{x}\right)\,,$$

is an exact differential. Hence, eq. (44) is equivalent to $d(y/x)=0$, whose solution is given by $x=C'y$ for some constant C'. Of course, this is the same solution obtained previously if we identify $C'=1/C$.

The example given above is in some sense too simple, since we have already seen that one can easily identify the integrating factor of any first order *linear* differential equation. Indeed, eq. (6) can be rewritten in the following form,

$$dy+\bigl[P(x)y-Q(x)\bigr]dx=0\,,$$

and so we can identify $M(x,y)=P(x)y-Q(x)$ and $N(x,y)=1$. Hence,

$$\frac{\partial M}{\partial y}=P(x)\,,\qquad\qquad \frac{\partial N}{\partial x}=0\,. \tag{50}$$

It follows that

$$\frac{1}{N(x,y)}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=P(x)\,, \tag{51}$$

which means that we can use eq. (39) to obtain the integrating factor,

$$u(x)=\exp\left\{\int^{x}P(x')dx'\right\}\,,$$

in agreement with the result of eq. (5). Thus, we have recovered the integrating factor of the first order linear differential equation obtained in Section 1.

Here is an example of a first order nonlinear differential equation for which the integrating factor can be found with the method outlined above. Consider,

$$y(xy+1)dx-xdy=0\,. \tag{52}$$

In this example, $M(x,y) = y(xy+1)$ and $N(x,y) = -x$, so that

$$\frac{\partial M}{\partial y} = 2xy + 1\,, \qquad\qquad \frac{\partial N}{\partial x} = -1\,. \tag{53}$$

That is, $y(xy+1)dx - xdy$ is not an exact differential. However, note that

$$\frac{1}{M(x,y)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2xy+2}{y(xy+1)} = \frac{2}{y}\,. \tag{54}$$

Hence, we can use eq. (43) to obtain the integrating factor [cf. eq. (49)],

$$u(y) = \frac{1}{y^2}\,.$$

Indeed, one can check that,

$$\frac{y(xy+1)dx - xdy}{y^2} = xdx + \frac{ydx - xdy}{y^2} = \tfrac{1}{2}dx^2 + d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2} + \frac{x}{y}\right)\,.$$

Hence, the solution to eq. (52) is

$$\frac{x^2}{2} + \frac{x}{y} = C\,,$$

for some constant C. Solving for y yields,

$$y = \frac{2x}{2C - x^2}\,.$$

Although the above analysis is not valid when $y = 0$, it is easy to check that the family of solutions (corresponding to different nonzero values of C) all pass through the origin where $x = y = 0$.

Unfortunately, there are many cases in which neither eq. (36) nor eq. (40) is applicable. Over the years, mathematicians have identified additional cases in which the integrating factor can be determined.[3] However, often it is somewhat of an art to identify the integrating factor of a specific first order nonlinear differential equation.

For example, consider the differential equation,

$$ydx + (x + x^3y^2)dy = 0\,. \tag{55}$$

In this example, $M(x,y) = y$ and $N(x,y) = x + x^3y^2$, so that

$$\frac{\partial M}{\partial y} = 1\,, \qquad\qquad \frac{\partial N}{\partial x} = 1 + 3x^2y^2\,. \tag{56}$$

Hence, $ydx + (x + x^3y^2)dy$ is not an exact differential. However, neither of the two special cases discussed above apply in this case. So, how should we proceed? Perhaps an inspired guess might lead you to propose that the integration factor is of the form x^py^r for some choice of p and r. If it works, then all you would have to do is to use eq. (34) to determine the exponents when applied to eq. (55).

[3] A useful list of some of these additional cases can be found on pp. 324–325 of David Zwillinger, *Handbook of Differential Equations*, 3rd Edition (Academic Press, San Diego, CA, 1998). See also pp. 28–34 of George Moseley Murphy, *Ordinary Differential Equations and Their Solutions* (Dover Publications, Inc., Mineola, NY, 2011).

However, here I shall propose an alternative strategy. You should always be on the lookout for exact differentials. In particular, you should immediate notice if any of the exact differentials listed in eqs. (15)–(20) appear. With this strategy in mind, you would immediately spot $d(xy) = xdy + ydx$ which is sitting inside eq. (55). Thus, one can rewrite eq. (55) as

$$d(xy) + x^3y^2dy = 0\,.$$

Dividing this equation by x^3y^3 then yields a separable differential equation,

$$\frac{dv}{v^3} + \frac{dy}{y} = 0\,,$$

where $v \equiv xy$. Integrating this equation yields the solution to eq. (55),

$$\ln y - \frac{1}{2x^2y^2} = C\,, \tag{57}$$

for some constant C. By this procedure, we have in fact uncovered the integrating factor

$$u(x,y) = \frac{1}{x^3y^3}\,. \tag{58}$$

As a check, after multiplying eq. (55) by $u(x,y)$ given in eq. (58), we obtain

$$\frac{dx}{x^3y^2} + \left(\frac{1}{x^2y^3} + \frac{1}{y}\right) dy = 0\,. \tag{59}$$

