Part IV

Ordinary Differential Equations

Chapter 14

First Order Differential Equations

Don't show me your technique. Show me your heart.

-Tetsuyasu Uekuma

14.1 Notation

A *differential equation* is an equation involving a function, it's derivatives, and independent variables. If there is only one independent variable, then it is an *ordinary differential equation*. Identities such as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f^2(x)\right) = 2f(x)f'(x), \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y} = 1$$

are not differential equations.

The *order* of a differential equation is the order of the highest derivative. The following equations for $y(x)$ are first, second and third order, respectively.

- $y' = xy^2$

- $y'' + 3xy' + 2y = x^2$
- $y''' = y''y$

The *degree* of a differential equation is the highest power of the highest derivative in the equation. The following equations are first, second and third degree, respectively.

- $y' - 3y^2 = \sin x$
- $(y'')^2 + 2x \cos y = \mathrm{e}^x$
- $(y')^3 + y^5 = 0$

An equation is said to be *linear* if it is linear in the dependent variable.

- $y'' \cos x + x^2 y = 0$ is a linear differential equation.
- $y' + xy^2 = 0$ is a nonlinear differential equation.

A differential equation is *homogeneous* if it has no terms that are functions of the independent variable alone. Thus an *inhomogeneous* equation is one in which there are terms that are functions of the independent variables alone.

- $y'' + xy + y = 0$ is a homogeneous equation.
- $y' + y + x^2 = 0$ is an inhomogeneous equation.

A first order differential equation may be written in terms of differentials. Recall that for the function $y(x)$ the differential $\mathrm{d}y$ is defined $\mathrm{d}y = y'(x)\,\mathrm{d}x$. Thus the differential equations

$$y' = x^2 y \quad \text{and} \quad y' + xy^2 = \sin(x)$$

can be denoted:

$$\mathrm{d}y = x^2 y\,\mathrm{d}x \quad \text{and} \quad \mathrm{d}y + xy^2\,\mathrm{d}x = \sin(x)\,\mathrm{d}x.$$

A *solution* of a differential equation is a function which when substituted into the equation yields an identity. For example, $y = x \ln|x|$ is a solution of

$$y' - \frac{y}{x} = 1.$$

We verify this by substituting it into the differential equation.

$$\ln|x| + 1 - \ln|x| = 1$$

We can also verify that $y = c\,\mathrm{e}^x$ is a solution of $y'' - y = 0$ for any value of the parameter c.

$$c\,\mathrm{e}^x - c\,\mathrm{e}^x = 0$$

14.2 Example Problems

In this section we will discuss physical and geometrical problems that lead to first order differential equations.

14.2.1 Growth and Decay

Example 14.2.1 *Consider a culture of bacteria in which each bacterium divides once per hour. Let $n(t) \in \mathcal{N}$ denote the population, let t denote the time in hours and let n_0 be the population at time $t = 0$. The population doubles every hour. Thus for integer t, the population is $n_0 2^t$. Figure 14.1 shows two possible populations when there is initially a single bacterium. In the first plot, each of the bacteria divide at times $t = m$ for $m \in \mathcal{N}$. In the second plot, they divide at times $t = m - 1/2$. For both plots the population is 2^t for integer t.*

We model this problem by considering a continuous population $y(t) \in \mathcal{R}$ which approximates the discrete population. In Figure 14.2 we first show the population when there is initially 8 bacteria. The divisions of bacteria is spread out over each one second interval. For integer t, the populations is $8 \cdot 2^t$. Next we show the population with a plot of the continuous function $y(t) = 8 \cdot 2^t$. We see that $y(t)$ is a reasonable approximation of the discrete population.

In the discrete problem, the growth of the population is proportional to its number; the population doubles every hour. For the continuous problem, we assume that this is true for $y(t)$. We write this as an equation:

$$y'(t) = \alpha y(t).$$

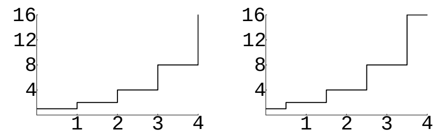

Figure 14.1: The population of bacteria.

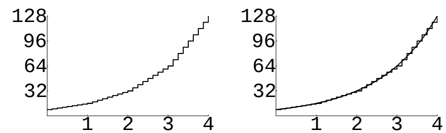

Figure 14.2: The discrete population of bacteria and a continuous population approximation.

That is, the rate of change $y'(t)$ *in the population is proportional to the population* $y(t)$*, (with constant of proportionality* α*). We specify the population at time* $t = 0$ *with the initial condition:* $y(0) = n_0$*. Note that* $y(t) = n_0 \, \mathrm{e}^{\alpha t}$ *satisfies the problem:*

$$y'(t) = \alpha y(t), \quad y(0) = n_0.$$

For our bacteria example, $\alpha = \ln 2$*.*

Result 14.2.1 A quantity $y(t)$ whose growth or decay is proportional to $y(t)$ is modelled by the problem:

$$y'(t) = \alpha y(t), \quad y(t_0) = y_0.$$

Here we assume that the quantity is known at time $t = t_0$. e^{α} is the factor by which the quantity grows/decays in unit time. The solution of this problem is $y(t) = y_0 \, \mathrm{e}^{\alpha(t-t_0)}$.

14.3 One Parameter Families of Functions

Consider the equation:

$$F(x, y(x), c) = 0, \tag{14.1}$$

which implicitly defines a one-parameter family of functions $y(x; c)$. Here y is a function of the variable x and the parameter c. For simplicity, we will write $y(x)$ and not explicitly show the parameter dependence.

Example 14.3.1 *The equation $y = cx$ defines family of lines with slope c, passing through the origin. The equation $x^2 + y^2 = c^2$ defines circles of radius c, centered at the origin.*

Consider a chicken dropped from a height h. The elevation y of the chicken at time t after its release is $y(t) = h - gt^2$, where g is the acceleration due to gravity. This is family of functions for the parameter h.

It turns out that the general solution of any first order differential equation is a one-parameter family of functions. This is not easy to prove. However, it is easy to verify the converse. We differentiate Equation 14.1 with respect to x.

$$F_x + F_y y' = 0$$

(We assume that F has a non-trivial dependence on y, that is $F_y \neq 0$.) This gives us two equations involving the independent variable x, the dependent variable $y(x)$ and its derivative and the parameter c. If we algebraically eliminate c between the two equations, the eliminant will be a first order differential equation for $y(x)$. Thus we see that every one-parameter family of functions $y(x)$ satisfies a first order differential equation. This $y(x)$ is the *primitive* of the differential equation. Later we will discuss why $y(x)$ is the *general solution* of the differential equation.

Example 14.3.2 *Consider the family of circles of radius c centered about the origin.*

$$x^2 + y^2 = c^2$$

Differentiating this yields:

$$2x + 2yy' = 0.$$

It is trivial to eliminate the parameter and obtain a differential equation for the family of circles.

$$x + yy' = 0$$

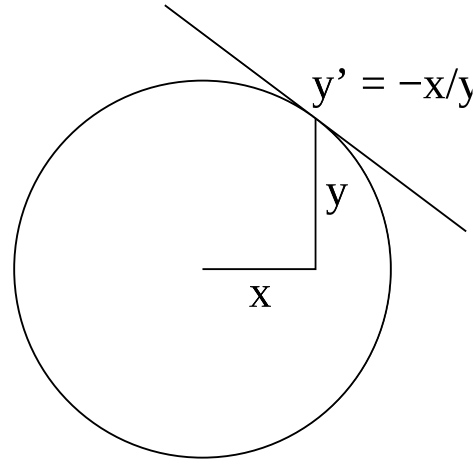

Figure 14.3: A circle and its tangent.

We can see the geometric meaning in this equation by writing it in the form:

$$y' = -\frac{x}{y}.$$

For a point on the circle, the slope of the tangent y' is the negative of the cotangent of the angle x/y. (See Figure 14.3.)

Example 14.3.3 *Consider the one-parameter family of functions:*

$$y(x) = f(x) + cg(x),$$

where $f(x)$ and $g(x)$ are known functions. The derivative is

$$y' = f' + cg'.$$

We eliminate the parameter.

$$gy' - g'y = gf' - g'f$$
$$y' - \frac{g'}{g}y = f' - \frac{g'f}{g}$$

Thus we see that $y(x) = f(x)+cg(x)$ satisfies a first order linear differential equation. Later we will prove the converse: the general solution of a first order linear differential equation has the form: $y(x) = f(x) + cg(x)$.

We have shown that every one-parameter family of functions satisfies a first order differential equation. We do not prove it here, but the converse is true as well.

Result 14.3.1 Every first order differential equation has a one-parameter family of solutions $y(x)$ defined by an equation of the form:

$$F(x, y(x); c) = 0.$$

This $y(x)$ is called the *general solution*. If the equation is linear then the general solution expresses the totality of solutions of the differential equation. If the equation is nonlinear, there may be other special *singular solutions*, which do not depend on a parameter.

This is strictly an existence result. It does not say that the general solution of a first order differential equation can be determined by some method, it just says that it exists. There is no method for solving the general first order differential equation. However, there are some special forms that are soluble. We will devote the rest of this chapter to studying these forms.

14.4 Integrable Forms

In this section we will introduce a few forms of differential equations that we may solve through integration.

14.4.1 Separable Equations

Any differential equation that can written in the form

$$P(x) + Q(y)y' = 0$$

is a *separable equation*, (because the dependent and independent variables are separated). We can obtain an implicit solution by integrating with respect to x.

$$\int P(x)\,\mathrm{d}x + \int Q(y)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = c$$
$$\int P(x)\,\mathrm{d}x + \int Q(y)\,\mathrm{d}y = c$$

Result 14.4.1 The separable equation $P(x) + Q(y)y' = 0$ may be solved by integrating with respect to x. The general solution is

$$\int P(x)\,\mathrm{d}x + \int Q(y)\,\mathrm{d}y = c.$$

Example 14.4.1 *Consider the differential equation $y' = xy^2$. We separate the dependent and independent variables* *and integrate to find the solution.*

$$
\begin{gathered}
\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 \\
y^{-2}\,\mathrm{d}y = x\,\mathrm{d}x \\
\int y^{-2}\,\mathrm{d}y = \int x\,\mathrm{d}x + c \\
-y^{-1} = \frac{x^2}{2} + c \\
\boxed{y = -\frac{1}{x^2/2 + c}}
\end{gathered}
$$

Example 14.4.2 *The equation $y' = y - y^2$ is separable.*

$$
\frac{y'}{y - y^2} = 1
$$

We expand in partial fractions and integrate.

$$
\begin{gathered}
\left(\frac{1}{y} - \frac{1}{y-1}\right) y' = 1 \\
\ln|y| - \ln|y-1| = x + c
\end{gathered}
$$

We have an implicit equation for $y(x)$. Now we solve for $y(x)$.

$$\ln\left|\frac{y}{y-1}\right| = x + c$$
$$\left|\frac{y}{y-1}\right| = \mathrm{e}^{x+c}$$
$$\frac{y}{y-1} = \pm\,\mathrm{e}^{x+c}$$
$$\frac{y}{y-1} = c\,\mathrm{e}^{x}\ 1$$
$$y = \frac{c\,\mathrm{e}^{x}}{c\,\mathrm{e}^{x} - 1}$$
$$\boxed{y = \frac{1}{1 + c\,\mathrm{e}^{x}}}$$

14.4.2 Exact Equations

Any first order ordinary differential equation of the first degree can be written as the total differential equation,

$$P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0.$$

If this equation can be integrated directly, that is if there is a primitive, $u(x, y)$, such that

$$du = P\,\mathrm{d}x + Q\,\mathrm{d}y,$$

then this equation is called *exact*. The (implicit) solution of the differential equation is

$$u(x, y) = c,$$

where c is an arbitrary constant. Since the differential of a function, $u(x, y)$, is

$$\mathrm{d}u \equiv \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y,$$

P and Q are the partial derivatives of u:

$$P(x,y) = \frac{\partial u}{\partial x}, \qquad Q(x,y) = \frac{\partial u}{\partial y}.$$

In an alternate notation, the differential equation

$$P(x,y) + Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \tag{14.2}$$

is exact if there is a primitive $u(x,y)$ such that

$$\frac{\mathrm{d}u}{\mathrm{d}x} \equiv \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = P(x,y) + Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The solution of the differential equation is $u(x,y) = c$.

Example 14.4.3

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is an exact differential equation since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}(x^2+y^2)\right) = x + y\frac{\mathrm{d}y}{\mathrm{d}x}$$

The solution of the differential equation is

$$\frac{1}{2}(x^2+y^2) = c.$$

Example 14.4.4 *, Let $f(x)$ and $g(x)$ be known functions.*

$$g(x)y' + g'(x)y = f(x)$$

is an exact differential equation since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)y(x)\right) = gy' + g'y.$$

The solution of the differential equation is

$$g(x)y(x) = \int f(x)\,\mathrm{d}x + c$$
$$y(x) = \frac{1}{g(x)} \int f(x)\,\mathrm{d}x + \frac{c}{g(x)}.$$

A necessary condition for exactness. The solution of the exact equation $P + Qy' = 0$ is $u = c$ where u is the primitive of the equation, $\frac{\mathrm{d}u}{\mathrm{d}x} = P + Qy'$. At present the only method we have for determining the primitive is guessing. This is fine for simple equations, but for more difficult cases we would like a method more concrete than divine inspiration. As a first step toward this goal we determine a criterion for determining if an equation is exact.

Consider the exact equation,

$$P + Qy' = 0,$$

with primitive u, where we assume that the functions P and Q are continuously differentiable. Since the mixed partial derivatives of u are equal,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x},$$

a necessary condition for exactness is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

A sufficient condition for exactness. This necessary condition for exactness is also a sufficient condition. We demonstrate this by deriving the general solution of (14.2). Assume that $P + Qy' = 0$ is not necessarily exact, but satisfies the condition $P_y = Q_x$. If the equation has a primitive,

$$\frac{\mathrm{d}u}{\mathrm{d}x} \equiv \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x},$$

then it satisfies

$$\frac{\partial u}{\partial x} = P, \qquad \frac{\partial u}{\partial y} = Q. \tag{14.3}$$

Integrating the first equation of (14.3), we see that the primitive has the form

$$u(x, y) = \int_{x_0}^{x} P(\xi, y)\,\mathrm{d}\xi + f(y),$$

for some $f(y)$. Now we substitute this form into the second equation of (14.3).

$$\frac{\partial u}{\partial y} = Q(x, y)$$
$$\int_{x_0}^{x} P_y(\xi, y)\,\mathrm{d}\xi + f'(y) = Q(x, y)$$

Now we use the condition $P_y = Q_x$.

$$\int_{x_0}^{x} Q_x(\xi, y)\,\mathrm{d}\xi + f'(y) = Q(x, y)$$
$$Q(x, y) - Q(x_0, y) + f'(y) = Q(x, y)$$
$$f'(y) = Q(x_0, y)$$
$$f(y) = \int_{y_0}^{y} Q(x_0, \psi)\,\mathrm{d}\psi$$

Thus we see that

$$u = \int_{x_0}^{x} P(\xi, y)\,\mathrm{d}\xi + \int_{y_0}^{y} Q(x_0, \psi)\,\mathrm{d}\psi$$

is a primitive of the derivative; the equation is exact. The solution of the differential equation is

$$\int_{x_0}^{x} P(\xi, y)\,\mathrm{d}\xi + \int_{y_0}^{y} Q(x_0, \psi)\,\mathrm{d}\psi = c.$$

Even though there are three arbitrary constants: x_0, y_0 and c, the solution is a one-parameter family. This is because changing x_0 or y_0 only changes the left side by an additive constant.

Result 14.4.2 Any first order differential equation of the first degree can be written in the form

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

This equation is exact if and only if

$$P_y = Q_x.$$

In this case the solution of the differential equation is given by

$$\int_{x_0}^{x} P(\xi, y)\,\mathrm{d}\xi + \int_{y_0}^{y} Q(x_0, \psi)\,\mathrm{d}\psi = c.$$

Exercise 14.1
Solve the following differential equations by inspection. That is, group terms into exact derivatives and then integrate. $f(x)$ and $g(x)$ are known functions.

1. $\frac{y'(x)}{y(x)} = f(x)$
2. $y^{\alpha}(x)y'(x) = f(x)$
3. $\frac{y'}{\cos x} + y\frac{\tan x}{\cos x} = \cos x$

Hint, Solution

14.4.3 Homogeneous Coefficient Equations

Homogeneous coefficient, first order differential equations form another class of soluble equations. We will find that a change of dependent variable will make such equations separable or we can determine an integrating factor that will make such equations exact. First we define homogeneous functions.

Euler's Theorem on Homogeneous Functions. The function $F(x, y)$ is *homogeneous of degree* n if

$$F(\lambda x, \lambda y) = \lambda^n F(x, y).$$

From this definition we see that

$$F(x, y) = x^n F\left(1, \frac{y}{x}\right).$$

(Just formally substitute $1/x$ for λ.) For example,

$$xy^2, \qquad \frac{x^2 y + 2y^3}{x + y}, \qquad x \cos(y/x)$$

are homogeneous functions of orders 3, 2 and 1, respectively.

Euler's theorem for a homogeneous function of order n is:

$$xF_x + yF_y = nF.$$

To prove this, we define $\xi = \lambda x$, $\psi = \lambda y$. From the definition of homogeneous functions, we have

$$F(\xi, \psi) = \lambda^n F(x, y).$$

We differentiate this equation with respect to λ.

$$\frac{\partial F(\xi, \psi)}{\partial \xi} \frac{\partial \xi}{\partial \lambda} + \frac{\partial F(\xi, \psi)}{\partial \psi} \frac{\partial \psi}{\partial \lambda} = n\lambda^{n-1} F(x, y)$$
$$xF_\xi + yF_\psi = n\lambda^{n-1} F(x, y)$$

Setting $\lambda = 1$, (and hence $\xi = x$, $\psi = y$), proves Euler's theorem.

Result 14.4.3 Euler's Theorem on Homogeneous Functions. If $F(x, y)$ is a homogeneous function of degree n, then

$$xF_x + yF_y = nF.$$

Homogeneous Coefficient Differential Equations. If the coefficient functions $P(x, y)$ and $Q(x, y)$ are homogeneous of degree n then the differential equation,

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \tag{14.4}$$

is called a *homogeneous coefficient equation*. They are often referred to simply as *homogeneous equations*.

Transformation to a Separable Equation. We can write the homogeneous equation in the form,

$$\begin{gathered} x^n P\left(1, \frac{y}{x}\right) + x^n Q\left(1, \frac{y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \\ P\left(1, \frac{y}{x}\right) + Q\left(1, \frac{y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0. \end{gathered}$$

This suggests the change of dependent variable $u(x) = \frac{y(x)}{x}$.

$$P(1, u) + Q(1, u)\left(u + x\frac{\mathrm{d}u}{\mathrm{d}x}\right) = 0$$

This equation is separable.

$$\begin{gathered} P(1, u) + uQ(1, u) + xQ(1, u)\frac{\mathrm{d}u}{\mathrm{d}x} = 0 \\ \frac{1}{x} + \frac{Q(1, u)}{P(1, u) + uQ(1, u)}\frac{\mathrm{d}u}{\mathrm{d}x} = 0 \\ \ln|x| + \int \frac{1}{u + P(1, u)/Q(1, u)}\,\mathrm{d}u = c \end{gathered}$$

By substituting $\ln|c|$ for c, we can write this in a simpler form.

$$\int \frac{1}{u + P(1, u)/Q(1, u)}\,\mathrm{d}u = \ln\left|\frac{c}{x}\right|.$$

Integrating Factor. One can show that

$$\mu(x, y) = \frac{1}{xP(x, y) + yQ(x, y)}$$

is an integrating factor for the Equation 14.4. The proof of this is left as an exercise for the reader. (See Exercise 14.2.)

Result 14.4.4 Homogeneous Coefficient Differential Equations. If $P(x, y)$ and $Q(x, y)$ are homogeneous functions of degree n, then the equation

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is made separable by the change of independent variable $u(x) = \frac{y(x)}{x}$. The solution is determined by

$$\int \frac{1}{u + P(1, u)/Q(1, u)}\,\mathrm{d}u = \ln\left|\frac{c}{x}\right|.$$

Alternatively, the homogeneous equation can be made exact with the integrating factor

$$\mu(x, y) = \frac{1}{xP(x, y) + yQ(x, y)}.$$

Example 14.4.5 *Consider the homogeneous coefficient equation*

$$x^2 - y^2 + xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

The solution for $u(x) = y(x)/x$ is determined by

$$\int \frac{1}{u + \frac{1-u^2}{u}} \, \mathrm{d}u = \ln\left|\frac{c}{x}\right|$$

$$\int u \, \mathrm{d}u = \ln\left|\frac{c}{x}\right|$$

$$\frac{1}{2}u^2 = \ln\left|\frac{c}{x}\right|$$

$$u = \pm\sqrt{2\ln|c/x|}$$

Thus the solution of the differential equation is

$$\boxed{y = \pm x\sqrt{2\ln|c/x|}}$$

Exercise 14.2
Show that

$$\mu(x, y) = \frac{1}{xP(x, y) + yQ(x, y)}$$

is an integrating factor for the homogeneous equation,

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hint, Solution

Exercise 14.3 (mathematica/ode/first_order/exact.nb)
Find the general solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2\frac{y}{t} + \left(\frac{y}{t}\right)^2.$$

Hint, Solution

14.5 The First Order, Linear Differential Equation

14.5.1 Homogeneous Equations

The first order, linear, homogeneous equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0.$$

Note that if we can find one solution, then any constant times that solution also satisfies the equation. If fact, all the solutions of this equation differ only by multiplicative constants. We can solve any equation of this type because it is separable.

$$\frac{y'}{y} = -p(x)$$
$$\ln|y| = -\int p(x)\,\mathrm{d}x + c$$
$$y = \pm\,\mathrm{e}^{-\int p(x)\,\mathrm{d}x + c}$$
$$y = c\,\mathrm{e}^{-\int p(x)\,\mathrm{d}x}$$

Result 14.5.1 First Order, Linear Homogeneous Differential Equations. The first order, linear, homogeneous differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0,$$

has the solution

$$y = c\,\mathrm{e}^{-\int p(x)\,\mathrm{d}x}. \tag{14.5}$$

The solutions differ by multiplicative constants.

Example 14.5.1 *Consider the equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x} y = 0.$$

We use Equation 14.5 to determine the solution.

$$y(x) = c\,\mathrm{e}^{-\int 1/x\,\mathrm{d}x}, \quad \textit{for } x \neq 0$$
$$y(x) = c\,\mathrm{e}^{-\ln|x|}$$
$$y(x) = \frac{c}{|x|}$$
$$\boxed{y(x) = \frac{c}{x}}$$

14.5.2 Inhomogeneous Equations

The first order, linear, inhomogeneous differential equation has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x). \tag{14.6}$$

This equation is not separable. Note that it is similar to the exact equation we solved in Example 14.4.4,

$$g(x)y'(x) + g'(x)y(x) = f(x).$$

To solve Equation 14.6, we multiply by an *integrating factor*. Multiplying a differential equation by its integrating factor changes it to an exact equation. Multiplying Equation 14.6 by the function, $I(x)$, yields,

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)I(x)y = f(x)I(x).$$

In order that $I(x)$ be an integrating factor, it must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}x} I(x) = p(x)I(x).$$

This is a first order, linear, homogeneous equation with the solution

$$I(x) = c\,\mathrm{e}^{\int p(x)\,\mathrm{d}x}.$$

This is an integrating factor for any constant c. For simplicity we will choose $c = 1$.

To solve Equation 14.6 we multiply by the integrating factor and integrate. Let $P(x) = \int p(x)\,\mathrm{d}x$.

$$\begin{gathered}
\mathrm{e}^{P(x)}\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)\,\mathrm{e}^{P(x)}\,y = \mathrm{e}^{P(x)}\,f(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{P(x)}\,y\right) = \mathrm{e}^{P(x)}\,f(x) \\
y = \mathrm{e}^{-P(x)}\int \mathrm{e}^{P(x)}\,f(x)\,\mathrm{d}x + c\,\mathrm{e}^{-P(x)} \\
y \equiv y_p + c\,y_h
\end{gathered}$$

Note that the *general solution* is the sum of a *particular solution*, y_p, that satisfies $y' + p(x)y = f(x)$, and an arbitrary constant times a *homogeneous solution*, y_h, that satisfies $y' + p(x)y = 0$.

Example 14.5.2 *Consider the differential equation*

$$y' + \frac{1}{x}y = x^2, \quad x > 0.$$

First we find the integrating factor.

$$I(x) = \exp\left(\int \frac{1}{x}\,\mathrm{d}x\right) = \mathrm{e}^{\ln x} = x$$

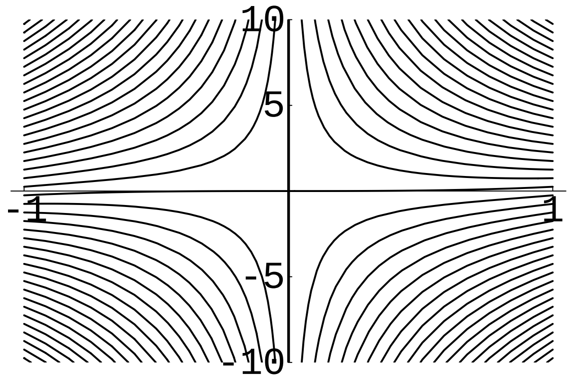

Figure 14.4: Solutions to $y' + y/x = x^2$.

We multiply by the integrating factor and integrate.

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x^3$$
$$xy = \frac{1}{4}x^4 + c$$
$$\boxed{y = \frac{1}{4}x^3 + \frac{c}{x}.}$$

The particular and homogeneous solutions are

$$y_p = \frac{1}{4}x^3 \qquad \text{and} \qquad y_h = \frac{1}{x}.$$

Note that the general solution to the differential equation is a one-parameter family of functions. The general solution is plotted in Figure 14.4 for various values of c.

Exercise 14.4 (mathematica/ode/first_order/linear.nb)
Solve the differential equation

$$y' - \frac{1}{x} y = x^\alpha, \quad x > 0.$$

Hint, Solution

14.5.3 Variation of Parameters.

We could also have found the particular solution with the method of variation of parameters. Although we can solve first order equations without this method, it will become important in the study of higher order inhomogeneous equations. We begin by assuming that the particular solution has the form $y_p = u(x) y_h(x)$ where $u(x)$ is an unknown function. We substitute this into the differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}x} y_p + p(x) y_p = f(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(u y_h) + p(x) u y_h = f(x)$$
$$u' y_h + u(y_h' + p(x) y_h) = f(x)$$

Since y_h is a homogeneous solution, $y_h' + p(x) y_h = 0$.

$$u' = \frac{f(x)}{y_h}$$
$$u = \int \frac{f(x)}{y_h(x)} \, \mathrm{d}x$$

Recall that the homogeneous solution is $y_h = \mathrm{e}^{-P(x)}$.

$$u = \int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x$$

Thus the particular solution is

$$y_p = \mathrm{e}^{-P(x)} \int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x.$$

14.6 Initial Conditions

In physical problems involving first order differential equations, the solution satisfies both the differential equation and a constraint which we call the *initial condition*. Consider a first order linear differential equation subject to the initial condition $y(x_0) = y_0$. The general solution is

$$y = y_p + c y_h = \mathrm{e}^{-P(x)} \int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x + c \, \mathrm{e}^{-P(x)} .$$

For the moment, we will assume that this problem is *well-posed*. A problem is well-posed if there is a unique solution to the differential equation that satisfies the constraint(s). Recall that $\int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x$ denotes any integral of $\mathrm{e}^{P(x)} f(x)$. For convenience, we choose $\int_{x_0}^{x} \mathrm{e}^{P(\xi)} f(\xi) \, \mathrm{d}\xi$. The initial condition requires that

$$y(x_0) = y_0 = \mathrm{e}^{-P(x_0)} \int_{x_0}^{x_0} \mathrm{e}^{P(\xi)} f(\xi) \, \mathrm{d}\xi + c \, \mathrm{e}^{-P(x_0)} = c \, \mathrm{e}^{-P(x_0)} .$$

Thus $c = y_0 \, \mathrm{e}^{P(x_0)}$. The solution subject to the initial condition is

$$y = \mathrm{e}^{-P(x)} \int_{x_0}^{x} \mathrm{e}^{P(\xi)} f(\xi) \, \mathrm{d}\xi + y_0 \, \mathrm{e}^{P(x_0) - P(x)} .$$

Example 14.6.1 *Consider the problem*

$$y' + (\cos x) y = x, \qquad y(0) = 2.$$

From Result 14.6.1, the solution subject to the initial condition is

$$\boxed{y = \mathrm{e}^{-\sin x} \int_0^x \xi \, \mathrm{e}^{\sin \xi} \, \mathrm{d}\xi + 2 \, \mathrm{e}^{-\sin x} .}$$

14.6.1 Piecewise Continuous Coefficients and Inhomogeneities

If the coefficient function $p(x)$ and the inhomogeneous term $f(x)$ in the first order linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x)$$

are continuous, then the solution is continuous and has a continuous first derivative. To see this, we note that the solution

$$y = \mathrm{e}^{-P(x)} \int \mathrm{e}^{P(x)} f(x)\,\mathrm{d}x + c\,\mathrm{e}^{-P(x)}$$

is continuous since the integral of a piecewise continuous function is continuous. The first derivative of the solution can be found directly from the differential equation.

$$y' = -p(x)y + f(x)$$

Since $p(x)$, y, and $f(x)$ are continuous, y' is continuous.

If $p(x)$ or $f(x)$ is only piecewise continuous, then the solution will be continuous since the integral of a piecewise continuous function is continuous. The first derivative of the solution will be piecewise continuous.

Example 14.6.2 *Consider the problem*

$$y' - y = H(x-1), \qquad y(0) = 1,$$

where $H(x)$ is the Heaviside function.

$$H(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

To solve this problem, we divide it into two equations on separate domains.

$$\begin{aligned} y_1' - y_1 &= 0, \qquad & y_1(0) &= 1, \qquad & &\text{for } x < 1 \\ y_2' - y_2 &= 1, \qquad & y_2(1) &= y_1(1), \qquad & &\text{for } x > 1 \end{aligned}$$

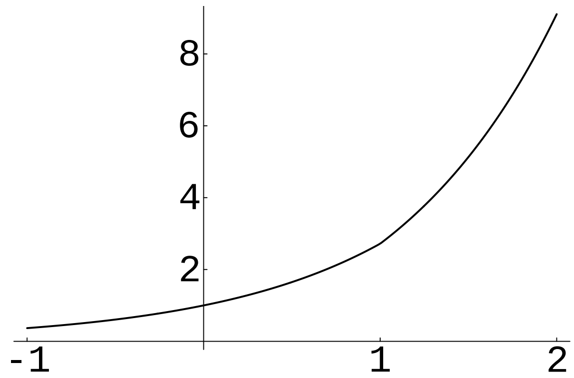

Figure 14.5: Solution to $y' - y = H(x-1)$.

With the condition $y_2(1) = y_1(1)$ on the second equation, we demand that the solution be continuous. The solution to the first equation is $y = \mathrm{e}^x$. The solution for the second equation is

$$y = \mathrm{e}^x \int_1^x \mathrm{e}^{-\xi}\,\mathrm{d}\xi + \mathrm{e}^1\,\mathrm{e}^{x-1} = -1 + \mathrm{e}^{x-1} + \mathrm{e}^x\,.$$

Thus the solution over the whole domain is

$$\boxed{y = \begin{cases} \mathrm{e}^x & \text{for } x < 1, \\ (1 + \mathrm{e}^{-1})\,\mathrm{e}^x - 1 & \text{for } x > 1. \end{cases}}$$

The solution is graphed in Figure 14.5.

Example 14.6.3 *Consider the problem,*

$$y' + \operatorname{sign}(x) y = 0, \qquad y(1) = 1.$$

Recall that

$$\operatorname{sign} x = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0. \end{cases}$$

Since $\operatorname{sign} x$ *is piecewise defined, we solve the two problems,*

$$\begin{aligned} y_+' + y_+ &= 0, \qquad y_+(1) = 1, \qquad && \text{for } x > 0 \\ y_-' - y_- &= 0, \qquad y_-(0) = y_+(0), \qquad && \text{for } x < 0, \end{aligned}$$

and define the solution, y*, to be*

$$y(x) = \begin{cases} y_+(x), & \text{for } x \geq 0, \\ y_-(x), & \text{for } x \leq 0. \end{cases}$$

The initial condition for y_- *demands that the solution be continuous.*

Solving the two problems for positive and negative x*, we obtain*

$$y(x) = \begin{cases} \mathrm{e}^{1-x}, & \text{for } x > 0, \\ \mathrm{e}^{1+x}, & \text{for } x < 0. \end{cases}$$

This can be simplified to

$$\boxed{y(x) = \mathrm{e}^{1-|x|}.}$$

This solution is graphed in Figure 14.6.

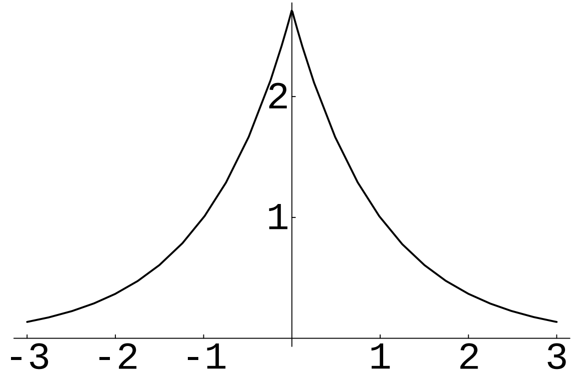

Figure 14.6: Solution to $y' + \operatorname{sign}(x) y = 0$.

Result 14.6.1 Existence, Uniqueness Theorem. Let $p(x)$ and $f(x)$ be piecewise continuous on the interval $[a, b]$ and let $x_0 \in [a, b]$. Consider the problem,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x) y = f(x), \qquad y(x_0) = y_0.$$

The general solution of the differential equation is

$$y = \mathrm{e}^{-P(x)} \int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x + c \, \mathrm{e}^{-P(x)} .$$

The unique, continuous solution of the differential equation subject to the initial condition is

$$y = \mathrm{e}^{-P(x)} \int_{x_0}^{x} \mathrm{e}^{P(\xi)} f(\xi) \, \mathrm{d}\xi + y_0 \, \mathrm{e}^{P(x_0) - P(x)},$$

where $P(x) = \int p(x) \, \mathrm{d}x$.

Exercise 14.5 (mathematica/ode/first_order/exact.nb)
Find the solutions of the following differential equations which satisfy the given initial conditions:

1. $\frac{\mathrm{d}y}{\mathrm{d}x} + xy = x^{2n+1}, \quad y(1) = 1, \quad n \in \mathbb{Z}$
2. $\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy = 1, \quad y(0) = 1$

Hint, Solution

Exercise 14.6 (mathematica/ode/first_order/exact.nb)
Show that if $\alpha > 0$ and $\lambda > 0$, then for any real β, every solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \alpha y(x) = \beta \,\mathrm{e}^{-\lambda x}$$

satisfies $\lim_{x \to +\infty} y(x) = 0$. (The case $\alpha = \lambda$ requires special treatment.) Find the solution for $\beta = \lambda = 1$ which satisfies $y(0) = 1$. Sketch this solution for $0 \leq x < \infty$ for several values of α. In particular, show what happens when $\alpha \to 0$ and $\alpha \to \infty$.
Hint, Solution

14.7 Well-Posed Problems

Example 14.7.1 *Consider the problem,*

$$y' - \frac{1}{x} y = 0, \qquad y(0) = 1.$$

The general solution is $y = cx$. Applying the initial condition demands that $1 = c \cdot 0$, which cannot be satisfied. The general solution for various values of c is plotted in Figure 14.7.

Example 14.7.2 *Consider the problem*

$$y' - \frac{1}{x} y = -\frac{1}{x}, \qquad y(0) = 1.$$

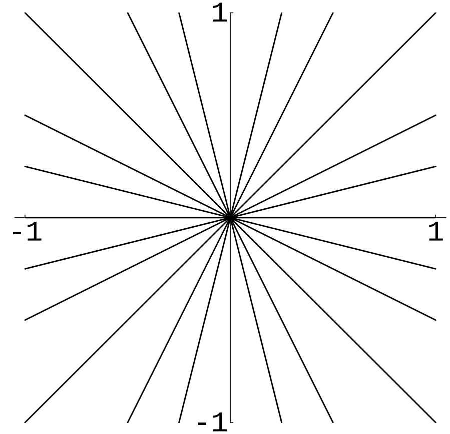

Figure 14.7: Solutions to $y' - y/x = 0$.

The general solution is

$$y = 1 + cx.$$

The initial condition is satisfied for any value of c so there are an infinite number of solutions.

Example 14.7.3 *Consider the problem*

$$y' + \frac{1}{x}y = 0, \qquad y(0) = 1.$$

The general solution is $y = \frac{c}{x}$. Depending on whether c is nonzero, the solution is either singular or zero at the origin and cannot satisfy the initial condition.

The above problems in which there were either no solutions or an infinite number of solutions are said to be *ill-posed*. If there is a unique solution that satisfies the initial condition, the problem is said to be *well-posed*. We should have suspected that we would run into trouble in the above examples as the initial condition was given at a singularity of the coefficient function, $p(x) = 1/x$.

Consider the problem,

$$y' + p(x)y = f(x), \qquad y(x_0) = y_0.$$

We assume that $f(x)$ bounded in a neighborhood of $x = x_0$. The differential equation has the general solution,

$$y = \mathrm{e}^{-P(x)} \int \mathrm{e}^{P(x)} f(x) \, \mathrm{d}x + c \, \mathrm{e}^{-P(x)}.$$

If the homogeneous solution, $\mathrm{e}^{-P(x)}$, is nonzero and finite at $x = x_0$, then there is a unique value of c for which the initial condition is satisfied. If the homogeneous solution vanishes at $x = x_0$ then either the initial condition cannot be satisfied or the initial condition is satisfied for all values of c. The homogeneous solution can vanish or be infinite only if $P(x) \to \pm\infty$ as $x \to x_0$. This can occur only if the coefficient function, $p(x)$, is unbounded at that point.

Result 14.7.1 If the initial condition is given where the homogeneous solution to a first order, linear differential equation is zero or infinite then the problem may be ill-posed. This may occur only if the coefficient function, $p(x)$, is unbounded at that point.

14.8 Equations in the Complex Plane

14.8.1 Ordinary Points

Consider the first order homogeneous equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0,$$

where $p(z)$, a function of a complex variable, is analytic in some domain D. The integrating factor,

$$I(z) = \exp\left(\int p(z) \, \mathrm{d}z\right),$$

is an analytic function in that domain. As with the case of real variables, multiplying by the integrating factor and integrating yields the solution,

$$w(z) = c \exp\left(-\int p(z)\,\mathrm{d}z\right).$$

We see that the solution is analytic in D.

Example 14.8.1 *It does not make sense to pose the equation*

$$\frac{\mathrm{d}w}{\mathrm{d}z} + |z|w = 0.$$

For the solution to exist, w and hence $w'(z)$ must be analytic. Since $p(z) = |z|$ is not analytic anywhere in the complex plane, the equation has no solution.

Any point at which $p(z)$ is analytic is called an *ordinary point* of the differential equation. Since the solution is analytic we can expand it in a Taylor series about an ordinary point. The radius of convergence of the series will be at least the distance to the nearest singularity of $p(z)$ in the complex plane.

Example 14.8.2 *Consider the equation*

$$\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{1}{1-z}w = 0.$$

The general solution is $w = \frac{c}{1-z}$. Expanding this solution about the origin,

$$w = \frac{c}{1-z} = c\sum_{n=0}^{\infty} z^n.$$

The radius of convergence of the series is,

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1,$$

which is the distance from the origin to the nearest singularity of $p(z) = \frac{1}{1-z}$.

We do not need to solve the differential equation to find the Taylor series expansion of the homogeneous solution. We could substitute a general Taylor series expansion into the differential equation and solve for the coefficients. Since we can always solve first order equations, this method is of limited usefulness. However, when we consider higher order equations in which we cannot solve the equations exactly, this will become an important method.

Example 14.8.3 *Again consider the equation*

$$\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{1}{1-z} w = 0.$$

Since we know that the solution has a Taylor series expansion about $z = 0$*, we substitute* $w = \sum_{n=0}^{\infty} a_n z^n$ *into the differential equation.*

$$(1-z)\frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=1}^{\infty} n a_n z^{n-1} - \sum_{n=1}^{\infty} n a_n z^n - \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n - \sum_{n=0}^{\infty} n a_n z^n - \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+1) a_{n+1} - (n+1) a_n\right) z^n = 0.$$

Now we equate powers of z *to zero. For* z^n*, the equation is* $(n+1)a_{n+1} - (n+1)a_n = 0$*, or* $a_{n+1} = a_n$*. Thus we have that* $a_n = a_0$ *for all* $n \geq 1$*. The solution is then*

$$\boxed{w = a_0 \sum_{n=0}^{\infty} z^n,}$$

which is the result we obtained by expanding the solution in Example 14.8.2.

Result 14.8.1 Consider the equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0.$$

If $p(z)$ is analytic at $z = z_0$ then z_0 is called an ordinary point of the differential equation. The Taylor series expansion of the solution can be found by substituting $w = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ into the equation and equating powers of $(z - z_0)$. The radius of convergence of the series is at least the distance to the nearest singularity of $p(z)$ in the complex plane.

Exercise 14.7
Find the Taylor series expansion about the origin of the solution to

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{1-z} w = 0$$

with the substitution $w = \sum_{n=0}^{\infty} a_n z^n$. What is the radius of convergence of the series? What is the distance to the nearest singularity of $\frac{1}{1-z}$?
Hint, Solution

14.8.2 Regular Singular Points

If the coefficient function $p(z)$ has a simple pole at $z = z_0$ then z_0 is a *regular singular point* of the first order differential equation.

Example 14.8.4 *Consider the equation*

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha}{z} w = 0, \qquad \alpha \neq 0.$$

This equation has a regular singular point at $z = 0$. The solution is $w = cz^{-\alpha}$. Depending on the value of α, the solution can have three different kinds of behavior.

α is a negative integer. *The solution is analytic in the finite complex plane.*

α is a positive integer *The solution has a pole at the origin. w is analytic in the annulus, $0 < |z|$.*

α is not an integer. *w has a branch point at $z = 0$. The solution is analytic in the cut annulus $0 < |z| < \infty$, $\theta_0 < \arg z < \theta_0 + 2\pi$.*

Consider the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0,$$

where $p(z)$ has a simple pole at the origin and is analytic in the annulus, $0 < |z| < r$, for some positive r. Recall that the solution is

$$\begin{aligned} w &= c \exp\left(-\int p(z)\,\mathrm{d}z\right) \\ &= c \exp\left(-\int \frac{b_0}{z} + p(z) - \frac{b_0}{z}\,\mathrm{d}z\right) \\ &= c \exp\left(-b_0 \log z - \int \frac{zp(z) - b_0}{z}\,\mathrm{d}z\right) \\ &= cz^{-b_0} \exp\left(-\int \frac{zp(z) - b_0}{z}\,\mathrm{d}z\right) \end{aligned}$$

The exponential factor has a removable singularity at $z = 0$ and is analytic in $|z| < r$. We consider the following cases for the z^{-b_0} factor:

$\mathbf{b_0}$ is a negative integer. Since z^{-b_0} is analytic at the origin, the solution to the differential equation is analytic in the circle $|z| < r$.