That is, $u(x,y)M(x,y) = 1/(x^3y^2)$ and $u(x,y)N(x,y) = 1/(x^2y^3) + 1/y$. Hence,

$$\frac{\partial}{\partial y}\big[u(x,y)M(x,y)\big] = \frac{\partial}{\partial x}\big[u(x,y)N(x,y)\big] = -\frac{2}{x^3y^3}\,.$$

This result implies that the differential on the left hand side of eq. (59) is exact,

$$\frac{dx}{x^3y^2} + \left(\frac{1}{x^2y^3} + \frac{1}{y}\right) dy = d\left(\ln y - \frac{1}{2x^2y^2}\right)\,. \tag{60}$$

Thus, we immediately recover the solution to eq. (55) given by eq. (57) above.

Our final example is taken from Problem 8.4–23 on p. 407 of Boas. If $M(x,y)$ and $N(x,y)$ are both homogeneous functions of x and y of degree n, then as shown in Appendix B, one can write $M(x,y) = x^n f(y/x)$, and similarly for $N(x,y)$. In this case, the integrating factor for $M(x,y)dx + N(x,y)dy$ is

$$u(x,y) = \frac{1}{xM(x,y) + yN(x,y)}\,, \tag{61}$$

under the assumption that $xM + yN \neq 0$. The proof of this assertion is given in Appendix C.

In the special case that $xM + yN = 0$, the differential equation $Mdx + Ndy = 0$ reduces to $M\big[dx - (x/y)dy\big] = 0$. Under the assumption that $M \neq 0$, we can divide by M to obtain $ydx = xdy$. This is a separable equation whose solution is $y = Cx$ for some constant C.

As example, consider the differential equation,

$$y(x^2+y^2)dx - x(x^2+2y^2)dy = 0\,. \tag{62}$$

In this case $M = y(x^2+y^2)$ and $N = -x(x^2+2y^2)$ are homogeneous functions of degree 3. Using the integrating factor given by eq. (61), $u(x,y) = (xM+yN)^{-1} = -1/(xy^3)$, we can conclude that there exists a function $F(x,y)$ such that

$$dF = -\frac{x^2+y^2}{xy^2}dx + \frac{x^2+2y^2}{y^3}dy\,.$$

Then, the solution to eq. (62) is given by $F(x,y) = C$, where C is an integration constant.

Comparing with the chain rule,

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy\,,$$

it follows that

$$\frac{\partial F}{\partial x} = -\frac{x^2+y^2}{xy^2}\,, \qquad \frac{\partial F}{\partial y} = \frac{x^2+2y^2}{y^3}\,. \tag{63}$$

Integrating the first equation in eq. (63) yields

$$F(x,y) = -\frac{x^2}{2y^2} - \ln|x| + f(y)\,, \tag{64}$$

where $f(y)$ is a function of y alone. From eq. (64), it follows that

$$\frac{\partial F}{\partial y} = \frac{x^2}{y^3} + \frac{df}{dy}\,.$$

Comparing with eq. (63), we obtain

$$\frac{df}{dy} = \frac{2}{y}\,, \qquad \Longrightarrow \qquad f(y) = 2\ln|y| + C'\,,$$

where C' is an integration constant. That is, $F(x,y) = \ln|y^2/x| - x^2/(2y^2) + C'$ and we have therefore verified that

$$dF = d\left(\ln\left|\frac{y^2}{x}\right| - \frac{x^2}{2y^2}\right) = -\frac{x^2+y^2}{xy^2}dx + \frac{x^2+2y^2}{y^3}dy\,.$$

Thus, the solution to eq. (62) is

$$\ln\left|\frac{y^2}{x}\right| - \frac{x^2}{2y^2} = C\,. \tag{65}$$

Boas suggests another method of solution on p. 406. In light of eq. (81), Boas notes that by making a change of variables by replacing y with $v = y/x$, one can convert eq. (62) into a separable equation in x and v. In particular, it is possible to convert the differential equation,

$$M(x,y)dx + N(x,y)dy = 0\,, \tag{66}$$

into a separable differential equation if both $M(x,y)$ and $N(x,y)$ are homogeneous functions of two variables x and y *of the same degree*. One can rewrite eq. (66) as