$\mathbf{b_0}$ is a positive integer. The solution has a pole of order $-b_0$ at the origin and is analytic in the annulus $0 < |z| < r$.

$\mathbf{b_0}$ is not an integer. The solution has a branch point at the origin and thus is not single-valued. The solution is analytic in the cut annulus $0 < |z| < r$, $\theta_0 < \arg z < \theta_0 + 2\pi$.

Since the exponential factor has a convergent Taylor series in $|z| < r$, the solution can be expanded in a series of the form

$$w = z^{-b_0} \sum_{n=0}^{\infty} a_n z^n, \quad \text{where } a_0 \neq 0 \text{ and } b_0 = \lim_{z \to 0} z\, p(z).$$

In the case of a regular singular point at $z = z_0$, the series is

$$w = (z - z_0)^{-b_0} \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \text{where } a_0 \neq 0 \text{ and } b_0 = \lim_{z \to z_0} (z - z_0)\, p(z).$$

Series of this form are known as *Frobenius series*. Since we can write the solution as

$$w = c(z - z_0)^{-b_0} \exp\left(- \int \left(p(z) - \frac{b_0}{z - z_0} \right) \mathrm{d}z \right),$$

we see that the Frobenius expansion of the solution will have a radius of convergence at least the distance to the nearest singularity of $p(z)$.

Result 14.8.2 Consider the equation,

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0,$$

where $p(z)$ has a simple pole at $z = z_0$, $p(z)$ is analytic in some annulus, $0 < |z - z_0| < r$, and $\lim_{z\to z_0}(z - z_0)p(z) = \beta$. The solution to the differential equation has a Frobenius series expansion of the form

$$w = (z - z_0)^{-\beta} \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_0 \neq 0.$$

The radius of convergence of the expansion will be at least the distance to the nearest singularity of $p(z)$.

Example 14.8.5 *We will find the first two nonzero terms in the series solution about $z = 0$ of the differential equation,*

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{\sin z} w = 0.$$

First we note that the coefficient function has a simple pole at $z = 0$ and

$$\lim_{z\to 0} \frac{z}{\sin z} = \lim_{z\to 0} \frac{1}{\cos z} = 1.$$

Thus we look for a series solution of the form

$$w = z^{-1} \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 \neq 0.$$

The nearest singularities of $1/\sin z$ in the complex plane are at $z = \pm\pi$. Thus the radius of convergence of the series will be at least π.

Substituting the first three terms of the expansion into the differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}z}(a_0 z^{-1} + a_1 + a_2 z) + \frac{1}{\sin z}(a_0 z^{-1} + a_1 + a_2 z) = O(z).$$

Recall that the Taylor expansion of $\sin z$ *is* $\sin z = z - \frac{1}{6}z^3 + O(z^5)$.

$$\left(z - \frac{z^3}{6} + O(z^5)\right)(-a_0 z^{-2} + a_2) + (a_0 z^{-1} + a_1 + a_2 z) = O(z^2)$$

$$-a_0 z^{-1} + \left(a_2 + \frac{a_0}{6}\right) z + a_0 z^{-1} + a_1 + a_2 z = O(z^2)$$

$$a_1 + \left(2a_2 + \frac{a_0}{6}\right) z = O(z^2)$$

a_0 *is arbitrary. Equating powers of* z,

$$z^0: \qquad a_1 = 0.$$

$$z^1: \qquad 2a_2 + \frac{a_0}{6} = 0.$$

Thus the solution has the expansion,

$$\boxed{w = a_0\left(z^{-1} - \frac{z}{12}\right) + O(z^2).}$$

In Figure 14.8 the exact solution is plotted in a solid line and the two term approximation is plotted in a dashed line. The two term approximation is very good near the point $x = 0$.

Example 14.8.6 *Find the first two nonzero terms in the series expansion about* $z = 0$ *of the solution to*

$$w' - i\frac{\cos z}{z} w = 0.$$

Since $\frac{\cos z}{z}$ *has a simple pole at* $z = 0$ *and* $\lim_{z\to 0} -i\cos z = -i$ *we see that the Frobenius series will have the form*

$$w = z^i \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 \neq 0.$$

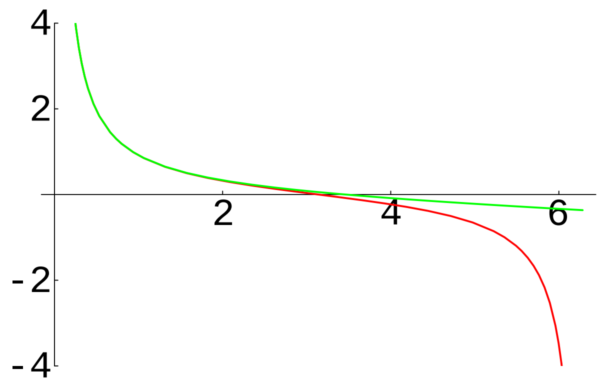

Figure 14.8: Plot of the exact solution and the two term approximation.

Recall that $\cos z$ *has the Taylor expansion* $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$. *Substituting the Frobenius expansion into the differential equation yields*

$$z\left(iz^{i-1}\sum_{n=0}^{\infty} a_n z^n + z^i \sum_{n=0}^{\infty} n a_n z^{n-1}\right) - i\left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right)\left(z^i \sum_{n=0}^{\infty} a_n z^n\right) = 0$$

$$\sum_{n=0}^{\infty}(n+i)a_n z^n - i\left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} a_n z^n\right) = 0.$$

Equating powers of z,

$$\begin{aligned}
z^0 &: \quad ia_0 - ia_0 = 0 && \to a_0 \text{ is arbitrary} \\
z^1 &: \quad (1+i)a_1 - ia_1 = 0 && \to a_1 = 0 \\
z^2 &: \quad (2+i)a_2 - ia_2 + \frac{i}{2}a_0 = 0 && \to a_2 = -\frac{i}{4}a_0.
\end{aligned}$$

Thus the solution is

$$\boxed{w = a_0 z^i \left(1 - \frac{i}{4} z^2 + O(z^3)\right).}$$

14.8.3 Irregular Singular Points

If a point is not an ordinary point or a regular singular point then it is called an *irregular singular point*. The following equations have irregular singular points at the origin.

- $w' + \sqrt{z} w = 0$
- $w' - z^{-2} w = 0$
- $w' + \exp(1/z) w = 0$

Example 14.8.7 *Consider the differential equation*

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \alpha z^\beta w = 0, \qquad \alpha \neq 0, \quad \beta \neq -1, 0, 1, 2, \ldots$$

This equation has an irregular singular point at the origin. Solving this equation,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\exp\left(\int \alpha z^\beta \, \mathrm{d}z\right) w\right) = 0$$

$$\boxed{w = c \exp\left(-\frac{\alpha}{\beta + 1} z^{\beta+1}\right) = c \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\alpha}{\beta + 1}\right)^n z^{(\beta+1)n}.}$$

If β is not an integer, then the solution has a branch point at the origin. If β is an integer, $\beta < -1$, then the solution has an essential singularity at the origin. The solution cannot be expanded in a Frobenius series, $w = z^\lambda \sum_{n=0}^{\infty} a_n z^n$.

Although we will not show it, this result holds for any irregular singular point of the differential equation. We cannot approximate the solution near an irregular singular point using a Frobenius expansion.

Now would be a good time to summarize what we have discovered about solutions of first order differential equations in the complex plane.

Result 14.8.3 Consider the first order differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0.$$

Ordinary Points If $p(z)$ is analytic at $z = z_0$ then z_0 is an ordinary point of the differential equation. The solution can be expanded in the Taylor series $w = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. The radius of convergence of the series is at least the distance to the nearest singularity of $p(z)$ in the complex plane.

Regular Singular Points If $p(z)$ has a simple pole at $z = z_0$ and is analytic in some annulus $0 < |z - z_0| < r$ then z_0 is a regular singular point of the differential equation. The solution at z_0 will either be analytic, have a pole, or have a branch point. The solution can be expanded in the Frobenius series $w = (z - z_0)^{-\beta} \sum_{n=0}^{\infty} a_n (z - z_0)^n$ where $a_0 \neq 0$ and $\beta = \lim_{z \to z_0} (z - z_0) p(z)$. The radius of convergence of the Frobenius series will be at least the distance to the nearest singularity of $p(z)$.

Irregular Singular Points If the point $z = z_0$ is not an ordinary point or a regular singular point, then it is an irregular singular point of the differential equation. The solution cannot be expanded in a Frobenius series about that point.

14.8.4 The Point at Infinity

Now we consider the behavior of first order linear differential equations at the point at infinity. Recall from complex variables that the complex plane together with the point at infinity is called the extended complex plane. To study the behavior of a function $f(z)$ at infinity, we make the transformation $z = \frac{1}{\zeta}$ and study the behavior of $f(1/\zeta)$ at $\zeta = 0$.

Example 14.8.8 *Let's examine the behavior of* $\sin z$ *at infinity. We make the substitution* $z = 1/\zeta$ *and find the Laurent expansion about* $\zeta = 0$.

$$\sin(1/\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,\zeta^{(2n+1)}}$$

Since $\sin(1/\zeta)$ *has an essential singularity at* $\zeta = 0$, $\sin z$ *has an essential singularity at infinity.*

We use the same approach if we want to examine the behavior at infinity of a differential equation. Starting with the first order differential equation,

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0,$$

we make the substitution

$$z = \frac{1}{\zeta}, \qquad \frac{\mathrm{d}}{\mathrm{d}z} = -\zeta^2 \frac{\mathrm{d}}{\mathrm{d}\zeta}, \qquad w(z) = u(\zeta)$$

to obtain

$$-\zeta^2 \frac{\mathrm{d}u}{\mathrm{d}\zeta} + p(1/\zeta)u = 0$$

$$\frac{\mathrm{d}u}{\mathrm{d}\zeta} - \frac{p(1/\zeta)}{\zeta^2} u = 0.$$

Result 14.8.4 The behavior at infinity of

$$\frac{\mathrm{d}w}{\mathrm{d}z} + p(z)w = 0$$

is the same as the behavior at $\zeta = 0$ of

$$\frac{\mathrm{d}u}{\mathrm{d}\zeta} - \frac{p(1/\zeta)}{\zeta^2}u = 0.$$

Example 14.8.9 *We classify the singular points of the equation*

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{z^2+9}w = 0.$$

We factor the denominator of the fraction to see that $z = \imath 3$ and $z = -\imath 3$ are regular singular points.

$$\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{(z-\imath 3)(z+\imath 3)}w = 0$$

We make the transformation $z = 1/\zeta$ to examine the point at infinity.

$$\frac{\mathrm{d}u}{\mathrm{d}\zeta} - \frac{1}{\zeta^2}\frac{1}{(1/\zeta)^2+9}u = 0$$
$$\frac{\mathrm{d}u}{\mathrm{d}\zeta} - \frac{1}{9\zeta^2+1}u = 0$$

Since the equation for u has a ordinary point at $\zeta = 0$, $z = \infty$ is a ordinary point of the equation for w.

14.9 Additional Exercises

Exact Equations

Exercise 14.8 (mathematica/ode/first_order/exact.nb)
Find the general solution $y = y(x)$ of the equations

1. $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + xy + y^2}{x^2},$$
2. $$(4y - 3x)\,\mathrm{d}x + (y - 2x)\,\mathrm{d}y = 0.$$

Hint, Solution

Exercise 14.9 (mathematica/ode/first_order/exact.nb)
Determine whether or not the following equations can be made exact. If so find the corresponding general solution.

1. $$(3x^2 - 2xy + 2)\,\mathrm{d}x + (6y^2 - x^2 + 3)\,\mathrm{d}y = 0$$
2. $$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ax + by}{bx + cy}$$

Hint, Solution

Exercise 14.10 (mathematica/ode/first_order/exact.nb)
Find the solutions of the following differential equations which satisfy the given initial condition. In each case determine the interval in which the solution is defined.

1. $$\frac{\mathrm{d}y}{\mathrm{d}x} = (1 - 2x)y^2, \quad y(0) = -1/6.$$
2. $$x\,\mathrm{d}x + y\,\mathrm{e}^{-x}\,\mathrm{d}y = 0, \quad y(0) = 1.$$

Hint, Solution

Exercise 14.11
Are the following equations exact? If so, solve them.

1. $(4y - x)y' - (9x^2 + y - 1) = 0$
2. $(2x - 2y)y' + (2x + 4y) = 0.$

Hint, Solution

Exercise 14.12 (mathematica/ode/first_order/exact.nb)
Find all functions $f(t)$ such that the differential equation

$$y^2 \sin t + y f(t) \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \tag{14.7}$$

is exact. Solve the differential equation for these $f(t)$.
Hint, Solution

The First Order, Linear Differential Equation

Exercise 14.13 (mathematica/ode/first_order/linear.nb)
Solve the differential equation

$$y' + \frac{y}{\sin x} = 0.$$

Hint, Solution

Initial Conditions
Well-Posed Problems

Exercise 14.14
Find the solutions of

$$t \frac{\mathrm{d}y}{\mathrm{d}t} + Ay = 1 + t^2, \quad t > 0$$

which are bounded at $t = 0$. Consider all (real) values of A.
Hint, Solution

Equations in the Complex Plane

Exercise 14.15

Classify the singular points of the following first order differential equations, (include the point at infinity).

1. $w' + \frac{\sin z}{z} w = 0$
2. $w' + \frac{1}{z-3} w = 0$
3. $w' + z^{1/2} w = 0$

Hint, Solution

Exercise 14.16

Consider the equation

$$w' + z^{-2} w = 0.$$

The point $z = 0$ is an irregular singular point of the differential equation. Thus we know that we cannot expand the solution about $z = 0$ in a Frobenius series. Try substituting the series solution

$$w = z^\lambda \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 \neq 0$$

into the differential equation anyway. What happens?

Hint, Solution

14.10 Hints

Hint 14.1

1. $\frac{d}{dx} \ln |u| = \frac{1}{u}$
2. $\frac{d}{dx} u^c = u^{c-1} u'$

Hint 14.2

Hint 14.3

The equation is homogeneous. Make the change of variables $u = y/t$.

Hint 14.4

Make sure you consider the case $\alpha = 0$.

Hint 14.5

Hint 14.6

Hint 14.7

The radius of convergence of the series and the distance to the nearest singularity of $\frac{1}{1-z}$ are not the same.

Exact Equations

Hint 14.8

1.
2.

Hint 14.9

1. The equation is exact. Determine the primitive u by solving the equations $u_x = P$, $u_y = Q$.

2. The equation can be made exact.

Hint 14.10

1. This equation is separable. Integrate to get the general solution. Apply the initial condition to determine the constant of integration.

2. Ditto. You will have to numerically solve an equation to determine where the solution is defined.

Hint 14.11

Hint 14.12

The First Order, Linear Differential Equation

Hint 14.13
Look in the appendix for the integral of $\csc x$.

Initial Conditions
Well-Posed Problems

Hint 14.14

Equations in the Complex Plane

Hint 14.15

Hint 14.16
Try to find the value of λ by substituting the series into the differential equation and equating powers of z.

14.11 Solutions

Solution 14.1

1.

$$\frac{y'(x)}{y(x)} = f(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \ln |y(x)| = f(x)$$
$$\ln |y(x)| = \int f(x) \,\mathrm{d}x + c$$
$$y(x) = \pm \mathrm{e}^{\int f(x) \,\mathrm{d}x + c}$$
$$\boxed{y(x) = c\,\mathrm{e}^{\int f(x) \,\mathrm{d}x}}$$

2.

$$y^\alpha(x) y'(x) = f(x)$$
$$\frac{y^{\alpha+1}(x)}{\alpha+1} = \int f(x) \,\mathrm{d}x + c$$
$$\boxed{y(x) = \left((\alpha+1) \int f(x) \,\mathrm{d}x + a \right)^{1/(\alpha+1)}}$$

3.

$$
\begin{aligned}
\frac{y'}{\cos x} + y \frac{\tan x}{\cos x} &= \cos x \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\cos x}\right) &= \cos x \\
\frac{y}{\cos x} &= \sin x + c \\
\Aboxed{y(x) &= \sin x \cos x + c \cos x}
\end{aligned}
$$

Solution 14.2
We consider the homogeneous equation,

$$
P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.
$$

That is, both P and Q are homogeneous of degree n. We hypothesize that multiplying by

$$
\mu(x, y) = \frac{1}{xP(x, y) + yQ(x, y)}
$$

will make the equation exact. To prove this we use the result that

$$
M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0
$$

is exact if and only if $M_y = N_x$.

$$
\begin{aligned}
M_y &= \frac{\partial}{\partial y}\left[\frac{P}{xP + yQ}\right] \\
&= \frac{P_y(xP + yQ) - P(xP_y + Q + yQ_y)}{(xP + yQ)^2}
\end{aligned}
$$

$$\begin{aligned} N_x &= \frac{\partial}{\partial x}\left[\frac{Q}{xP+yQ}\right] \\ &= \frac{Q_x(xP+yQ) - Q(P + xP_x + yQ_x)}{(xP+yQ)^2} \end{aligned}$$

$$\begin{aligned} M_y &= N_x \\ P_y(xP+yQ) - P(xP_y + Q + yQ_y) &= Q_x(xP+yQ) - Q(P + xP_x + yQ_x) \\ yP_yQ - yPQ_y &= xPQ_x - xP_xQ \\ xP_xQ + yP_yQ &= xPQ_x + yPQ_y \\ (xP_x + yP_y)Q &= P(xQ_x + yQ_y) \end{aligned}$$

With Euler's theorem, this reduces to an identity.

$$nPQ = PnQ$$

Thus the equation is exact. $\mu(x, y)$ is an integrating factor for the homogeneous equation.

Solution 14.3

We note that this is a homogeneous differential equation. The coefficient of $\mathrm{d}y/\mathrm{d}t$ and the inhomogeneity are homogeneous of degree zero.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2\left(\frac{y}{t}\right) + \left(\frac{y}{t}\right)^2.$$

We make the change of variables $u = y/t$ to obtain a separable equation.

$$\begin{aligned} tu' + u &= 2u + u^2 \\ \frac{u'}{u^2+u} &= \frac{1}{t} \end{aligned}$$

Now we integrate to solve for u.

$$\frac{u'}{u(u+1)} = \frac{1}{t}$$
$$\frac{u'}{u} - \frac{u'}{u+1} = \frac{1}{t}$$
$$\ln|u| - \ln|u+1| = \ln|t| + c$$
$$\ln\left|\frac{u}{u+1}\right| = \ln|ct|$$
$$\frac{u}{u+1} = \pm ct$$
$$\frac{u}{u+1} = ct$$
$$u = \frac{ct}{1-ct}$$
$$u = \frac{t}{c-t}$$
$$\boxed{y = \frac{t^2}{c-t}}$$

Solution 14.4
We consider

$$y' - \frac{1}{x}y = x^\alpha, \quad x > 0.$$

First we find the integrating factor.

$$I(x) = \exp\left(\int -\frac{1}{x}\,dx\right) = \exp(-\ln x) = \frac{1}{x}.$$

We multiply by the integrating factor and integrate.

$$\frac{1}{x}y' - \frac{1}{x^2}y = x^{\alpha-1}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}y\right) = x^{\alpha-1}$$
$$\frac{1}{x}y = \int x^{\alpha-1}\,\mathrm{d}x + c$$
$$y = x\int x^{\alpha-1}\,\mathrm{d}x + cx$$
$$\boxed{y = \begin{cases} \frac{x^{\alpha+1}}{\alpha} + cx & \text{for } \alpha \neq 0, \\ x\ln x + cx & \text{for } \alpha = 0. \end{cases}}$$

Solution 14.5

1.

$$y' + xy = x^{2n+1}, \quad y(1) = 1, \quad n \in \mathbb{Z}$$

We find the integrating factor.

$$I(x) = \mathrm{e}^{\int x\,\mathrm{d}x} = \mathrm{e}^{x^2/2}$$

We multiply by the integrating factor and integrate. Since the initial condition is given at $x = 1$, we will take the lower bound of integration to be that point.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2/2}\,y\right) = x^{2n+1}\,\mathrm{e}^{x^2/2}$$
$$y = \mathrm{e}^{-x^2/2}\int_1^x \xi^{2n+1}\,\mathrm{e}^{\xi^2/2}\,\mathrm{d}\xi + c\,\mathrm{e}^{-x^2/2}$$

We choose the constant of integration to satisfy the initial condition.

$$\boxed{y = \mathrm{e}^{-x^2/2}\int_1^x \xi^{2n+1}\,\mathrm{e}^{\xi^2/2}\,\mathrm{d}\xi + \mathrm{e}^{(1-x^2)/2}}$$

If $n \geq 0$ then we can use integration by parts to write the integral as a sum of terms. If $n < 0$ we can write the integral in terms of the exponential integral function. However, the integral form above is as nice as any other and we leave the answer in that form.

2.

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy(x) = 1, \quad y(0) = 1.$$

We determine the integrating factor and then integrate the equation.

$$I(x) = \mathrm{e}^{\int -2x \,\mathrm{d}x} = \mathrm{e}^{-x^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x^2} y\right) = \mathrm{e}^{-x^2}$$
$$y = \mathrm{e}^{x^2} \int_0^x \mathrm{e}^{-\xi^2} \,\mathrm{d}\xi + c\,\mathrm{e}^{x^2}$$

We choose the constant of integration to satisfy the initial condition.

$$\boxed{y = \mathrm{e}^{x^2}\left(1 + \int_0^x \mathrm{e}^{-\xi^2} \,\mathrm{d}\xi\right)}$$

We can write the answer in terms of the *Error function*,

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-\xi^2} \,\mathrm{d}\xi.$$
$$\boxed{y = \mathrm{e}^{x^2}\left(1 + \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)\right)}$$

Solution 14.6

We determine the integrating factor and then integrate the equation.

$$I(x) = \mathrm{e}^{\int \alpha \, \mathrm{d}x} = \mathrm{e}^{\alpha x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\mathrm{e}^{\alpha x} y \right) = \beta \, \mathrm{e}^{(\alpha - \lambda) x}$$

$$y = \beta \, \mathrm{e}^{-\alpha x} \int \mathrm{e}^{(\alpha - \lambda) x} \, \mathrm{d}x + c \, \mathrm{e}^{-\alpha x}$$

First consider the case $\alpha \neq \lambda$.

$$y = \beta \, \mathrm{e}^{-\alpha x} \frac{\mathrm{e}^{(\alpha - \lambda) x}}{\alpha - \lambda} + c \, \mathrm{e}^{-\alpha x}$$

$$\boxed{y = \frac{\beta}{\alpha - \lambda} \mathrm{e}^{-\lambda x} + c \, \mathrm{e}^{-\alpha x}}$$

Clearly the solution vanishes as $x \to \infty$.