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}\,, \tag{67}$$

If both $M(x,y)$ and $N(x,y)$ are homogeneous functions of degree n, then eq. (81) implies that $M(x,y) = x^n f(y/x)$ and $N(x,y) = x^n g(y/x)$, where the functions f and g depend only on the ratio y/x and not on x and y separately. Thus, eq. (67) is of the form,

$$\frac{dy}{dx} = h\left(\frac{y}{x}\right)\,, \tag{68}$$

where $h = -f/g$. We shall therefore introduce a new variable, $v = y/x$ or equivalently,

$$y = xv\,, \tag{69}$$

Differentiating this equation yields,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v\,.$$

This allows us to rewrite eq. (68) as,

$$x\frac{dv}{dx} + v = h(v)\,.$$

This is a separable differential equation, which can be rearranged into the following form

$$\frac{dv}{h(v) - v} = \frac{dx}{x}\,.$$

Integrating the above equation yields,

$$\ln|x| = \int^v \frac{dv'}{h(v') - v'} + C\,, \tag{70}$$

where C is an integration constant.

Let us try out this method in solving eq. (62). In this example,

$$-\frac{M(x,y)}{N(x,y)} = \frac{y(x^2+y^2)}{x(x^2+2y^2)} = \frac{v(1+v^2)}{1+2v^2} = h(v)\,. \tag{71}$$

Then eqs. (70) and (71) yields

$$\ln|x| = -\int^v \frac{1+2v'^2}{v'^3}\,dv' + C = \frac{1}{2v^2} - 2\ln|v| + C\,. \tag{72}$$

Using eq. (69), we substitute $v = y/x$ back into eq. (72), and recover our previous result given in eq. (65).

APPENDIX A: The integrability condition for exact differentials

In Section 2, we showed that $M(x,y)dx + N(x,y)dy$ is an exact differential if the following integrability condition is satisfied,

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}\,. \tag{73}$$

The term *integrability condition* arises from the following considerations. In vector calculus, Green's theorem relates the line integral over a closed counterclockwise path C to a double integral over a region A that lies inside the closed curve C,[4]

$$\oint M(x,y)dx + N(x,y)dy = \iint\limits_A \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy\,. \tag{74}$$

In light of eq. (73), we see that if $M(x,y)dx+N(x,y)dy$ is an exact differential, then Green's theorem yields,

$$\oint M(x,y)dx + N(x,y)dy = 0\,. \tag{75}$$

That is, the integrability condition ensures that the line integral above over a closed path is equal to zero.

Eq. (75) must be satisfied when $M(x,y)dx+N(x,y)dy$ is an exact differential, since the definition of an exact differential states that a function $F(x,y)$ exists such that,

$$dF(x,y) = M(x,y)dx + N(x,y)dy\,. \tag{76}$$

Consequently, for a closed counterclockwise path that starts and ends at the same point, denoted below by (x_0, y_0), the fundamental theorem of integral calculus yields,

$$\oint M(x,y)dx + N(x,y)dy = \oint dF(x,y) = F(x_0,x_0) - F(x_0,x_0) = 0\,. \tag{77}$$

Likewise, the line integral along a path L from the point (x_0, y_0) to the point (x_1, y_1) is given by,

$$\int_L M(x,y)dx + N(x,y)dy = \int_{(x_0,y_0)}^{(x_1,y_1)} dF(x,y) = F(x_1,y_1) - F(x_0,y_0)\,, \tag{78}$$

which depends only on the value of the function F at the two endpoints of the path L.

If the integrability condition is not satisfied, then $M(x,y)dx + N(x,y)dy$ is *not* an exact differential, and

$$\oint M(x,y)dx + N(x,y)dy \neq 0\,. \tag{79}$$

Moreover, eq. (79) also implies that the line integral of $M(x,y)dx + N(x,y)dy$ along a path L depends on which path is taken in the x–y plane from (x_0, y_0) to (x_1, y_1), in contrast to eq. (78) which just depends on the value of F at the two endpoints.

[4]See eq. (9.7) in Section 9 of Chapter 6 on p. 310 of Boas.

APPENDIX B: Euler's theorem for homogeneous functions

A homogeneous function $F(x, y)$ of two variables x and y is said to be homogeneous of degree n in x and y if and only if

$$F(cx, cy) = c^n F(x, y)\,, \tag{80}$$

for any dimensionless nonzero constant c. For example, in a physical problem, if x and y are variables with the dimensions of length L and the function $F(x, y)$ is a quantity of dimension L^n, then $F(x, y)$ is a homogeneous function of x and y. One consequence of eq. (80) is that

$$F(x, y) = x^n f\left(\frac{y}{x}\right)\,, \tag{81}$$

where $f(y/x)$ is a function of the quantity y/x alone (and not separately dependent on x and y). On p. 406 of Boas, the following example of a homogeneous function of degree of 3 is given: $F(x, y) = x^3 - xy^2 = x^3\big[1 - (y/x)^2\big]$. Notice that in this example, if both x and y are variables with dimension of length L, then each term in $F(x, y)$ has dimension L^3.