Next consider $\alpha = \lambda$.

$$y = \beta \, \mathrm{e}^{-\alpha x} x + c \, \mathrm{e}^{-\alpha x}$$

$$\boxed{y = (c + \beta x) \, \mathrm{e}^{-\alpha x}}$$

We use L'Hospital's rule to show that the solution vanishes as $x \to \infty$.

$$\lim_{x \to \infty} \frac{c + \beta x}{\mathrm{e}^{\alpha x}} = \lim_{x \to \infty} \frac{\beta}{\alpha \, \mathrm{e}^{\alpha x}} = 0$$

For $\beta = \lambda = 1$, the solution is

$$y = \begin{cases} \frac{1}{\alpha - 1} \mathrm{e}^{-x} + c \, \mathrm{e}^{-\alpha x} & \text{for } \alpha \neq 1, \\ (c + x) \, \mathrm{e}^{-x} & \text{for } \alpha = 1. \end{cases}$$

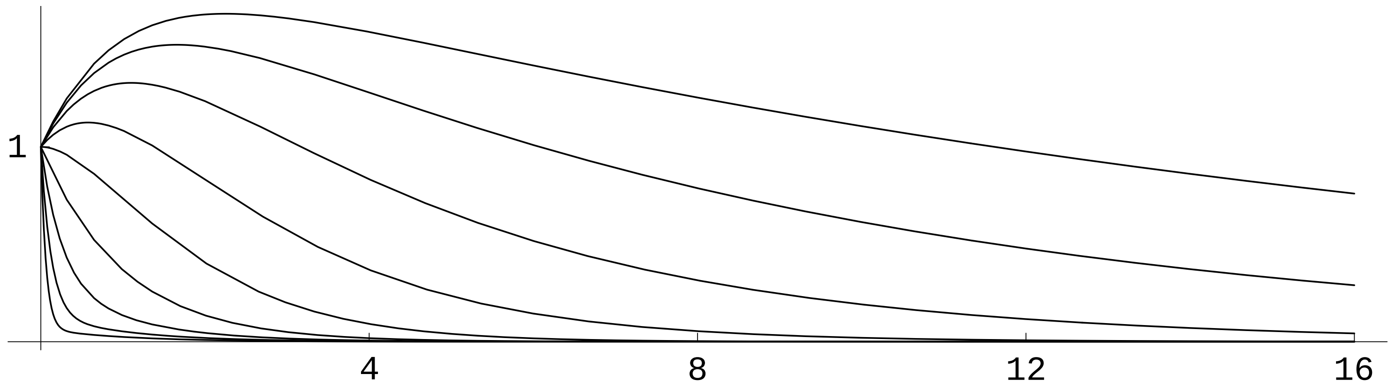

Figure 14.9: The Solution for a Range of α

The solution which satisfies the initial condition is

$$\boxed{y = \begin{cases} \frac{1}{\alpha-1}\left(\mathrm{e}^{-x} + (\alpha-2)\,\mathrm{e}^{-\alpha x}\right) & \text{for } \alpha \neq 1, \\ (1+x)\,\mathrm{e}^{-x} & \text{for } \alpha = 1. \end{cases}}$$

In Figure 14.9 the solution is plotted for $\alpha = 1/16, 1/8, \ldots, 16$.

Consider the solution in the limit as $\alpha \to 0$.

$$\begin{aligned} \lim_{\alpha \to 0} y(x) &= \lim_{\alpha \to 0} \frac{1}{\alpha - 1}\left(\mathrm{e}^{-x} + (\alpha-2)\,\mathrm{e}^{-\alpha x}\right) \\ &= 2 - \mathrm{e}^{-x} \end{aligned}$$

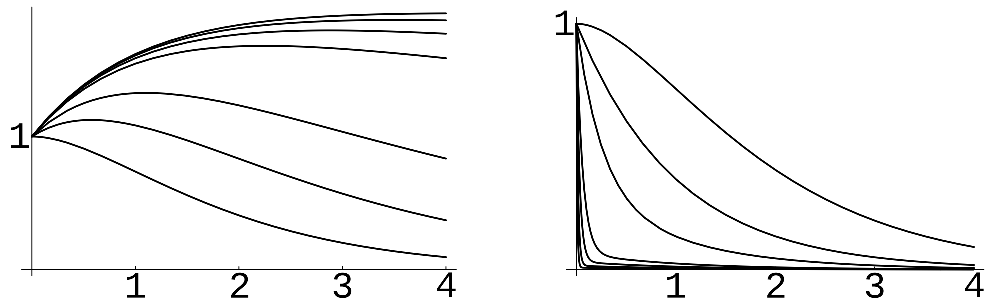

Figure 14.10: The Solution as $\alpha \to 0$ and $\alpha \to \infty$

In the limit as $\alpha \to \infty$ we have,

$$\begin{aligned}
\lim_{\alpha\to\infty} y(x) &= \lim_{\alpha\to\infty} \frac{1}{\alpha-1}\left(e^{-x} + (\alpha-2)\,e^{-\alpha x}\right) \\
&= \lim_{\alpha\to\infty} \frac{\alpha-2}{\alpha-1}\,e^{-\alpha x} \\
&= \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x > 0. \end{cases}
\end{aligned}$$

This behavior is shown in Figure 14.10. The first graph plots the solutions for $\alpha = 1/128, 1/64, \ldots, 1$. The second graph plots the solutions for $\alpha = 1, 2, \ldots, 128$.

Solution 14.7

We substitute $w = \sum_{n=0}^{\infty} a_n z^n$ into the equation $\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{1-z} w = 0$.

$$\frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} a_n z^n + \frac{1}{1-z} \sum_{n=0}^{\infty} a_n z^n = 0$$

$$(1-z) \sum_{n=1}^{\infty} n a_n z^{n-1} + \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n - \sum_{n=0}^{\infty} n a_n z^n + \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+1) a_{n+1} - (n-1) a_n\right) z^n = 0$$

Equating powers of z to zero, we obtain the relation,

$$a_{n+1} = \frac{n-1}{n+1} a_n.$$

a_0 is arbitrary. We can compute the rest of the coefficients from the recurrence relation.

$$a_1 = \frac{-1}{1} a_0 = -a_0$$
$$a_2 = \frac{0}{2} a_1 = 0$$

We see that the coefficients are zero for $n \geq 2$. Thus the Taylor series expansion, (and the exact solution), is

$$\boxed{w = a_0(1-z).}$$

The radius of convergence of the series in infinite. The nearest singularity of $\frac{1}{1-z}$ is at $z = 1$. Thus we see the radius of convergence can be greater than the distance to the nearest singularity of the coefficient function, $p(z)$.

Solution 14.8

1.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + xy + y^2}{x^2}$$

Since the right side is a homogeneous function of order zero, this is a homogeneous differential equation. We make the change of variables $u = y/x$ and then solve the differential equation for u.

$$xu' + u = 1 + u + u^2$$
$$\frac{\mathrm{d}u}{1 + u^2} = \frac{\mathrm{d}x}{x}$$
$$\arctan(u) = \ln|x| + c$$
$$u = \tan(\ln(|cx|))$$
$$\boxed{y = x\tan(\ln(|cx|))}$$

2.

$$(4y - 3x)\,\mathrm{d}x + (y - 2x)\,\mathrm{d}y = 0$$

Since the coefficients are homogeneous functions of order one, this is a homogeneous differential equation. We make the change of variables $u = y/x$ and then solve the differential equation for u.

$$\left(4\frac{y}{x} - 3\right) \mathrm{d}x + \left(\frac{y}{x} - 2\right) \mathrm{d}y = 0$$
$$(4u - 3)\,\mathrm{d}x + (u - 2)(u\,\mathrm{d}x + x\,\mathrm{d}u) = 0$$
$$(u^2 + 2u - 3)\,\mathrm{d}x + x(u - 2)\,\mathrm{d}u = 0$$
$$\frac{\mathrm{d}x}{x} + \frac{u - 2}{(u + 3)(u - 1)}\,\mathrm{d}u = 0$$
$$\frac{\mathrm{d}x}{x} + \left(\frac{5/4}{u + 3} - \frac{1/4}{u - 1}\right) \mathrm{d}u = 0$$
$$\ln(x) + \frac{5}{4}\ln(u + 3) - \frac{1}{4}\ln(u - 1) = c$$
$$\frac{x^4(u + 3)^5}{u - 1} = c$$
$$\frac{x^4(y/x + 3)^5}{y/x - 1} = c$$
$$\boxed{\frac{(y + 3x)^5}{y - x} = c}$$

Solution 14.9

1.

$$(3x^2 - 2xy + 2)\,\mathrm{d}x + (6y^2 - x^2 + 3)\,\mathrm{d}y = 0$$

We check if this form of the equation, $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$, is exact.

$$P_y = -2x, \qquad Q_x = -2x$$

Since $P_y = Q_x$, the equation is exact. Now we find the primitive $u(x, y)$ which satisfies

$$\mathrm{d}u = (3x^2 - 2xy + 2)\,\mathrm{d}x + (6y^2 - x^2 + 3)\,\mathrm{d}y.$$

The primitive satisfies the partial differential equations

$$u_x = P, \quad u_y = Q. \tag{14.8}$$

We integrate the first equation of 14.8 to determine u up to a function of integration.

$$\begin{gathered}
u_x = 3x^2 - 2xy + 2 \\
u = x^3 - x^2 y + 2x + f(y)
\end{gathered}$$

We substitute this into the second equation of 14.8 to determine the function of integration up to an additive constant.

$$\begin{gathered}
-x^2 + f'(y) = 6y^2 - x^2 + 3 \\
f'(y) = 6y^2 + 3 \\
f(y) = 2y^3 + 3y
\end{gathered}$$

The solution of the differential equation is determined by the implicit equation $u = c$.

$$\boxed{x^3 - x^2 y + 2x + 2y^3 + 3y = c}$$

2.

$$\begin{gathered}
\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ax + by}{bx + cy} \\
(ax + by)\,\mathrm{d}x + (bx + cy)\,\mathrm{d}y = 0
\end{gathered}$$

We check if this form of the equation, $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$, is exact.

$$P_y = b, \qquad Q_x = b$$

Since $P_y = Q_x$, the equation is exact. Now we find the primitive $u(x, y)$ which satisfies

$$\mathrm{d}u = (ax + by)\,\mathrm{d}x + (bx + cy)\,\mathrm{d}y$$

The primitive satisfies the partial differential equations

$$u_x = P, \quad u_y = Q. \tag{14.9}$$

We integrate the first equation of 14.9 to determine u up to a function of integration.

$$u_x = ax + by$$
$$u = \frac{1}{2}ax^2 + bxy + f(y)$$

We substitute this into the second equation of 14.9 to determine the function of integration up to an additive constant.

$$bx + f'(y) = bx + cy$$
$$f'(y) = cy$$
$$f(y) = \frac{1}{2}cy^2$$

The solution of the differential equation is determined by the implicit equation $u = d$.

$$\boxed{ax^2 + 2bxy + cy^2 = d}$$

Solution 14.10

Note that since these equations are nonlinear, we cannot predict where the solutions will be defined from the equation alone.

1. This equation is separable. We integrate to get the general solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1 - 2x)y^2$$
$$\frac{\mathrm{d}y}{y^2} = (1 - 2x)\,\mathrm{d}x$$
$$-\frac{1}{y} = x - x^2 + c$$
$$y = \frac{1}{x^2 - x - c}$$

Now we apply the initial condition.

$$y(0) = \frac{1}{-c} = -\frac{1}{6}$$
$$y = \frac{1}{x^2 - x - 6}$$
$$\boxed{y = \frac{1}{(x+2)(x-3)}}$$

The solution is defined on the interval $(-2 \ldots 3)$.

2. This equation is separable. We integrate to get the general solution.

$$x \, \mathrm{d}x + y \, \mathrm{e}^{-x} \, \mathrm{d}y = 0$$
$$x \, \mathrm{e}^{x} \, \mathrm{d}x + y \, \mathrm{d}y = 0$$
$$(x-1) \, \mathrm{e}^{x} + \frac{1}{2} y^2 = c$$
$$y = \sqrt{2(c + (1-x) \, \mathrm{e}^{x})}$$

We apply the initial condition to determine the constant of integration.

$$y(0) = \sqrt{2(c+1)} = 1$$
$$c = -\frac{1}{2}$$
$$\boxed{y = \sqrt{2(1-x) \, \mathrm{e}^{x} - 1}}$$

The function $2(1-x) \, \mathrm{e}^{x} - 1$ is plotted in Figure 14.11. We see that the argument of the square root in the solution is non-negative only on an interval about the origin. Because $2(1-x) \, \mathrm{e}^{x} - 1 == 0$ is a mixed algebraic / transcendental equation, we cannot solve it analytically. The solution of the differential equation is defined on the interval $(-1.67835 \ldots 0.768039)$.

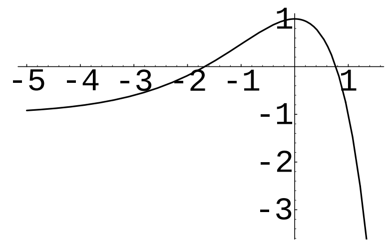

Figure 14.11: The function $2(1-x)\,\mathrm{e}^x - 1$.

Solution 14.11

1. We consider the differential equation,

$$(4y - x)y' - (9x^2 + y - 1) = 0.$$

$$P_y = \frac{\partial}{\partial y}\left(1 - y - 9x^2\right) = -1$$
$$Q_x = \frac{\partial}{\partial x}(4y - x) = -1$$

This equation is exact. It is simplest to solve the equation by rearranging terms to form exact derivatives.

$$4yy' - xy' - y + 1 - 9x^2 = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[2y^2 - xy\right] + 1 - 9x^2 = 0$$
$$2y^2 - xy + x - 3x^3 + c = 0$$
$$\boxed{y = \frac{1}{4}\left(x \pm \sqrt{x^2 - 8(c + x - 3x^3)}\right)}$$

2. We consider the differential equation,

$$(2x - 2y)y' + (2x + 4y) = 0.$$

$$P_y = \frac{\partial}{\partial y}(2x + 4y) = 4$$
$$Q_x = \frac{\partial}{\partial x}(2x - 2y) = 2$$

Since $P_y \neq Q_x$, this is not an exact equation.

Solution 14.12
Recall that the differential equation
$$P(x, y) + Q(x, y)y' = 0$$
is exact if and only if $P_y = Q_x$. For Equation 14.7, this criterion is
$$2y \sin t = y f'(t)$$
$$f'(t) = 2 \sin t$$
$$\boxed{f(t) = 2(a - \cos t).}$$
In this case, the differential equation is
$$y^2 \sin t + 2yy'(a - \cos t) = 0.$$
We can integrate this exact equation by inspection.
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y^2(a - \cos t)\right) = 0$$
$$y^2(a - \cos t) = c$$
$$\boxed{y = \pm \frac{c}{\sqrt{a - \cos t}}}$$

The First Order, Linear Differential Equation

Solution 14.13
Consider the differential equation
$$y' + \frac{y}{\sin x} = 0.$$

We use Equation 14.5 to determine the solution.

$$y = c\,\mathrm{e}^{\int -1/\sin x \,\mathrm{d}x}$$
$$y = c\,\mathrm{e}^{-\ln|\tan(x/2)|}$$
$$y = c\left|\cot\left(\frac{x}{2}\right)\right|$$
$$\boxed{y = c\cot\left(\frac{x}{2}\right)}$$

Initial Conditions

Well-Posed Problems

Solution 14.14
First we write the differential equation in the standard form.

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{A}{t}y = \frac{1}{t} + t, \quad t > 0$$

We determine the integrating factor.

$$I(t) = \mathrm{e}^{\int A/t \,\mathrm{d}t} = \mathrm{e}^{A\ln t} = t^A$$

We multiply the differential equation by the integrating factor and integrate.

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{A}{t}y = \frac{1}{t} + t$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^A y\right) = t^{A-1} + t^{A+1}$$

$$t^A y = \begin{cases} \frac{t^A}{A} + \frac{t^{A+2}}{A+2} + c, & A \neq 0, -2 \\ \ln t + \frac{1}{2}t^2 + c, & A = 0 \\ -\frac{1}{2}t^{-2} + \ln t + c, & A = -2 \end{cases}$$

$$y = \begin{cases} \frac{1}{A} + \frac{t^2}{A+2} + ct^{-A}, & A \neq -2 \\ \ln t + \frac{1}{2}t^2 + c, & A = 0 \\ -\frac{1}{2} + t^2 \ln t + ct^2, & A = -2 \end{cases}$$

For positive A, the solution is bounded at the origin only for $c = 0$. For $A = 0$, there are no bounded solutions. For negative A, the solution is bounded there for any value of c and thus we have a one-parameter family of solutions.

In summary, the solutions which are bounded at the origin are:

$$\boxed{y = \begin{cases} \frac{1}{A} + \frac{t^2}{A+2}, & A > 0 \\ \frac{1}{A} + \frac{t^2}{A+2} + ct^{-A}, & A < 0,\ A \neq -2 \\ -\frac{1}{2} + t^2 \ln t + ct^2, & A = -2 \end{cases}}$$

Equations in the Complex Plane

Solution 14.15

1. Consider the equation $w' + \frac{\sin z}{z} w = 0$. The point $z = 0$ is the only point we need to examine in the finite plane. Since $\frac{\sin z}{z}$ has a removable singularity at $z = 0$, there are no singular points in the finite plane. The substitution $z = \frac{1}{\zeta}$ yields the equation

$$u' - \frac{\sin(1/\zeta)}{\zeta} u = 0.$$

Since $\frac{\sin(1/\zeta)}{\zeta}$ has an essential singularity at $\zeta = 0$, the point at infinity is an irregular singular point of the original differential equation.

2. Consider the equation $w' + \frac{1}{z-3}w = 0$. Since $\frac{1}{z-3}$ has a simple pole at $z = 3$, the differential equation has a regular singular point there. Making the substitution $z = 1/\zeta$, $w(z) = u(\zeta)$

$$u' - \frac{1}{\zeta^2(1/\zeta - 3)}u = 0$$
$$u' - \frac{1}{\zeta(1-3\zeta)}u = 0.$$

Since this equation has a simple pole at $\zeta = 0$, the original equation has a regular singular point at infinity.

3. Consider the equation $w' + z^{1/2}w = 0$. There is an irregular singular point at $z = 0$. With the substitution $z = 1/\zeta$, $w(z) = u(\zeta)$,

$$u' - \frac{\zeta^{-1/2}}{\zeta^2}u = 0$$
$$u' - \zeta^{-5/2}u = 0.$$

We see that the point at infinity is also an irregular singular point of the original differential equation.

Solution 14.16

We start with the equation

$$w' + z^{-2}w = 0.$$

Substituting $w = z^\lambda \sum_{n=0}^\infty a_n z^n$, $a_0 \neq 0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^\lambda \sum_{n=0}^\infty a_n z^n\right) + z^{-2} z^\lambda \sum_{n=0}^\infty a_n z^n = 0$$
$$\lambda z^{\lambda-1} \sum_{n=0}^\infty a_n z^n + z^\lambda \sum_{n=1}^\infty n a_n z^{n-1} + z^\lambda \sum_{n=0}^\infty a_n z^{n-2} = 0$$

The lowest power of z in the expansion is $z^{\lambda-2}$. The coefficient of this term is a_0. Equating powers of z demands that $a_0 = 0$ which contradicts our initial assumption that it was nonzero. Thus we cannot find a λ such that the solution can be expanded in the form,

$$w = z^\lambda \sum_{n=0}^{\infty} a_n z^n, \qquad a_0 \neq 0.$$

14.12 Quiz

Problem 14.1
What is the *general solution* of a first order differential equation?
Solution

Problem 14.2
Write a statement about the functions P and Q to make the following statement correct.
The first order differential equation

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is exact if and only if ________. It is separable if ______________________.
Solution

Problem 14.3
Derive the general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x).$$

Solution

Problem 14.4
Solve $y' = y - y^2$.
Solution

14.13 Quiz Solutions

Solution 14.1

The general solution of a first order differential equation is a one-parameter family of functions which satisfies the equation.

Solution 14.2

The first order differential equation

$$P(x, y) + Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is exact if and only if $P_y = Q_x$. It is separable if $P = P(x)$ and $Q = Q(y)$.

Solution 14.3

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = f(x)$$

We multiply by the integrating factor $\mu(x) = \exp(P(x)) = \exp\left(\int p(x)\,\mathrm{d}x\right)$, and integrate.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{e}^{P(x)} + p(x)y\,\mathrm{e}^{P(x)} &= \mathrm{e}^{P(x)}\,f(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{P(x)}\right) &= \mathrm{e}^{P(x)}\,f(x) \\
y\,\mathrm{e}^{P(x)} &= \int \mathrm{e}^{P(x)}\,f(x)\,\mathrm{d}x + c \\
y &= \mathrm{e}^{-P(x)}\int \mathrm{e}^{P(x)}\,f(x)\,\mathrm{d}x + c\,\mathrm{e}^{-P(x)}
\end{aligned}$$

Solution 14.4
$y' = y - y^2$ is separable.

$$
\begin{aligned}
y' &= y - y^2 \\
\frac{y'}{y - y^2} &= 1 \\
\frac{y'}{y} - \frac{y'}{y-1} &= 1 \\
\ln y - \ln(y-1) &= x + c
\end{aligned}
$$

We do algebraic simplifications and rename the constant of integration to write the solution in a nice form.

$$
\begin{aligned}
\frac{y}{y-1} &= c\,\mathrm{e}^x \\
y &= (y-1)c\,\mathrm{e}^x \\
y &= \frac{-c\,\mathrm{e}^x}{1 - c\,\mathrm{e}^x} \\
y &= \frac{\mathrm{e}^x}{\mathrm{e}^x - c} \\
y &= \frac{1}{1 - c\,\mathrm{e}^{-x}}
\end{aligned}
$$

Chapter 15

First Order Linear Systems of Differential Equations

We all agree that your theory is crazy, but is it crazy enough?

- Niels Bohr

15.1 Introduction

In this chapter we consider first order linear systems of differential equations. That is, we consider equations of the form,

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t),$$

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

Initially we will consider the homogeneous problem, $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$. (Later we will find particular solutions with variation of parameters.) The best way to solve these equations is through the use of the matrix exponential. Unfortunately, using the matrix exponential requires knowledge of the Jordan canonical form and matrix functions. Fortunately, we can solve a certain class of problems using only the concepts of eigenvalues and eigenvectors of a matrix. We present this simple method in the next section. In the following section we will take a detour into matrix theory to cover Jordan canonical form and its applications. Then we will be able to solve the general case.

15.2 Using Eigenvalues and Eigenvectors to find Homogeneous Solutions

If you have forgotten what eigenvalues and eigenvectors are and how to compute them, go find a book on linear algebra and spend a few minutes re-aquainting yourself with the rudimentary material.

Recall that the single differential equation $x'(t) = Ax$ has the general solution $x = c\,\mathrm{e}^{At}$. Maybe the system of differential equations

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) \tag{15.1}$$

has similiar solutions. Perhaps it has a solution of the form $\mathbf{x}(t) = \mathbf{x}i\,\mathrm{e}^{\lambda t}$ for some constant vector $\mathbf{x}i$ and some value λ. Let's substitute this into the differential equation and see what happens.

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{A}\mathbf{x}(t) \\ \mathbf{x}i\lambda\,\mathrm{e}^{\lambda t} &= \mathbf{A}\mathbf{x}i\,\mathrm{e}^{\lambda t} \\ \mathbf{A}\mathbf{x}i &= \lambda\mathbf{x}i \end{aligned}$$

We see that if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{x}i$ then $\mathbf{x}(t) = \mathbf{x}i\,\mathrm{e}^{\lambda t}$ satisfies the differential equation. Since the differential equation is linear, $c\mathbf{x}i\,\mathrm{e}^{\lambda t}$ is a solution.

Suppose that the $n \times n$ matrix $\mathbf{A}$ has the eigenvalues $\{\lambda_k\}$ with a complete set of linearly independent eigenvectors $\{\mathbf{x}i_k\}$. Then each of $\mathbf{x}i_k\,\mathrm{e}^{\lambda_k t}$ is a homogeneous solution of Equation 15.1. We note that each of these solutions is linearly independent. Without any kind of justification I will tell you that the general solution of the differential equation is a linear combination of these n linearly independent solutions.

Result 15.2.1 Suppose that the $n \times n$ matrix $\mathbf{A}$ has the eigenvalues $\{\lambda_k\}$ with a complete set of linearly independent eigenvectors $\{\mathbf{x}i_k\}$. The system of differential equations,

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),$$

has the general solution,

$$\mathbf{x}(t) = \sum_{k=1}^{n} c_k \mathbf{x}i_k \, \mathrm{e}^{\lambda_k t}$$

Example 15.2.1 (mathematica/ode/systems/systems.nb) *Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.*

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The matrix has the distinct eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$. The corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} \mathrm{e}^{3t}.$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \mathrm{e}^{3t}}$$

For large t, the solution looks like

$$\mathbf{x} \approx \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \mathrm{e}^{3t}.$$

Both coordinates tend to infinity.

Figure 15.1 shows some homogeneous solutions in the phase plane.

Example 15.2.2 (mathematica/ode/systems/systems.nb) *Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.*

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

The matrix has the distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. The corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

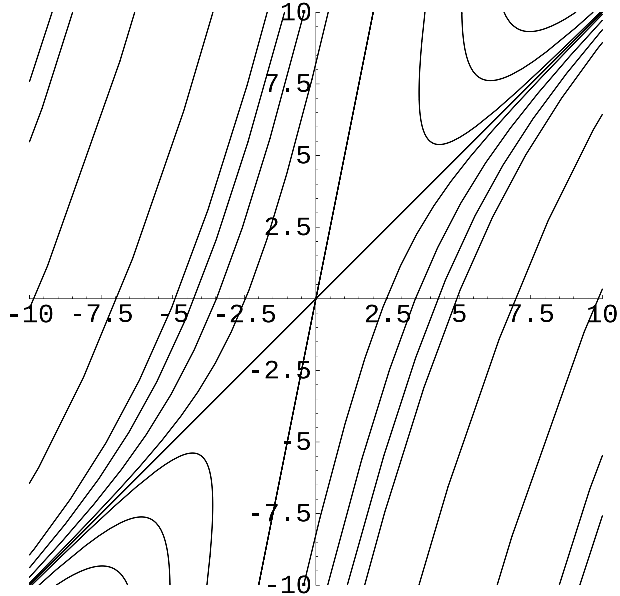

Figure 15.1: Homogeneous solutions in the phase plane.

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{2t} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \mathrm{e}^{3t}.$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$c_1 = 1, \quad c_2 = 2, \quad c_3 = 0$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{t} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{2t}.}$$

As $t \to \infty$, all coordinates tend to infinity.

Exercise 15.1 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hint, Solution

Exercise 15.2 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Hint, Solution

Exercise 15.3
Use the matrix form of the method of variation of parameters to find the general solution of

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \quad t > 0.$$

Hint, Solution

15.3 Matrices and Jordan Canonical Form

Functions of Square Matrices. Consider a function $f(x)$ with a Taylor series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

We can define the function to take square matrices as arguments. The function of the square matrix $\mathbf{A}$ is defined in terms of the Taylor series.

$$f(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \mathbf{A}^n$$

(Note that this definition is usually not the most convenient method for computing a function of a matrix. Use the Jordan canonical form for that.)

Eigenvalues and Eigenvectors. Consider a square matrix $\mathbf{A}$. A nonzero vector $\mathbf{x}$ is an *eigenvector* of the matrix with *eigenvalue* λ if

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}.$$

Note that we can write this equation as

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}.$$

This equation has solutions for nonzero $\mathbf{x}$ if and only if $\mathbf{A} - \lambda\mathbf{I}$ is singular, ($\det(\mathbf{A} - \lambda\mathbf{I}) = 0$). We define the *characteristic polynomial* of the matrix $\chi(\lambda)$ as this determinant.

$$\chi(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$$

The roots of the characteristic polynomial are the eigenvalues of the matrix. The eigenvectors of distinct eigenvalues are linearly independent. Thus if a matrix has distinct eigenvalues, the eigenvectors form a basis.

If λ is a root of $\chi(\lambda)$ of multiplicity m then there are up to m linearly independent eigenvectors corresponding to that eigenvalue. That is, it has from 1 to m eigenvectors.

Diagonalizing Matrices. Consider an $n \times n$ matrix $\mathbf{A}$ that has a complete set of n linearly independent eigenvectors. $\mathbf{A}$ may or may not have distinct eigenvalues. Consider the matrix $\mathbf{S}$ with eigenvectors as columns.

$$\mathbf{S} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{pmatrix}$$

$\mathbf{A}$ is diagonalized by the similarity transformation:

$$\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}.$$

$\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ as the diagonal elements. Furthermore, the k^{th} diagonal element is λ_k, the eigenvalue corresponding to the the eigenvector, $\mathbf{x}_k$.

Generalized Eigenvectors. A vector $\mathbf{x}_k$ is a *generalized eigenvector of rank* k if

$$(\mathbf{A} - \lambda\mathbf{I})^k \mathbf{x}_k = \mathbf{0} \quad \text{but} \quad (\mathbf{A} - \lambda\mathbf{I})^{k-1} \mathbf{x}_k \neq \mathbf{0}.$$

Eigenvectors are generalized eigenvectors of rank 1. An $n \times n$ matrix has n linearly independent generalized eigenvectors. A *chain* of generalized eigenvectors generated by the rank m generalized eigenvector $\mathbf{x}_m$ is the set: $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$, where

$$\mathbf{x}_k = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x}_{k+1}, \quad \text{for} \quad k = m-1, \ldots, 1.$$

Computing Generalized Eigenvectors. Let λ be an eigenvalue of multiplicity m. Let n be the smallest integer such that

$$\operatorname{rank}\left(\operatorname{nullspace}\left((A-\lambda I)^n\right)\right) = m.$$

Let N_k denote the number of eigenvalues of rank k. These have the value:

$$N_k = \operatorname{rank}\left(\operatorname{nullspace}\left((A-\lambda I)^k\right)\right) - \operatorname{rank}\left(\operatorname{nullspace}\left((A-\lambda I)^{k-1}\right)\right).$$

One can compute the generalized eigenvectors of a matrix by looping through the following three steps until all the the N_k are zero:

1. Select the largest k for which N_k is positive. Find a generalized eigenvector $\mathbf{x}_k$ of rank k which is linearly independent of all the generalized eigenvectors found thus far.

2. From $\mathbf{x}_k$ generate the chain of eigenvectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$. Add this chain to the known generalized eigenvectors.

3. Decrement each positive N_k by one.

Example 15.3.1 *Consider the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

The characteristic polynomial of the matrix is

$$\begin{aligned}
\chi(\lambda) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{vmatrix} \\
&= (1-\lambda)^2(4-\lambda) + 3 + 4 + 3(1-\lambda) - 2(4-\lambda) + 2(1-\lambda) \\
&= -(\lambda-2)^3.
\end{aligned}$$

Thus we see that $\lambda = 2$ *is an eigenvalue of multiplicity 3.* $\mathbf{A} - 2\mathbf{I}$ *is*

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}$$

The rank of the nullspace space of $\mathbf{A} - 2\mathbf{I}$ *is less than 3.*

$$(\mathbf{A} - 2\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

The rank of $\mathrm{nullspace}((\mathbf{A} - 2\mathbf{I})^2)$ *is less than 3 as well, so we have to take one more step.*

$$(\mathbf{A} - 2\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of $\mathrm{nullspace}((\mathbf{A} - 2\mathbf{I})^3)$ *is 3. Thus there are generalized eigenvectors of ranks 1, 2 and 3. The generalized eigenvector of rank 3 satisfies:*

$$(\mathbf{A} - 2\mathbf{I})^3 \mathbf{x}_3 = \mathbf{0}$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x}_3 = \mathbf{0}$$

We choose the solution

$$\mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Now to compute the chain generated by $\mathbf{x}_3$*.*

$$\mathbf{x}_2 = (\mathbf{A} - 2\mathbf{I})\mathbf{x}_3 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

$$\mathbf{x}_1 = (\mathbf{A} - 2\mathbf{I})\mathbf{x}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Thus a set of generalized eigenvectors corresponding to the eigenvalue $\lambda = 2$ *are*

$$\boxed{\mathbf{x}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.}$$

Jordan Block. A Jordan block is a square matrix which has the constant, λ, on the diagonal and ones on the first super-diagonal:

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

Jordan Canonical Form. A matrix $\mathbf{J}$ is in Jordan canonical form if all the elements are zero except for Jordan blocks $\mathbf{J}_k$ along the diagonal.

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{J}_{n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{J}_n \end{pmatrix}$$

The Jordan canonical form of a matrix is obtained with the similarity transformation:

$$\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S},$$

where $\mathbf{S}$ is the matrix of the generalized eigenvectors of $\mathbf{A}$ and the generalized eigenvectors are grouped in chains.

Example 15.3.2 *Again consider the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

Since $\lambda = 2$ is an eigenvalue of multiplicity 3, the Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

In Example 15.3.1 we found the generalized eigenvectors of $\mathbf{A}$. We define the matrix with generalized eigenvectors as columns:

$$\mathbf{S} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix}.$$

We can verify that $\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

$$\begin{aligned}\mathbf{J} &= \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \\ &= \begin{pmatrix} 0 & -3 & -2 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}\end{aligned}$$

Functions of Matrices in Jordan Canonical Form. The function of an $n \times n$ Jordan block is the upper-triangular matrix:

$$f(\mathbf{J}_k) = \begin{pmatrix} f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} & \cdots & \frac{f^{(n-2)}(\lambda)}{(n-2)!} & \frac{f^{(n-1)}(\lambda)}{(n-1)!} \\ 0 & f(\lambda) & \frac{f'(\lambda)}{1!} & \cdots & \frac{f^{(n-3)}(\lambda)}{(n-3)!} & \frac{f^{(n-2)}(\lambda)}{(n-2)!} \\ 0 & 0 & f(\lambda) & \ddots & \frac{f^{(n-4)}(\lambda)}{(n-4)!} & \frac{f^{(n-3)}(\lambda)}{(n-3)!} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & f(\lambda) & \frac{f'(\lambda)}{1!} \\ 0 & 0 & 0 & \cdots & 0 & f(\lambda) \end{pmatrix}$$

The function of a matrix in Jordan canonical form is

$$f(\mathbf{J}) = \begin{pmatrix} f(\mathbf{J}_1) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & f(\mathbf{J}_2) & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ddots & f(\mathbf{J}_{n-1}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & f(\mathbf{J}_n) \end{pmatrix}$$

The Jordan canonical form of a matrix satisfies:

$$f(\mathbf{J}) = \mathbf{S}^{-1} f(\mathbf{A}) \mathbf{S},$$

where $\mathbf{S}$ is the matrix of the generalized eigenvectors of $\mathbf{A}$. This gives us a convenient method for computing functions of matrices.

Example 15.3.3 *Consider the matrix exponential function* $\mathrm{e}^{\mathbf{A}}$ *for our old friend:*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

In Example 15.3.2 we showed that the Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since all the derivatives of e^{λ} *are just* e^{λ}*, it is especially easy to compute* $\mathrm{e}^{\mathbf{J}}$*.*

$$\mathrm{e}^{\mathbf{J}} = \begin{pmatrix} \mathrm{e}^2 & \mathrm{e}^2 & \mathrm{e}^2/2 \\ 0 & \mathrm{e}^2 & \mathrm{e}^2 \\ 0 & 0 & \mathrm{e}^2 \end{pmatrix}$$

We find $\mathrm{e}^{\mathbf{A}}$ *with a similarity transformation of* $\mathrm{e}^{\mathbf{J}}$*. We use the matrix of generalized eigenvectors found in Example 15.3.2.*

$$\mathrm{e}^{\mathbf{A}} = \mathbf{S}\,\mathrm{e}^{\mathbf{J}}\,\mathbf{S}^{-1}$$

$$\mathrm{e}^{\mathbf{A}} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^2 & \mathrm{e}^2 & \mathrm{e}^2/2 \\ 0 & \mathrm{e}^2 & \mathrm{e}^2 \\ 0 & 0 & \mathrm{e}^2 \end{pmatrix} \begin{pmatrix} 0 & -3 & -2 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\boxed{\mathrm{e}^{\mathbf{A}} = \begin{pmatrix} 0 & 2 & 2 \\ 3 & 1 & -1 \\ -5 & 3 & 5 \end{pmatrix} \frac{\mathrm{e}^2}{2}}$$

15.4 Using the Matrix Exponential

The homogeneous differential equation

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

has the solution

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\,\mathbf{c}$$

where $\mathbf{c}$ is a vector of constants. The solution subject to the initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\,\mathbf{x}_0.$$

The homogeneous differential equation

$$\mathbf{x}'(t) = \frac{1}{t}\mathbf{A}\mathbf{x}(t)$$

has the solution

$$\mathbf{x}(t) = t^{\mathbf{A}}\mathbf{c} \equiv \mathrm{e}^{\mathbf{A}\,\mathrm{Log}\,t}\,\mathbf{c},$$

where $\mathbf{c}$ is a vector of constants. The solution subject to the initial condition, $\mathbf{x}(t_0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = \left(\frac{t}{t_0}\right)^{\mathbf{A}}\mathbf{x}_0 \equiv \mathrm{e}^{\mathbf{A}\,\mathrm{Log}(t/t_0)}\,\mathbf{x}_0.$$

The inhomogeneous problem

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has the solution

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}(t-t_0)}\,\mathbf{x}_0 + \mathrm{e}^{\mathbf{A}t}\int_{t_0}^{t}\mathrm{e}^{-\mathbf{A}\tau}\,\mathbf{f}(\tau)\,\mathrm{d}\tau.$$

Example 15.4.1 *Consider the system*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}\mathbf{x}.$$

The general solution of the system of differential equations is

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\,\mathbf{c}.$$

In Example 15.3.3 we found $\mathrm{e}^{\mathbf{A}}$. $\mathbf{A}t$ *is just a constant times* $\mathbf{A}$. *The eigenvalues of* $\mathbf{A}t$ *are* $\{\lambda_k t\}$ *where* $\{\lambda_k\}$ *are the eigenvalues of* $\mathbf{A}$. *The generalized eigenvectors of* $\mathbf{A}t$ *are the same as those of* $\mathbf{A}$.

Consider $\mathrm{e}^{\mathbf{J}t}$. *The derivatives of* $f(\lambda) = \mathrm{e}^{\lambda t}$ *are* $f'(\lambda) = t\,\mathrm{e}^{\lambda t}$ *and* $f''(\lambda) = t^2\,\mathrm{e}^{\lambda t}$. *Thus we have*

$$\mathrm{e}^{\mathbf{J}t} = \begin{pmatrix} \mathrm{e}^{2t} & t\,\mathrm{e}^{2t} & t^2\,\mathrm{e}^{2t}/2 \\ 0 & \mathrm{e}^{2t} & t\,\mathrm{e}^{2t} \\ 0 & 0 & \mathrm{e}^{2t} \end{pmatrix}$$

$$\mathrm{e}^{\mathbf{J}t} = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \mathrm{e}^{2t}$$

We find e^{At} *with a similarity transformation.*

$$\mathrm{e}^{\mathbf{A}t} = \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\,\mathbf{S}^{-1}$$

$$\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \mathrm{e}^{2t} \begin{pmatrix} 0 & -3 & -2 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} 1-t & t & t \\ 2t - t^2/2 & 1 - t + t^2/2 & -t + t^2/2 \\ -3t + t^2/2 & 2t - t^2/2 & 1 + 2t - t^2/2 \end{pmatrix} \mathrm{e}^{2t}$$

The solution of the system of differential equations is

$$\boxed{\mathbf{x}(t) = \left(c_1 \begin{pmatrix} 1-t \\ 2t - t^2/2 \\ -3t + t^2/2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 - t + t^2/2 \\ 2t - t^2/2 \end{pmatrix} + c_3 \begin{pmatrix} t \\ -t + t^2/2 \\ 1 + 2t - t^2/2 \end{pmatrix} \right) \mathrm{e}^{2t}}$$

Example 15.4.2 *Consider the Euler equation system*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{1}{t}\mathbf{A}\mathbf{x} \equiv \frac{1}{t}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$$

The solution is $\mathbf{x}(t) = t^{\mathbf{A}}\mathbf{c}$. *Note that* $\mathbf{A}$ *is almost in Jordan canonical form. It has a one on the sub-diagonal instead of the super-diagonal. It is clear that a function of* $\mathbf{A}$ *is defined*

$$f(\mathbf{A}) = \begin{pmatrix} f(1) & 0 \\ f'(1) & f(1) \end{pmatrix}.$$

The function $f(\lambda) = t^\lambda$ *has the derivative* $f'(\lambda) = t^\lambda \log t$. *Thus the solution of the system is*

$$\boxed{\mathbf{x}(t) = \begin{pmatrix} t & 0 \\ t\log t & t \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\begin{pmatrix} t \\ t\log t \end{pmatrix} + c_2\begin{pmatrix} 0 \\ t \end{pmatrix}}$$

Example 15.4.3 *Consider an inhomogeneous system of differential equations.*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{f}(t) \equiv \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}, \quad t > 0.$$

The general solution is

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\,\mathbf{c} + \mathrm{e}^{\mathbf{A}t}\int \mathrm{e}^{-\mathbf{A}t}\,f(t)\,\mathrm{d}t.$$

First we find homogeneous solutions. The characteristic equation for the matrix is

$$\chi(\lambda) = \begin{vmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{vmatrix} = \lambda^2 = 0$$

$\lambda = 0$ *is an eigenvalue of multiplicity 2. Thus the Jordan canonical form of the matrix is*

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since $\mathrm{rank}(\mathrm{nullspace}(\mathbf{A}-0\mathbf{I}))=1$ *there is only one eigenvector. A generalized eigenvector of rank 2 satisfies*

$$(\mathbf{A}-0\mathbf{I})^2\mathbf{x}_2=\mathbf{0}$$
$$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\mathbf{x}_2=\mathbf{0}$$

We choose

$$\mathbf{x}_2=\begin{pmatrix}1\\ 0\end{pmatrix}$$

Now we generate the chain from $\mathbf{x}_2$*.*

$$\mathbf{x}_1=(\mathbf{A}-0\mathbf{I})\mathbf{x}_2=\begin{pmatrix}4\\ 8\end{pmatrix}$$

We define the matrix of generalized eigenvectors $\mathbf{S}$*.*

$$\mathbf{S}=\begin{pmatrix}4 & 1\\ 8 & 0\end{pmatrix}$$

The derivative of $f(\lambda)=\mathrm{e}^{\lambda t}$ *is* $f'(\lambda)=t\,\mathrm{e}^{\lambda t}$*. Thus*

$$\mathrm{e}^{\mathbf{J}t}=\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$$

The homogeneous solution of the differential equation system is $\mathbf{x}_h=\mathrm{e}^{\mathbf{A}t}\,\mathbf{c}$ *where*

$$\mathrm{e}^{\mathbf{A}t}=\mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\,\mathbf{S}^{-1}$$
$$\mathrm{e}^{\mathbf{A}t}=\begin{pmatrix}4 & 1\\ 8 & 0\end{pmatrix}\cdot\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1/8\\ 1 & -1/2\end{pmatrix}$$
$$\mathrm{e}^{\mathbf{A}t}=\begin{pmatrix}1+4t & -2t\\ 8t & 1-4t\end{pmatrix}$$

The general solution of the inhomogeneous system of equations is

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\,\mathbf{c} + \mathrm{e}^{\mathbf{A}t} \int \mathrm{e}^{-\mathbf{A}t}\, f(t)\,\mathrm{d}t$$

$$\mathbf{x}(t) = \begin{pmatrix} 1+4t & -2t \\ 8t & 1-4t \end{pmatrix} \mathbf{c} + \begin{pmatrix} 1+4t & -2t \\ 8t & 1-4t \end{pmatrix} \int \begin{pmatrix} 1-4t & 2t \\ -8t & 1+4t \end{pmatrix} \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix} \mathrm{d}t$$

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1+4t \\ 8t \end{pmatrix} + c_2 \begin{pmatrix} -2t \\ 1-4t \end{pmatrix} + \begin{pmatrix} 2 - 2\,\mathrm{Log}\,t + \frac{6}{t} - \frac{1}{2t^2} \\ 4 - 4\,\mathrm{Log}\,t + \frac{13}{t} \end{pmatrix}$$

We can tidy up the answer a little bit. First we take linear combinations of the homogeneous solutions to obtain a simpler form.