Euler's theorem states that if $F(x, y)$ is a homogeneous of degree n in x and y, then

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF\,. \tag{82}$$

The proof of eq. (82) is an exercise of partial differentiation. Starting from eq. (81) and using the chain rule,

$$\frac{\partial F}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) - x^n \cdot \frac{y}{x^2} f'\left(\frac{y}{x}\right)\,, \tag{83}$$

$$\frac{\partial F}{\partial y} = x^n \cdot \frac{1}{x} f'\left(\frac{y}{x}\right)\,, \tag{84}$$

where

$$f'\left(\frac{y}{x}\right) \equiv \left.\frac{df}{dz}\right|_{z=y/x}\,.$$

Hence eqs. (83) and (84) yields,

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = x^n\left\{nf\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right) + \frac{y}{x}f'\left(\frac{y}{x}\right)\right\} = nx^n f\left(\frac{y}{x}\right) = nF\,.$$

Thus, Euler's theorem is established.

APPENDIX C: The integrating factor for $Mdx + Ndy$ in the case that M and N are homogeneous functions of the same degree

In light of eq. (34), to prove that $(xM + yN)^{-1}$ is the correct integrating factor for $Mdx + Ndy$ in the case that M and N are homogeneous functions of the same degree, one must verify that

$$\frac{\partial}{\partial x}\left[\frac{N}{Mx + N}\right] \stackrel{?}{=} \frac{\partial}{\partial y}\left[\frac{M}{Mx + N}\right]\,. \tag{85}$$

Evaluating the partial derivatives above is an exercise in partial differentiation. With the help of the chain rule.

$$\begin{aligned}\frac{\partial}{\partial x}\left[\frac{N}{Mx+N}\right] &= \frac{1}{Mx+Ny}\frac{\partial N}{\partial x} - \frac{N}{(Mx+Ny)^2}\frac{\partial}{\partial x}(Mx+Ny) \\ &= \frac{1}{Mx+Ny}\frac{\partial N}{\partial x} - \frac{N}{(Mx+Ny)^2}\left[M + x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x}\right], \qquad (86)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial y}\left[\frac{M}{Mx+N}\right] &= \frac{1}{Mx+Ny}\frac{\partial M}{\partial y} - \frac{M}{(Mx+Ny)^2}\frac{\partial}{\partial y}(Mx+Ny) \\ &= \frac{1}{Mx+Ny}\frac{\partial M}{\partial y} - \frac{M}{(Mx+Ny)^2}\left[N + y\frac{\partial N}{\partial y} + x\frac{\partial M}{\partial y}\right]. \qquad (87)\end{aligned}$$

Subtracting the two equations above, one must obtain zero if eq. (85) is satisfied. That is, we must check whether

$$\begin{aligned}&\frac{1}{Mx+Ny}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) - \frac{N}{(Mx+Ny)^2}\left[M + x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x}\right] \\ &\qquad + \frac{M}{(Mx+Ny)^2}\left[N + y\frac{\partial N}{\partial y} + x\frac{\partial M}{\partial y}\right] \stackrel{?}{=} 0. \qquad (88)\end{aligned}$$

Multiplying both sides of the above equation by $(Mx+Ny)^2$ followed by a slight algebraic simplification yields,

$$(Mx+Ny)\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) - N\left[x\frac{\partial M}{\partial x} + y\frac{\partial N}{\partial x}\right] + M\left[y\frac{\partial N}{\partial y} + x\frac{\partial M}{\partial y}\right] \stackrel{?}{=} 0. \qquad (89)$$

After some further algebraic simplification, we end up with

$$M\left(x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y}\right) - N\left(x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y}\right) \stackrel{?}{=} 0. \qquad (90)$$

By assumption, both $M(x,y)$ and $N(x,y)$ are homogeneous functions of x and y of the same degree (which we denote as n below). Using Euler's theorem given in Appendix C, it follows that

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = nM, \qquad x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = nN. \qquad (91)$$

Hence, eq. (90) yields

$$nMN - nNM = 0. \quad \checkmark$$

Thus, we have confirmed eq. (85), which means that $u(x,y) = (Mx+Ny)^{-1}$ is the correct integrating factor for $M(x,y)dx + N(x,y)$ if both $M(x,y)$ and $N(x,y)$ are homogeneous functions of x and y of degree n and $Mx + Ny \neq 0$.