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2t \\ 4t-1 \end{pmatrix} + \begin{pmatrix} 2 - 2\,\mathrm{Log}\,t + \frac{6}{t} - \frac{1}{2t^2} \\ 4 - 4\,\mathrm{Log}\,t + \frac{13}{t} \end{pmatrix}$$

Then we subtract 2 times the first homogeneous solution from the particular solution.

$$\boxed{\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2t \\ 4t-1 \end{pmatrix} + \begin{pmatrix} -2\,\mathrm{Log}\,t + \frac{6}{t} - \frac{1}{2t^2} \\ -4\,\mathrm{Log}\,t + \frac{13}{t} \end{pmatrix}}$$

15.5 Exercises

Exercise 15.4 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Hint, Solution

Exercise 15.5 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Hint, Solution

Exercise 15.6 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hint, Solution

Exercise 15.7 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Hint, Solution

Exercise 15.8 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Hint, Solution

Exercise 15.9 (mathematica/ode/systems/systems.nb)
Find the solution of the following initial value problem. Describe the behavior of the solution as $t \to \infty$.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$$

Hint, Solution

Exercise 15.10

1. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x}. \tag{15.2}$$

   (a) Show that $\lambda = 2$ is an eigenvalue of multiplicity 3 of the coefficient matrix $\mathbf{A}$, and that there is only one corresponding eigenvector, namely

$$\mathbf{x}i^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

   (b) Using the information in part (i), write down one solution $\mathbf{x}^{(1)}(t)$ of the system (15.2). There is no other solution of a purely exponential form $\mathbf{x} = \mathbf{x}i\, \mathrm{e}^{\lambda t}$.

   (c) To find a second solution use the form $\mathbf{x} = \mathbf{x}it\, \mathrm{e}^{2t} + \boldsymbol{\eta}\, \mathrm{e}^{2t}$, and find appropriate vectors $\mathbf{x}i$ and $\boldsymbol{\eta}$. This gives a solution of the system (15.2) which is independent of the one obtained in part (ii).

(d) To find a third linearly independent solution use the form $\mathbf{x} = \mathbf{x}i(t^2/2)\,\mathrm{e}^{2t} + \boldsymbol{\eta}t\,\mathrm{e}^{2t} + \boldsymbol{\zeta}\,\mathrm{e}^{2t}$. Show that $\mathbf{x}i$, $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ satisfy the equations

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x}i = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \mathbf{x}i, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}.$$

The first two equations can be taken to coincide with those obtained in part (iii). Solve the third equation, and write down a third independent solution of the system (15.2).

2. Consider the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{x}. \tag{15.3}$$

(a) Show that $\lambda = 1$ is an eigenvalue of multiplicity 3 of the coefficient matrix $\mathbf{A}$, and that there are only two linearly independent eigenvectors, which we may take as

$$\mathbf{x}i^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{x}i^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

Find two independent solutions of equation (15.3).

(b) To find a third solution use the form $\mathbf{x} = \mathbf{x}it\,\mathrm{e}^{t} + \boldsymbol{\eta}e^{t}$; then show that $\mathbf{x}i$ and $\boldsymbol{\eta}$ must satisfy

$$(\mathbf{A} - \mathbf{I})\mathbf{x}i = \mathbf{0}, \quad (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \mathbf{x}i.$$

Show that the most general solution of the first of these equations is $\mathbf{x}i = c_1\mathbf{x}i_1 + c_2\mathbf{x}i_2$, where c_1 and c_2 are arbitrary constants. Show that, in order to solve the second of these equations it is necessary to take $c_1 = c_2$. Obtain such a vector $\boldsymbol{\eta}$, and use it to obtain a third independent solution of the system (15.3).

Hint, Solution

Exercise 15.11 (mathematica/ode/systems/systems.nb)
Consider the system of ODE's

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{A}$ is the constant 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$$

1. Find the eigenvalues and associated eigenvectors of $\mathbf{A}$. [HINT: notice that $\lambda = -1$ is a root of the characteristic polynomial of $\mathbf{A}$.]

2. Use the results from part (a) to construct $\mathrm{e}^{\mathbf{A}t}$ and therefore the solution to the initial value problem above.

3. Use the results of part (a) to find the general solution to

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{1}{t}\mathbf{A}\mathbf{x}.$$

Hint, Solution

Exercise 15.12 (mathematica/ode/systems/systems.nb)

1. Find the general solution to

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x}$$

   where

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

2. Solve

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{A}\mathbf{x} + \mathbf{g}(t), \quad \mathbf{x}(0) = \mathbf{0}$$

using $\mathbf{A}$ from part (a).

Hint, Solution

Exercise 15.13

Let $\mathbf{A}$ be an $n \times n$ matrix of constants. The system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{1}{t}\mathbf{A}\mathbf{x}, \tag{15.4}$$

is analogous to the Euler equation.

1. Verify that when $\mathbf{A}$ is a 2×2 constant matrix, elimination of (15.4) yields a second order Euler differential equation.

2. Now assume that $\mathbf{A}$ is an $n \times n$ matrix of constants. Show that this system, in analogy with the Euler equation has solutions of the form $\mathbf{x} = \mathbf{a}t^{\lambda}$ where $\mathbf{a}$ is a constant vector provided $\mathbf{a}$ and λ satisfy certain conditions.

3. Based on your experience with the treatment of multiple roots in the solution of constant coefficient systems, what form will the general solution of (15.4) take if λ is a multiple eigenvalue in the eigenvalue problem derived in part (b)?

4. Verify your prediction by deriving the general solution for the system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{1}{t}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$$

Hint, Solution

15.6 Hints

Hint 15.1

Hint 15.2

Hint 15.3

Hint 15.4

Hint 15.5

Hint 15.6

Hint 15.7

Hint 15.8

Hint 15.9

Hint 15.10

Hint 15.11

Hint 15.12

Hint 15.13

15.7 Solutions

Solution 15.1

We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The matrix has the distinct eigenvalues $\lambda_1 = -1 - \imath$, $\lambda_2 = -1 + \imath$. The corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 2 - \imath \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 + \imath \\ 1 \end{pmatrix}.$$

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 - \imath \\ 1 \end{pmatrix} \mathrm{e}^{(-1-\imath)t} + c_2 \begin{pmatrix} 2 + \imath \\ 1 \end{pmatrix} \mathrm{e}^{(-1+\imath)t}.$$

We can take the real and imaginary parts of either of these solution to obtain real-valued solutions.

$$\begin{pmatrix} 2 + \imath \\ 1 \end{pmatrix} \mathrm{e}^{(-1+\imath)t} = \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} \mathrm{e}^{-t} + \imath \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix} \mathrm{e}^{-t}$$

$$\mathbf{x} = c_1 \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} \mathrm{e}^{-t} + c_2 \begin{pmatrix} \cos(t) + 2\sin(t) \\ \sin(t) \end{pmatrix} \mathrm{e}^{-t}$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$c_1 = 1, \quad c_2 = -1$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \begin{pmatrix} \cos(t) - 3\sin(t) \\ \cos(t) - \sin(t) \end{pmatrix} \mathrm{e}^{-t}.}$$

Plotted in the phase plane, the solution spirals in to the origin as t increases. Both coordinates tend to zero as $t \to \infty$.

Solution 15.2
We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The matrix has the distinct eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1 - \imath\sqrt{2}$, $\lambda_3 = -1 + \imath\sqrt{2}$. The corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 2 + \imath\sqrt{2} \\ -1 + \imath\sqrt{2} \\ 3 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 - \imath\sqrt{2} \\ -1 - \imath\sqrt{2} \\ 3 \end{pmatrix}.$$

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{-2t} + c_2 \begin{pmatrix} 2 + \imath\sqrt{2} \\ -1 + \imath\sqrt{2} \\ 3 \end{pmatrix} \mathrm{e}^{(-1-\imath\sqrt{2})t} + c_3 \begin{pmatrix} 2 - \imath\sqrt{2} \\ -1 - \imath\sqrt{2} \\ 3 \end{pmatrix} \mathrm{e}^{(-1+\imath\sqrt{2})t}.$$

We can take the real and imaginary parts of the second or third solution to obtain two real-valued solutions.

$$\begin{pmatrix} 2 + \imath\sqrt{2} \\ -1 + \imath\sqrt{2} \\ 3 \end{pmatrix} \mathrm{e}^{(-1-\imath\sqrt{2})t} = \begin{pmatrix} 2\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ 3\cos(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t} + \imath \begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) - 2\sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) + \sin(\sqrt{2}t) \\ -3\sin(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t}$$

$$\mathbf{x} = c_1 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{-2t} + c_2 \begin{pmatrix} 2\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ 3\cos(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t} + c_3 \begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) - 2\sin(\sqrt{2}t) \\ \sqrt{2}\cos(\sqrt{2}t) + \sin(\sqrt{2}t) \\ -3\sin(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t}$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} 2 & 2 & \sqrt{2} \\ -2 & -1 & \sqrt{2} \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$c_1 = \frac{1}{3}, \quad c_2 = -\frac{1}{9}, \quad c_3 = \frac{5}{9\sqrt{2}}$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^{-2t} + \frac{1}{6} \begin{pmatrix} 2\cos(\sqrt{2}t) - 4\sqrt{2}\sin(\sqrt{2}t) \\ 4\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ -2\cos(\sqrt{2}t) - 5\sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} \mathrm{e}^{-t}.}$$

As $t \to \infty$, all coordinates tend to infinity. Plotted in the phase plane, the solution would spiral in to the origin.

Solution 15.3

Homogeneous Solution, Method 1. We designate the inhomogeneous system of differential equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t).$$

First we find homogeneous solutions. The characteristic equation for the matrix is

$$\chi(\lambda) = \begin{vmatrix} 4-\lambda & -2 \\ 8 & -4-\lambda \end{vmatrix} = \lambda^2 = 0$$

$\lambda = 0$ is an eigenvalue of multiplicity 2. The eigenvectors satisfy

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus we see that there is only one linearly independent eigenvector. We choose

$$\mathbf{x}i = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

One homogeneous solution is then

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathrm{e}^{0t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We look for a second homogeneous solution of the form

$$\mathbf{x}_2 = \mathbf{x}it + \boldsymbol{\eta}.$$

We substitute this into the homogeneous equation.

$$\mathbf{x}_2' = \mathbf{A}\mathbf{x}_2$$
$$\mathbf{x}i = \mathbf{A}(\mathbf{x}it + \boldsymbol{\eta})$$

We see that $\mathbf{x}i$ and $\boldsymbol{\eta}$ satisfy

$$\mathbf{A}\mathbf{x}i = 0, \qquad \mathbf{A}\boldsymbol{\eta} = \mathbf{x}i.$$

We choose $\mathbf{x}i$ to be the eigenvector that we found previously. The equation for $\boldsymbol{\eta}$ is then

$$\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$\boldsymbol{\eta}$ is determined up to an additive multiple of $\mathbf{x}i$. We choose

$$\boldsymbol{\eta} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}.$$

Thus a second homogeneous solution is

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}.$$

The general homogeneous solution of the system is

$$\mathbf{x}_h = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 2t - 1/2 \end{pmatrix}$$

We can write this in matrix notation using the fundamental matrix $\mathbf{\Psi}(t)$.

$$\mathbf{x}_h = \mathbf{\Psi}(t)\mathbf{c} = \begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Homogeneous Solution, Method 2. The similarity transform $\mathbf{c}^{-1}\mathbf{A}\mathbf{c}$ with

$$\mathbf{c} = \begin{pmatrix} 1 & 0 \\ 2 & -1/2 \end{pmatrix}$$

will convert the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$$

to Jordan canonical form. We make the change of variables,

$$\mathbf{y} = \begin{pmatrix} 1 & 0 \\ 2 & -1/2 \end{pmatrix} \mathbf{x}.$$

The homogeneous system becomes

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \begin{pmatrix} 1 & 0 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1/2 \end{pmatrix} \mathbf{y}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

The equation for y_2 is

$$y_2' = 0.$$
$$y_2 = c_2$$

The equation for y_1 becomes

$$y_1' = c_2.$$
$$y_1 = c_1 + c_2 t$$

The solution for $\mathbf{y}$ is then

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

We multiply this by $\mathbf{c}$ to obtain the homogeneous solution for $\mathbf{x}$.

$$\mathbf{x}_h = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 2t - 1/2 \end{pmatrix}$$

Inhomogeneous Solution. By the method of variation of parameters, a particular solution is

$$\mathbf{x}_p = \mathbf{\Psi}(t) \int \mathbf{\Psi}^{-1}(t) \mathbf{g}(t) \, \mathrm{d}t.$$

$$\mathbf{x}_p = \begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix} \int \begin{pmatrix} 1 - 4t & 2t \\ 4 & -2 \end{pmatrix} \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix} \mathrm{d}t$$

$$\mathbf{x}_p = \begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix} \int \begin{pmatrix} -2t^{-1} - 4t^{-2} + t^{-3} \\ 2t^{-2} + 4t^{-3} \end{pmatrix} \mathrm{d}t$$

$$\mathbf{x}_p = \begin{pmatrix} 1 & t \\ 2 & 2t - 1/2 \end{pmatrix} \begin{pmatrix} -2 \log t + 4t^{-1} - \frac{1}{2} t^{-2} \\ -2t^{-1} - 2t^{-2} \end{pmatrix}$$

$$\mathbf{x}_p = \begin{pmatrix} -2 - 2 \log t + 2t^{-1} - \frac{1}{2} t^{-2} \\ -4 - 4 \log t + 5t^{-1} \end{pmatrix}$$

By adding 2 times our first homogeneous solution, we obtain

$$\mathbf{x}_p = \begin{pmatrix} -2 \log t + 2t^{-1} - \frac{1}{2} t^{-2} \\ -4 \log t + 5t^{-1} \end{pmatrix}$$

The general solution of the system of differential equations is

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 2t - 1/2 \end{pmatrix} + \begin{pmatrix} -2 \log t + 2t^{-1} - \frac{1}{2} t^{-2} \\ -4 \log t + 5t^{-1} \end{pmatrix}}$$

Solution 15.4
We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0$.

$$\begin{aligned}
\mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0 \\
&= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\,\mathbf{S}^{-1}\mathbf{x}_0 \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{3t} \end{pmatrix} \frac{1}{4} \begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} \mathrm{e}^{-t} + \mathrm{e}^{3t} \\ \mathrm{e}^{-t} + 5\,\mathrm{e}^{3t} \end{pmatrix}
\end{aligned}$$

$$\boxed{\mathbf{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \mathrm{e}^{3t}}$$

Solution 15.5
We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0$.

$$\begin{aligned}
\mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0 \\
&= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\,\mathbf{S}^{-1}\mathbf{x}_0 \\
&= \begin{pmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^t & 0 & 0 \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & 0 & \mathrm{e}^{3t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 4 & -2 & -4 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 2\,\mathrm{e}^{2t} \\ -2\,\mathrm{e}^t + 2\,\mathrm{e}^{2t} \\ \mathrm{e}^t \end{pmatrix}
\end{aligned}$$

$$\boxed{\mathbf{x} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \mathrm{e}^t + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \mathrm{e}^{2t}.}$$

Solution 15.6

We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} -1 - \imath & 0 \\ 0 & -1 + \imath \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0$.

$$
\begin{aligned}
\mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0 \\
&= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\,\mathbf{S}^{-1}\mathbf{x}_0 \\
&= \begin{pmatrix} 2-\imath & 2+\imath \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{(-1-\imath)t} & 0 \\ 0 & \mathrm{e}^{(-1+\imath)t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} \imath & 1-\imath 2 \\ -\imath & 1+\imath 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (\cos(t) - 3\sin(t))\,\mathrm{e}^{-t} \\ (\cos(t) - \sin(t))\,\mathrm{e}^{-t} \end{pmatrix}
\end{aligned}
$$

$$
\boxed{\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-t}\cos(t) - \begin{pmatrix} 3 \\ 1 \end{pmatrix} \mathrm{e}^{-t}\sin(t)}
$$

Solution 15.7

We consider an initial value problem.

$$
\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
$$

The Jordan canonical form of the matrix is

$$
\mathbf{J} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1-\imath\sqrt{2} & 0 \\ 0 & 0 & -1+\imath\sqrt{2} \end{pmatrix}.
$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0$.

$$
\begin{aligned}
\mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0 \\
&= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\,\mathbf{S}^{-1}\mathbf{x}_0 \\
&= \frac{1}{3}\begin{pmatrix} 6 & 2+\imath\sqrt{2} & 2-\imath\sqrt{2} \\ -6 & -1+\imath\sqrt{2} & -1-\imath\sqrt{2} \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-2t} & 0 & 0 \\ 0 & \mathrm{e}^{(-1-\imath\sqrt{2})t} & 0 \\ 0 & 0 & \mathrm{e}^{(-1+\imath\sqrt{2})t} \end{pmatrix} \\
&\quad \frac{1}{6}\begin{pmatrix} 2 & -2 & -2 \\ -1-\imath 5\sqrt{2}/2 & 1-\imath 2\sqrt{2} & 4+\imath\sqrt{2} \\ -1+\imath 5\sqrt{2}/2 & 1+\imath 2\sqrt{2} & 4-\imath\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

$$
\boxed{\mathbf{x} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\mathrm{e}^{-2t} + \frac{1}{6}\begin{pmatrix} 2\cos(\sqrt{2}t) - 4\sqrt{2}\sin(\sqrt{2}t) \\ 4\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \\ -2\cos(\sqrt{2}t) - 5\sqrt{2}\sin(\sqrt{2}t) \end{pmatrix}\mathrm{e}^{-t}.}
$$

Solution 15.8

We consider an initial value problem.

$$
\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} 3 \\ 2 \end{pmatrix}
$$

Method 1. Find Homogeneous Solutions. The matrix has the double eigenvalue $\lambda_1 = \lambda_2 = -3$. There is only one corresponding eigenvector. We compute a chain of generalized eigenvectors.

$$(\mathbf{A}+3\mathbf{I})^2\mathbf{x}_2 = \mathbf{0}$$
$$\mathbf{0}\mathbf{x}_2 = \mathbf{0}$$
$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$(\mathbf{A}+3\mathbf{I})\mathbf{x}_2 = \mathbf{x}_1$$
$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{e}^{-3t} + c_2 \left(\begin{pmatrix} 4 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \mathrm{e}^{-3t}.$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$
$$c_1 = 2, \quad c_2 = 1$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \begin{pmatrix} 3+4t \\ 2+4t \end{pmatrix} \mathrm{e}^{-3t}.}$$

Both coordinates tend to zero as $t \to \infty$.

Method 2. Use the Exponential Matrix. The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0$.

$$\begin{aligned}
\mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0 \\
&= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\,\mathbf{S}^{-1}\mathbf{x}_0 \\
&= \begin{pmatrix} 1 & 1/4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-3t} & t\,\mathrm{e}^{-3t} \\ 0 & \mathrm{e}^{-3t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}
\end{aligned}$$

$$\boxed{\mathbf{x} = \begin{pmatrix} 3+4t \\ 2+4t \end{pmatrix} \mathrm{e}^{-3t}.}$$

Solution 15.9
We consider an initial value problem.

$$\mathbf{x}' = \mathbf{A}x \equiv \begin{pmatrix} -1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \equiv \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$$

Method 1. Find Homogeneous Solutions. The matrix has the distinct eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$, $\lambda_3 = 2$. The corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The general solution of the system of differential equations is

$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \mathrm{e}^{-t} + c_2 \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \mathrm{e}^{t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{2t}.$$

We apply the initial condition to determine the constants.

$$\begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ 5 & 6 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix}$$

$$c_1 = 1, \quad c_2 = -4, \quad c_3 = -11$$

The solution subject to the initial condition is

$$\boxed{\mathbf{x} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \mathrm{e}^{-t} - 4 \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \mathrm{e}^{t} - 11 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{2t}.}$$

As $t \to \infty$, the first coordinate vanishes, the second coordinate tends to ∞ and the third coordinate tends to $-\infty$

Method 2. Use the Exponential Matrix. The Jordan canonical form of the matrix is

$$\mathbf{J} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The solution of the initial value problem is $\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0$.

$$\begin{aligned} \mathbf{x} &= \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}_0 \\ &= \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\,\mathbf{S}^{-1}\mathbf{x}_0 \\ &= \begin{pmatrix} -1 & 0 & 0 \\ -2 & -1 & 0 \\ 5 & 6 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-t} & 0 & 0 \\ 0 & \mathrm{e}^{t} & 0 \\ 0 & 0 & \mathrm{e}^{2t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ -7 & 6 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix} \end{aligned}$$

$$\boxed{\mathbf{x} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \mathrm{e}^{-t} - 4 \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix} \mathrm{e}^{t} - 11 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{2t}.}$$

Solution 15.10

1. (a) We compute the eigenvalues of the matrix.

$$\chi(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -3 & 2 & 4-\lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda - 2)^3$$

$\lambda = 2$ is an eigenvalue of multiplicity 3. The rank of the null space of $\mathbf{A} - 2\mathbf{I}$ is 1. (The first two rows are linearly independent, but the third is a linear combination of the first two.)

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}$$

Thus there is only one eigenvector.

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \mathbf{0}$$

$$\mathbf{x}i^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(b) One solution of the system of differential equations is

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \mathrm{e}^{2t}.$$

(c) We substitute the form $\mathbf{x} = \mathbf{x}it\,\mathrm{e}^{2t} + \boldsymbol{\eta}\,\mathrm{e}^{2t}$ into the differential equation.

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

$$\mathbf{x}i\,\mathrm{e}^{2t} + 2\mathbf{x}it\,\mathrm{e}^{2t} + 2\boldsymbol{\eta}\,\mathrm{e}^{2t} = \mathbf{A}\mathbf{x}it\,\mathrm{e}^{2t} + \mathbf{A}\boldsymbol{\eta}\,\mathrm{e}^{2t}$$

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x}i = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \mathbf{x}i$$

We already have a solution of the first equation, we need the generalized eigenvector $\boldsymbol{\eta}$. Note that $\boldsymbol{\eta}$ is only determined up to a constant times $\mathbf{x}i$. Thus we look for the solution whose second component vanishes to simplify the algebra.

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \mathbf{x}i$$
$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ 0 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$
$$-\eta_1 + \eta_3 = 0, \quad 2\eta_1 - \eta_3 = 1, \quad -3\eta_1 + 2\eta_3 = -1$$
$$\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

A second linearly independent solution is

$$\boxed{\mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t\,\mathrm{e}^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{2t}.}$$

(d) To find a third solution we substutite the form $\mathbf{x} = \mathbf{x}i(t^2/2)\,\mathrm{e}^{2t} + \boldsymbol{\eta}t\,\mathrm{e}^{2t} + \boldsymbol{\zeta}\,\mathrm{e}^{2t}$ into the differential equation.

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$
$$2\mathbf{x}i(t^2/2)\,\mathrm{e}^{2t} + (\mathbf{x}i + 2\boldsymbol{\eta})t\,\mathrm{e}^{2t} + (\boldsymbol{\eta} + 2\boldsymbol{\zeta})\,\mathrm{e}^{2t} = \mathbf{A}\mathbf{x}i(t^2/2)\,\mathrm{e}^{2t} + \mathbf{A}\boldsymbol{\eta}t\,\mathrm{e}^{2t} + \mathbf{A}\boldsymbol{\zeta}\,\mathrm{e}^{2t}$$
$$(\mathbf{A} - 2\mathbf{I})\mathbf{x}i = \mathbf{0}, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \mathbf{x}i, \quad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$$

We have already solved the first two equations, we need the generalized eigenvector $\boldsymbol{\zeta}$. Note that $\boldsymbol{\zeta}$ is only determined up to a constant times $\mathbf{x}i$. Thus we look for the solution whose second component vanishes to simplify the algebra.

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\zeta} = \boldsymbol{\eta}$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ 0 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$-\zeta_1 + \zeta_3 = 1, \quad 2\zeta_1 - \zeta_3 = 0, \quad -3\zeta_1 + 2\zeta_3 = 1$$

$$\boldsymbol{\zeta} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

A third linearly independent solution is

$$\boxed{\mathbf{x}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} (t^2/2)\,\mathrm{e}^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t\,\mathrm{e}^{2t} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \mathrm{e}^{2t}}$$

2. (a) We compute the eigenvalues of the matrix.

$$\chi(\lambda) = \begin{vmatrix} 5-\lambda & -3 & -2 \\ 8 & -5-\lambda & -4 \\ -4 & 3 & 3-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3$$

$\lambda = 1$ is an eigenvalue of multiplicity 3. The rank of the null space of $\mathbf{A} - \mathbf{I}$ is 2. (The second and third rows are multiples of the first.)

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix}$$

Thus there are two eigenvectors.

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \mathbf{0}$$

$$\mathbf{x}i^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{x}i^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

Two linearly independent solutions of the differential equation are

$$\boxed{\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \mathrm{e}^t, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \mathrm{e}^t.}$$

(b) We substitute the form $\mathbf{x} = \mathbf{x}it\,\mathrm{e}^t + \boldsymbol{\eta}\,\mathrm{e}^t$ into the differential equation.

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

$$\mathbf{x}i\,\mathrm{e}^t + \mathbf{x}it\,\mathrm{e}^t + \boldsymbol{\eta}\,\mathrm{e}^t = \mathbf{A}\mathbf{x}it\,\mathrm{e}^t + \mathbf{A}\boldsymbol{\eta}\,\mathrm{e}^t$$

$$(\mathbf{A} - \mathbf{I})\mathbf{x}i = \mathbf{0}, \quad (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \mathbf{x}i$$

The general solution of the first equation is a linear combination of the two solutions we found in the previous part.

$$\mathbf{x}i = c_1\mathbf{x}i_1 + c_2\mathbf{x}i_2$$

Now we find the generalized eigenvector, $\boldsymbol{\eta}$. Note that $\boldsymbol{\eta}$ is only determined up to a linear combination of $\mathbf{x}i_1$ and $\mathbf{x}i_2$. Thus we can take the first two components of $\boldsymbol{\eta}$ to be zero.

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \eta_3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

$$-2\eta_3 = c_1, \quad -4\eta_3 = 2c_2, \quad 2\eta_3 = 2c_1 - 3c_2$$

$$c_1 = c_2, \quad \eta_3 = -\frac{c_1}{2}$$

We see that we must take $c_1 = c_2$ in order to obtain a solution. We choose $c_1 = c_2 = 2$ A third linearly independent solution of the differential equation is

$$\boxed{\mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} t\,\mathrm{e}^t + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{e}^t.}$$

Solution 15.11

1. The characteristic polynomial of the matrix is

$$\begin{aligned}
\chi(\lambda) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & -1 \\ -8 & -5 & -3-\lambda \end{vmatrix} \\
&= (1-\lambda)^2(-3-\lambda) + 8 - 10 - 5(1-\lambda) - 2(-3-\lambda) - 8(1-\lambda) \\
&= -\lambda^3 - \lambda^2 + 4\lambda + 4 \\
&= -(\lambda+2)(\lambda+1)(\lambda-2)
\end{aligned}$$

Thus we see that the eigenvalues are $\lambda = -2, -1, 2$. The eigenvectors $\mathbf{x}i$ satisfy

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x}i = \mathbf{0}.$$

For $\lambda = -2$, we have

$$(\mathbf{A} + 2\mathbf{I})\mathbf{x}i = \mathbf{0}.$$

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & -1 \\ -8 & -5 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If we take $\xi_3 = 1$ then the first two rows give us the system,

$$\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

which has the solution $\xi_1 = -4/7$, $\xi_2 = 5/7$. For the first eigenvector we choose:

$$\mathbf{x}i = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}$$

For $\lambda = -1$, we have

$$(\mathbf{A} + \mathbf{I})\mathbf{x}i = \mathbf{0}.$$
$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ -8 & -5 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If we take $\xi_3 = 1$ then the first two rows give us the system,

$$\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

which has the solution $\xi_1 = -3/2$, $\xi_2 = 2$. For the second eigenvector we choose:

$$\mathbf{x}i = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$$

For $\lambda = 2$, we have

$$(\mathbf{A} + \mathbf{I})\mathbf{x}i = \mathbf{0}.$$
$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -8 & -5 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

If we take $\xi_3 = 1$ then the first two rows give us the system,

$$\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

which has the solution $\xi_1 = 0$, $\xi_2 = -1$. For the third eigenvector we choose:

$$\mathbf{x}i = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

In summary, the eigenvalues and eigenvectors are

$$\boxed{\lambda = \{-2, -1, 2\}, \qquad \mathbf{x}i = \left\{ \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}}$$

2. The matrix is diagonalized with the similarity transformation

$$\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S},$$

where $\mathbf{S}$ is the matrix with eigenvectors as columns:

$$\mathbf{S} = \begin{pmatrix} -4 & -3 & 0 \\ 5 & 4 & -1 \\ 7 & 2 & 1 \end{pmatrix}$$

The matrix exponential, $\mathrm{e}^{\mathbf{A}t}$ is given by

$$\mathrm{e}^{\mathbf{A}} = \mathbf{S}\,\mathrm{e}^{\mathbf{J}}\,\mathbf{S}^{-1}.$$

$$\mathrm{e}^{\mathbf{A}} = \begin{pmatrix} -4 & -3 & 0 \\ 5 & 4 & -1 \\ 7 & 2 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-2t} & 0 & 0 \\ 0 & \mathrm{e}^{-t} & 0 \\ 0 & 0 & \mathrm{e}^{2t} \end{pmatrix} \frac{1}{12} \begin{pmatrix} 6 & 3 & 3 \\ -12 & -4 & -4 \\ -18 & -13 & -1 \end{pmatrix}.$$

$$
\boxed{e^{\mathbf{A}t} = \begin{pmatrix} -2\,e^{-2t} + 3\,e^{-t} & -\,e^{-2t} + e^{-t} & -\,e^{-2t} + e^{-t} \\ \frac{5\,e^{-2t} - 8\,e^{-t} + 3\,e^{t}}{2} & \frac{15\,e^{-2t} - 16\,e^{-t} + 13\,e^{t}}{12} & \frac{15\,e^{-2t} - 16\,e^{-t} + e^{t}}{12} \\ \frac{7\,e^{-2t} - 4\,e^{-t} - 3\,e^{t}}{2} & \frac{21\,e^{-2t} - 8\,e^{-t} - 13\,e^{t}}{12} & \frac{21\,e^{-2t} - 8\,e^{-t} - e^{t}}{12} \end{pmatrix}}
$$

The solution of the initial value problem is $e^{\mathbf{A}t}\,\mathbf{x}_0$.

3. The general solution of the Euler equation is

$$
\boxed{c_1 \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix} t^{-2} + c_2 \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} t^{-1} + c_3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t^2.}
$$

We could also write the solution as

$$
\mathbf{x} = t^{\mathbf{A}}\mathbf{c} \equiv e^{\mathbf{A}\log t}\,\mathbf{c},
$$

Solution 15.12

1. The characteristic polynomial of the matrix is

$$
\begin{aligned}
\chi(\lambda) &= \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} \\
&= (2-\lambda)^2(3-\lambda)
\end{aligned}
$$

Thus we see that the eigenvalues are $\lambda = 2, 2, 3$. Consider

$$
\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{pmatrix}.
$$

Since $\operatorname{rank}(\operatorname{nullspace}(\mathbf{A} - 2\mathbf{I})) = 1$ there is one eigenvector and one generalized eigenvector of rank two for

$\lambda = 2$. The generalized eigenvector of rank two satisfies

$$(\mathbf{A} - 2\mathbf{I})^2 \mathbf{x}i_2 = \mathbf{0}$$
$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{x}i_2 = \mathbf{0}$$

We choose the solution

$$\mathbf{x}i_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

The eigenvector for $\lambda = 2$ is

$$\mathbf{x}i_1 = (\mathbf{A} - 2\mathbf{I})\mathbf{x}i_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The eigenvector for $\lambda = 3$ satisfies

$$(\mathbf{A} - 3\mathbf{I})^2 \mathbf{x}i = \mathbf{0}$$
$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{x}i = \mathbf{0}$$

We choose the solution

$$\mathbf{x}i = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvalues and generalized eigenvectors are

$$\lambda = \{2, 2, 3\}, \qquad \mathbf{x}i = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The matrix of eigenvectors and its inverse is

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \mathbf{S}^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The Jordan canonical form of the matrix, which satisfies $\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is

$$\mathbf{J} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Recall that the function of a Jordan block is:

$$f\left(\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}\right) = \begin{pmatrix} f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} & \frac{f'''(\lambda)}{3!} \\ 0 & f(\lambda) & \frac{f'(\lambda)}{1!} & \frac{f''(\lambda)}{2!} \\ 0 & 0 & f(\lambda) & \frac{f'(\lambda)}{1!} \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix},$$

and that the function of a matrix in Jordan canonical form is

$$f\left(\begin{pmatrix} \mathbf{J}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}_4 \end{pmatrix}\right) = \begin{pmatrix} f(\mathbf{J}_1) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & f(\mathbf{J}_2) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & f(\mathbf{J}_3) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & f(\mathbf{J}_4) \end{pmatrix}.$$

We want to compute $\mathrm{e}^{\mathbf{J}t}$ so we consider the function $f(\lambda) = \mathrm{e}^{\lambda t}$, which has the derivative $f'(\lambda) = t\,\mathrm{e}^{\lambda t}$. Thus we see that

$$\mathrm{e}^{\mathbf{J}t} = \begin{pmatrix} \mathrm{e}^{2t} & t\,\mathrm{e}^{2t} & 0 \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & 0 & \mathrm{e}^{3t} \end{pmatrix}$$

The exponential matrix is

$$\mathrm{e}^{\mathbf{A}t} = \mathbf{S}\,\mathrm{e}^{\mathbf{J}t}\,\mathbf{S}^{-1},$$

$$\boxed{\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} \mathrm{e}^{2t} & -(1+t)\,\mathrm{e}^{2t} + \mathrm{e}^{3t} & -\mathrm{e}^{2t} + \mathrm{e}^{3t} \\ 0 & \mathrm{e}^{2t} & 0 \\ 0 & -\mathrm{e}^{2t} + \mathrm{e}^{3t} & \mathrm{e}^{3t} \end{pmatrix}.}$$

The general solution of the homogeneous differential equation is

$$\boxed{\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\,\mathbf{c}.}$$

2. The solution of the inhomogeneous differential equation subject to the initial condition is

$$\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\,\mathbf{0} + \mathrm{e}^{\mathbf{A}t}\int_0^t \mathrm{e}^{-\mathbf{A}\tau}\,\mathbf{g}(\tau)\,\mathrm{d}\tau$$

$$\boxed{\mathbf{x} = \mathrm{e}^{\mathbf{A}t}\int_0^t \mathrm{e}^{-\mathbf{A}\tau}\,\mathbf{g}(\tau)\,\mathrm{d}\tau}$$

Solution 15.13

1.

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{1}{t}\mathbf{A}\mathbf{x}$$

$$t\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The first component of this equation is

$$t x_1' = a x_1 + b x_2.$$

We differentiate and multiply by t to obtain a second order coupled equation for x_1. We use (15.4) to eliminate the dependence on x_2.

$$\begin{gathered}
t^2 x_1'' + t x_1' = a t x_1' + b t x_2' \\
t^2 x_1'' + (1-a) t x_1' = b(c x_1 + d x_2) \\
t^2 x_1'' + (1-a) t x_1' - bc x_1 = d(t x_1' - a x_1) \\
t^2 x_1'' + (1-a-d) t x_1' + (ad - bc) x_1 = 0
\end{gathered}$$

Thus we see that x_1 satisfies a second order, Euler equation. By symmetry we see that x_2 satisfies,

$$t^2 x_2'' + (1-b-c) t x_2' + (bc - ad) x_2 = 0.$$

2. We substitute $\mathbf{x} = \mathbf{a} t^\lambda$ into (15.4).

$$\begin{gathered}
\lambda \mathbf{a} t^{\lambda-1} = \frac{1}{t} \mathbf{A} \mathbf{a} t^\lambda \\
\mathbf{A} \mathbf{a} = \lambda \mathbf{a}
\end{gathered}$$

Thus we see that $\mathbf{x} = \mathbf{a} t^\lambda$ is a solution if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{a}$.

3. Suppose that $\lambda = \alpha$ is an eigenvalue of multiplicity 2. If $\lambda = \alpha$ has two linearly independent eigenvectors, $\mathbf{a}$ and $\mathbf{b}$ then $\mathbf{a} t^\alpha$ and $\mathbf{b} t^\alpha$ are linearly independent solutions. If $\lambda = \alpha$ has only one linearly independent eigenvector, $\mathbf{a}$, then $\mathbf{a} t^\alpha$ is a solution. We look for a second solution of the form

$$\mathbf{x} = \mathbf{x}i t^\alpha \log t + \boldsymbol{\eta} t^\alpha.$$

Substituting this into the differential equation yields

$$\alpha \mathbf{x}i t^{\alpha-1} \log t + \mathbf{x}i t^{\alpha-1} + \alpha \boldsymbol{\eta} t^{\alpha-1} = \mathbf{A} \mathbf{x}i t^{\alpha-1} \log t + \mathbf{A} \boldsymbol{\eta} t^{\alpha-1}$$

We equate coefficients of $t^{\alpha-1} \log t$ and $t^{\alpha-1}$ to determine $\mathbf{x}i$ and $\boldsymbol{\eta}$.

$$(\mathbf{A} - \alpha \mathbf{I}) \mathbf{x}i = \mathbf{0}, \qquad (\mathbf{A} - \alpha \mathbf{I}) \boldsymbol{\eta} = \mathbf{x}i$$

These equations have solutions because $\lambda = \alpha$ has generalized eigenvectors of first and second order.

Note that the change of independent variable $\tau = \log t$, $\mathbf{y}(\tau) = \mathbf{x}(t)$, will transform (15.4) into a constant coefficient system.

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\tau} = \mathbf{A}\mathbf{y}$$

Thus all the methods for solving constant coefficient systems carry over directly to solving (15.4). In the case of eigenvalues with multiplicity greater than one, we will have solutions of the form,

$$\mathbf{x}i t^\alpha, \quad \mathbf{x}i t^\alpha \log t + \boldsymbol{\eta} t^\alpha, \quad \mathbf{x}i t^\alpha (\log t)^2 + \boldsymbol{\eta} t^\alpha \log t + \boldsymbol{\zeta} t^\alpha, \quad \ldots,$$

analogous to the form of the solutions for a constant coefficient system,

$$\mathbf{x}i \,\mathrm{e}^{\alpha\tau}, \quad \mathbf{x}i\tau \,\mathrm{e}^{\alpha\tau} + \boldsymbol{\eta}\,\mathrm{e}^{\alpha\tau}, \quad \mathbf{x}i\tau^2 \,\mathrm{e}^{\alpha\tau} + \boldsymbol{\eta}\tau\,\mathrm{e}^{\alpha\tau} + \boldsymbol{\zeta}\,\mathrm{e}^{\alpha\tau}, \quad \ldots.$$

4. **Method 1.** Now we consider

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{1}{t}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$$

The characteristic polynomial of the matrix is

$$\chi(\lambda) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2.$$

$\lambda = 1$ is an eigenvalue of multiplicity 2. The equation for the associated eigenvectors is

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There is only one linearly independent eigenvector, which we choose to be

$$\mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One solution of the differential equation is

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t.$$

We look for a second solution of the form

$$\mathbf{x}_2 = \mathbf{a} t \log t + \boldsymbol{\eta} t.$$

$\boldsymbol{\eta}$ satisfies the equation

$$(\mathbf{A} - I)\boldsymbol{\eta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution is determined only up to an additive multiple of $\mathbf{a}$. We choose

$$\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus a second linearly independent solution is

$$\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t \log t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t.$$

The general solution of the differential equation is

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} t + c_2 \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} t \log t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t \right).}$$

Method 2. Note that the matrix is lower triangular.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{1}{t} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{15.5}$$

We have an uncoupled equation for x_1.

$$x_1' = \frac{1}{t} x_1$$
$$x_1 = c_1 t$$

By substituting the solution for x_1 into (15.5), we obtain an uncoupled equation for x_2.

$$
\begin{aligned}
x_2' &= \frac{1}{t}\left(c_1 t + x_2\right) \\
x_2' - \frac{1}{t} x_2 &= c_1 \\
\left(\frac{1}{t} x_2\right)' &= \frac{c_1}{t} \\
\frac{1}{t} x_2 &= c_1 \log t + c_2 \\
x_2 &= c_1 t \log t + c_2 t
\end{aligned}
$$

Thus the solution of the system is

$$
\mathbf{x} = \begin{pmatrix} c_1 t \\ c_1 t \log t + c_2 t \end{pmatrix},
$$

$$
\boxed{\mathbf{x} = c_1 \begin{pmatrix} t \\ t \log t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ t \end{pmatrix},}
$$

which is equivalent to the solution we obtained previously.

Chapter 16

Theory of Linear Ordinary Differential Equations

A little partyin' is good for the soul.

-Matt Metz

16.1 Exact Equations

Exercise 16.1
Consider a second order, linear, homogeneous differential equation:

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{16.1}$$

Show that $P'' - Q' + R = 0$ is a necessary and sufficient condition for this equation to be exact.
Hint, Solution

Exercise 16.2
Determine an equation for the integrating factor $\mu(x)$ for Equation 16.1.

Hint, Solution

Exercise 16.3
Show that

$$y'' + xy' + y = 0$$

is exact. Find the solution.
Hint, Solution

16.2 Nature of Solutions

Result 16.2.1 Consider the n^{th} order ordinary differential equation of the form

$$L[y] = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x)y = f(x). \tag{16.2}$$

If the coefficient functions $p_{n-1}(x), \ldots, p_0(x)$ and the inhomogeneity $f(x)$ are continuous on some interval $a < x < b$ then the differential equation subject to the conditions,

$$y(x_0) = v_0, \quad y'(x_0) = v_1, \quad \ldots \quad y^{(n-1)}(x_0) = v_{n-1}, \qquad a < x_0 < b,$$

has a unique solution on the interval.

Exercise 16.4
On what intervals do the following problems have unique solutions?

1. $xy'' + 3y = x$

2. $x(x-1)y'' + 3xy' + 4y = 2$

3. $e^x y'' + x^2 y' + y = \tan x$

Hint, Solution

Linearity of the Operator. The differential operator L is linear. To verify this,

$$\begin{aligned}
L[cy] &= \frac{d^n}{dx^n}(cy) + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}(cy) + \cdots + p_1(x)\frac{d}{dx}(cy) + p_0(x)(cy) \\
&= c\frac{d^n}{dx^n}y + cp_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}y + \cdots + cp_1(x)\frac{d}{dx}y + cp_0(x)y \\
&= cL[y] \\
L[y_1 + y_2] &= \frac{d^n}{dx^n}(y_1 + y_2) + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}(y_1 + y_2) + \cdots + p_1(x)\frac{d}{dx}(y_1 + y_2) + p_0(x)(y_1 + y_2) \\
&= \frac{d^n}{dx^n}(y_1) + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}(y_1) + \cdots + p_1(x)\frac{d}{dx}(y_1) + p_0(x)(y_1) \\
&\quad + \frac{d^n}{dx^n}(y_2) + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}(y_2) + \cdots + p_1(x)\frac{d}{dx}(y_2) + p_0(x)(y_2) \\
&= L[y_1] + L[y_2].
\end{aligned}$$

Homogeneous Solutions. The general homogeneous equation has the form

$$L[y] = \frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y = 0.$$

From the linearity of L, we see that if y_1 and y_2 are solutions to the homogeneous equation then $c_1 y_1 + c_2 y_2$ is also a solution, ($L[c_1 y_1 + c_2 y_2] = 0$).

On any interval where the coefficient functions are continuous, the n^{th} order linear homogeneous equation has n linearly independent solutions, $y_1, y_2, \ldots, y_n$. (We will study linear independence in Section 16.4.) The general solution to the homogeneous problem is then

$$y_h = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

Particular Solutions. Any function, y_p, that satisfies the inhomogeneous equation, $L[y_p] = f(x)$, is called a particular solution or particular integral of the equation. Note that for linear differential equations the particular solution is not unique. If y_p is a particular solution then $y_p + y_h$ is also a particular solution where y_h is any homogeneous solution.

The general solution to the problem $L[y] = f(x)$ is the sum of a particular solution and a linear combination of the homogeneous solutions

$$y = y_p + c_1 y_1 + \cdots + c_n y_n.$$

Example 16.2.1 *Consider the differential equation*

$$y'' - y' = 1.$$

You can verify that two homogeneous solutions are e^x *and* 1*. A particular solution is* $-x$*. Thus the general solution is*

$$y = -x + c_1 \mathrm{e}^x + c_2.$$

Exercise 16.5
Suppose you are able to find three linearly independent particular solutions $u_1(x)$, $u_2(x)$ and $u_3(x)$ of the second order linear differential equation $L[y] = f(x)$. What is the general solution?
Hint, Solution

Real-Valued Solutions. If the coefficient function and the inhomogeneity in Equation 16.2 are real-valued, then the general solution can be written in terms of real-valued functions. Let y be any, homogeneous solution, (perhaps complex-valued). By taking the complex conjugate of the equation $L[y] = 0$ we show that $\bar{y}$ is a homogeneous solution as well.

$$\begin{gathered}
L[y] = 0 \\
\overline{L[y]} = 0 \\
\overline{y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y} = 0 \\
\bar{y}^{(n)} + p_{n-1} \bar{y}^{(n-1)} + \cdots + p_0 \bar{y} = 0 \\
L\left[\bar{y}\right] = 0
\end{gathered}$$

For the same reason, if y_p is a particular solution, then $\overline{y_p}$ is a particular solution as well.

Since the real and imaginary parts of a function y are linear combinations of y and $\bar{y}$,

$$\Re(y) = \frac{y + \bar{y}}{2}, \quad \Im(y) = \frac{y - \bar{y}}{\imath 2},$$

if y is a homogeneous solution then both $\Re y$ and $\Im(y)$ are homogeneous solutions. Likewise, if y_p is a particular solution then $\Re(y_p)$ is a particular solution.

$$L\left[\Re(y_p)\right] = L\left[\frac{y_p + \overline{y_p}}{2}\right] = \frac{f}{2} + \frac{f}{2} = f$$

Thus we see that the homogeneous solution, the particular solution and the general solution of a linear differential equation with real-valued coefficients and inhomogeneity can be written in terms of real-valued functions.

Result 16.2.2 The differential equation

$$L[y] = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x)y = f(x)$$

with continuous coefficients and inhomogeneity has a general solution of the form

$$y = y_p + c_1 y_1 + \cdots + c_n y_n$$

where y_p is a particular solution, $L[y_p] = f$, and the y_k are linearly independent homogeneous solutions, $L[y_k] = 0$. If the coefficient functions and inhomogeneity are real-valued, then the general solution can be written in terms of real-valued functions.

16.3 Transformation to a First Order System

Any linear differential equation can be put in the form of a system of first order differential equations. Consider

$$y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_0 y = f(x).$$

We introduce the functions,

$$y_1 = y, \quad y_2 = y', \quad , \ldots, \quad y_n = y^{(n-1)}.$$

The differential equation is equivalent to the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ \vdots &= \vdots \\ y_n' &= f(x) - p_{n-1} y_n - \cdots - p_0 y_1. \end{aligned}$$

The first order system is more useful when numerically solving the differential equation.

Example 16.3.1 *Consider the differential equation*

$$y'' + x^2 y' + \cos x \; y = \sin x.$$

The corresponding system of first order equations is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \sin x - x^2 y_2 - \cos x \; y_1. \end{aligned}$$

16.4 The Wronskian

16.4.1 Derivative of a Determinant.

Before investigating the Wronskian, we will need a preliminary result from matrix theory. Consider an $n \times n$ matrix A whose elements $a_{ij}(x)$ are functions of x. We will denote the determinant by $\Delta[A(x)]$. We then have the following theorem.

Result 16.4.1 Let $a_{ij}(x)$, the elements of the matrix A, be differentiable functions of x. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\Delta[A(x)] = \sum_{k=1}^{n} \Delta_k[A(x)]$$

where $\Delta_k[A(x)]$ is the determinant of the matrix A with the k^{th} row replaced by the derivative of the k^{th} row.

Example 16.4.1 *Consider the the matrix*

$$A(x) = \begin{pmatrix} x & x^2 \\ x^2 & x^4 \end{pmatrix}$$

The determinant is $x^5 - x^4$ *thus the derivative of the determinant is* $5x^4 - 4x^3$. *To check the theorem,*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\Delta[A(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\begin{vmatrix} x & x^2 \\ x^2 & x^4 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 2x \\ x^2 & x^4 \end{vmatrix} + \begin{vmatrix} x & x^2 \\ 2x & 4x^3 \end{vmatrix} \\
&= x^4 - 2x^3 + 4x^4 - 2x^3 \\
&= 5x^4 - 4x^3.
\end{aligned}$$

16.4.2 The Wronskian of a Set of Functions.

A set of functions $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent on an interval if there are constants $c_1, \ldots, c_n$ not all zero such that

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0 \tag{16.3}$$

identically on the interval. The set is linearly independent if all of the constants must be zero to satisfy $c_1 y_1 + \cdots c_n y_n = 0$ on the interval.

Consider a set of functions $\{y_1, y_2, \dots, y_n\}$ that are linearly dependent on a given interval and $n-1$ times differentiable. There are a set of constants, not all zero, that satisfy equation 16.3

Differentiating equation 16.3 $n-1$ times gives the equations,

$$c_1 y_1' + c_2 y_2' + \cdots + c_n y_n' = 0$$
$$c_1 y_1'' + c_2 y_2'' + \cdots + c_n y_n'' = 0$$
$$\cdots$$
$$c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \cdots + c_n y_n^{(n-1)} = 0.$$

We could write the problem to find the constants as

$$\begin{pmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ y_1'' & y_2'' & \dots & y_n'' \\ \vdots & \vdots & \ddots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix} = 0$$

From linear algebra, we know that this equation has a solution for a nonzero constant vector only if the determinant of the matrix is zero. Here we define the **Wronskian** ,$W(x)$, of a set of functions.

$$W(x) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

Thus if a set of functions is linearly dependent on an interval, then the Wronskian is identically zero on that interval. Alternatively, if the Wronskian is identically zero, then the above matrix equation has a solution for a nonzero constant vector. This implies that the the set of functions is linearly dependent.

Result 16.4.2 The Wronskian of a set of functions vanishes identically over an interval if and only if the set of functions is linearly dependent on that interval. The Wronskian of a set of linearly independent functions does not vanish except possibly at isolated points.

Example 16.4.2 *Consider the set,* $\{x, x^2\}$*. The Wronskian is*

$$W(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2.$$

Thus the functions are independent.

Example 16.4.3 *Consider the set* $\{\sin x, \cos x, \mathrm{e}^{\imath x}\}$*. The Wronskian is*

$$W(x) = \begin{vmatrix} \sin x & \cos x & \mathrm{e}^{\imath x} \\ \cos x & -\sin x & \imath\,\mathrm{e}^{\imath x} \\ -\sin x & -\cos x & -\mathrm{e}^{\imath x} \end{vmatrix}.$$

Since the last row is a constant multiple of the first row, the determinant is zero. The functions are dependent. We could also see this with the identity $\mathrm{e}^{\imath x} = \cos x + \imath \sin x$*.*

16.4.3 The Wronskian of the Solutions to a Differential Equation

Consider the n^{th} order linear homogeneous differential equation

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_0(x) y = 0.$$

Let $\{y_1, y_2, \ldots, y_n\}$ be any set of n linearly independent solutions. Let $Y(x)$ be the matrix such that $W(x) = \Delta[Y(x)]$. Now let's differentiate $W(x)$.

$$W'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \Delta[Y(x)] = \sum_{k=1}^{n} \Delta_k[Y(x)]$$

We note that the all but the last term in this sum is zero. To see this, let's take a look at the first term.

$$\Delta_1[Y(x)] = \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

The first two rows in the matrix are identical. Since the rows are dependent, the determinant is zero.

The last term in the sum is

$$\Delta_n[Y(x)] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}.$$

In the last row of this matrix we make the substitution $y_i^{(n)} = -p_{n-1}(x)y_i^{(n-1)} - \cdots - p_0(x)y_i$. Recalling that we can add a multiple of a row to another without changing the determinant, we add $p_0(x)$ times the first row, and $p_1(x)$ times the second row, etc., to the last row. Thus we have the determinant,

$$\begin{aligned} W'(x) &= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -p_{n-1}(x)y_1^{(n-1)} & -p_{n-1}(x)y_2^{(n-1)} & \cdots & -p_{n-1}(x)y_n^{(n-1)} \end{vmatrix} \\ &= -p_{n-1}(x) \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \\ &= -p_{n-1}(x)W(x) \end{aligned}$$

Thus the Wronskian satisfies the first order differential equation,

$$W'(x) = -p_{n-1}(x)W(x).$$

Solving this equation we get a result known as **Abel's formula**.

$$W(x) = c\,\exp\left(-\int p_{n-1}(x)\,\mathrm{d}x\right)$$

Thus regardless of the particular set of solutions that we choose, we can compute their Wronskian up to a constant factor.

Result 16.4.3 The Wronskian of any linearly independent set of solutions to the equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0$$

is, (up to a multiplicative constant), given by

$$W(x) = \exp\left(-\int p_{n-1}(x)\,\mathrm{d}x\right).$$

Example 16.4.4 *Consider the differential equation*

$$y'' - 3y' + 2y = 0.$$

The Wronskian of the two independent solutions is

$$\begin{aligned} W(x) &= c\exp\left(-\int -3\,\mathrm{d}x\right) \\ &= c\,\mathrm{e}^{3x}. \end{aligned}$$

For the choice of solutions $\{\mathrm{e}^x, \mathrm{e}^{2x}\}$, *the Wronskian is*

$$W(x) = \begin{vmatrix} \mathrm{e}^x & \mathrm{e}^{2x} \\ \mathrm{e}^x & 2\,\mathrm{e}^{2x} \end{vmatrix} = 2\,\mathrm{e}^{3x} - \mathrm{e}^{3x} = \mathrm{e}^{3x}.$$

16.5 Well-Posed Problems

Consider the initial value problem for an n^{th} order linear differential equation.

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x)y = f(x)$$
$$y(x_0) = v_1, \quad y'(x_0) = v_2, \quad \ldots, \quad y^{(n-1)}(x_0) = v_n$$

Since the general solution to the differential equation is a linear combination of the n homogeneous solutions plus the particular solution

$$y = y_p + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

the problem to find the constants c_i can be written

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \ldots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} + \begin{pmatrix} y_p(x_0) \\ y_p'(x_0) \\ \vdots \\ y_p^{(n-1)}(x_0) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

From linear algebra we know that this system of equations has a unique solution only if the determinant of the matrix is nonzero. Note that the determinant of the matrix is just the Wronskian evaluated at x_0. Thus if the Wronskian vanishes at x_0, the initial value problem for the differential equation either has no solutions or infinitely many solutions. Such problems are said to be ill-posed. From Abel's formula for the Wronskian

$$W(x) = \exp\left(-\int p_{n-1}(x)\,\mathrm{d}x\right),$$

we see that the only way the Wronskian can vanish is if the value of the integral goes to ∞.

Example 16.5.1 *Consider the initial value problem*

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0, \qquad y(0) = y'(0) = 1.$$

The Wronskian

$$W(x) = \exp\left(-\int -\frac{2}{x}\,\mathrm{d}x\right) = \exp\left(2\log x\right) = x^2$$

vanishes at $x = 0$*. Thus this problem is not well-posed.*

The general solution of the differential equation is

$$y = c_1 x + c_2 x^2.$$

We see that the general solution cannot satisfy the initial conditions. If instead we had the initial conditions $y(0) = 0$, $y'(0) = 1$*, then there would be an infinite number of solutions.*

Example 16.5.2 *Consider the initial value problem*

$$y'' - \frac{2}{x^2} y = 0, \qquad y(0) = y'(0) = 1.$$

The Wronskian

$$W(x) = \exp\left(-\int 0\,\mathrm{d}x\right) = 1$$

does not vanish anywhere. However, this problem is not well-posed.

The general solution,

$$y = c_1 x^{-1} + c_2 x^2,$$

cannot satisfy the initial conditions. Thus we see that a non-vanishing Wronskian does not imply that the problem is well-posed.

Result 16.5.1 Consider the initial value problem

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x)y = 0$$
$$y(x_0) = v_1, \quad y'(x_0) = v_2, \quad \ldots, \quad y^{(n-1)}(x_0) = v_n.$$

If the Wronskian

$$W(x) = \exp\left(-\int p_{n-1}(x)\,dx\right)$$

vanishes at $x = x_0$ then the problem is ill-posed. The problem may be ill-posed even if the Wronskian does not vanish.

16.6 The Fundamental Set of Solutions

Consider a set of linearly independent solutions $\{u_1, u_2, \ldots, u_n\}$ to an n^{th} order linear homogeneous differential equation. This is called the **fundamental set of solutions at $\mathbf{x_0}$** if they satisfy the relations

$$\begin{array}{cccc} u_1(x_0) = 1 & u_2(x_0) = 0 & \ldots & u_n(x_0) = 0 \\ u_1'(x_0) = 0 & u_2'(x_0) = 1 & \ldots & u_n'(x_0) = 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)}(x_0) = 0 & u_2^{(n-1)}(x_0) = 0 & \ldots & u_n^{(n-1)}(x_0) = 1 \end{array}$$

Knowing the fundamental set of solutions is handy because it makes the task of solving an initial value problem trivial. Say we are given the initial conditions,

$$y(x_0) = v_1, \quad y'(x_0) = v_2, \quad \ldots, \quad y^{(n-1)}(x_0) = v_n.$$

If the u_i's are a fundamental set then the solution that satisfies these constraints is just

$$y = v_1 u_1(x) + v_2 u_2(x) + \cdots + v_n u_n(x).$$

Of course in general, a set of solutions is not the fundamental set. If the Wronskian of the solutions is nonzero and finite we can generate a fundamental set of solutions that are linear combinations of our original set. Consider the case of a second order equation Let $\{y_1, y_2\}$ be two linearly independent solutions. We will generate the fundamental set of solutions, $\{u_1, u_2\}$.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

For $\{u_1, u_2\}$ to satisfy the relations that define a fundamental set, it must satisfy the matrix equation

$$\begin{pmatrix} u_1(x_0) & u_1'(x_0) \\ u_2(x_0) & u_2'(x_0) \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} y_1(x_0) & y_1'(x_0) \\ y_2(x_0) & y_2'(x_0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} y_1(x_0) & y_1'(x_0) \\ y_2(x_0) & y_2'(x_0) \end{pmatrix}^{-1}$$

If the Wronskian is non-zero and finite, we can solve for the constants, c_{ij}, and thus find the fundamental set of solutions. To generalize this result to an equation of order n, simply replace all the 2×2 matrices and vectors of length 2 with $n \times n$ matrices and vectors of length n. I presented the case of $n = 2$ simply to save having to write out all the ellipses involved in the general case. (It also makes for easier reading.)

Example 16.6.1 *Two linearly independent solutions to the differential equation $y'' + y = 0$ are $y_1 = \mathrm{e}^{\imath x}$ and $y_2 = \mathrm{e}^{-\imath x}$.*

$$\begin{pmatrix} y_1(0) & y_1'(0) \\ y_2(0) & y_2'(0) \end{pmatrix} = \begin{pmatrix} 1 & \imath \\ 1 & -i \end{pmatrix}$$

To find the fundamental set of solutions, $\{u_1, u_2\}$, at $x = 0$ we solve the equation

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 1 & \imath \\ 1 & -\imath \end{pmatrix}^{-1}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \frac{1}{\imath 2} \begin{pmatrix} \imath & \imath \\ 1 & -1 \end{pmatrix}$$

The fundamental set is

$$u_1 = \frac{e^{\imath x} + e^{-\imath x}}{2}, \qquad u_2 = \frac{e^{\imath x} - e^{-\imath x}}{\imath 2}.$$

Using trigonometric identities we can rewrite these as

$$u_1 = \cos x, \qquad u_2 = \sin x.$$

Result 16.6.1 The fundamental set of solutions at $x = x_0$, $\{u_1, u_2, \ldots, u_n\}$, to an n^{th} order linear differential equation, satisfy the relations

$$\begin{array}{cccc} u_1(x_0) = 1 & u_2(x_0) = 0 & \ldots & u_n(x_0) = 0 \\ u_1'(x_0) = 0 & u_2'(x_0) = 1 & \ldots & u_n'(x_0) = 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)}(x_0) = 0 & u_2^{(n-1)}(x_0) = 0 & \ldots & u_n^{(n-1)}(x_0) = 1. \end{array}$$

If the Wronskian of the solutions is nonzero and finite at the point x_0 then you can generate the fundamental set of solutions from any linearly independent set of solutions.

Exercise 16.6
Two solutions of $y'' - y = 0$ are e^x and e^{-x}. Show that the solutions are independent. Find the fundamental set of solutions at $x = 0$.
Hint, Solution

16.7 Adjoint Equations

For the n^{th} order linear differential operator

$$L[y] = p_n \frac{d^n y}{dx^n} + p_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_0 y$$

(where the p_j are complex-valued functions) we define the adjoint of L

$$L^*[y] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\overline{p_n} y) + (-1)^{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(\overline{p_{n-1}} y) + \cdots + \overline{p_0} y.$$

Here $\overline{f}$ denotes the complex conjugate of f.

Example 16.7.1

$$L[y] = xy'' + \frac{1}{x} y' + y$$

has the adjoint

$$\begin{aligned} L^*[y] &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}[xy] - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x} y\right] + y \\ &= xy'' + 2y' - \frac{1}{x} y' + \frac{1}{x^2} y + y \\ &= xy'' + \left(2 - \frac{1}{x}\right) y' + \left(1 + \frac{1}{x^2}\right) y. \end{aligned}$$

Taking the adjoint of L^* yields

$$\begin{aligned} L^{**}[y] &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}[xy] - \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(2 - \frac{1}{x}\right) y\right] + \left(1 + \frac{1}{x^2}\right) y \\ &= xy'' + 2y' - \left(2 - \frac{1}{x}\right) y' - \left(\frac{1}{x^2}\right) y + \left(1 + \frac{1}{x^2}\right) y \\ &= xy'' + \frac{1}{x} y' + y. \end{aligned}$$

Thus by taking the adjoint of L^*, we obtain the original operator.

In general, $L^{**} = L$.

Consider $L[y] = p_n y^{(n)} + \cdots + p_0 y$. If each of the p_k is k times continuously differentiable and u and v are n times continuously differentiable on some interval, then on that interval

$$\overline{v} L[u] - u\overline{L^*[v]} = \frac{\mathrm{d}}{\mathrm{d}x} B[u, v]$$

where $B[u, v]$, the **bilinear concomitant**, is the bilinear form

$$B[u, v] = \sum_{m=1}^{n} \sum_{\substack{j+k=m-1 \\ j\geq 0, k\geq 0}} (-1)^j u^{(k)} (p_m \overline{v})^{(j)}.$$

This equation is known as **Lagrange's identity**. If L is a second order operator then

$$\begin{aligned}\overline{v} L[u] - u\overline{L^*[v]} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[u p_1 \overline{v} + u' p_2 \overline{v} - u(p_2 \overline{v})'\right] \\ &= u'' p_2 \overline{v} + u' p_1 \overline{v} + u\left[-p_2 \overline{v}'' + (-2p_2' + p_1)\overline{v}' + (-p_2'' + p_1')\overline{v}\right].\end{aligned}$$

Example 16.7.2 *Verify Lagrange's identity for the second order operator,* $L[y] = p_2 y'' + p_1 y' + p_0 y$.

$$\begin{aligned}\overline{v} L[u] - u\overline{L^*[v]} &= \overline{v}(p_2 u'' + p_1 u' + p_0 u) - u\overline{\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\overline{p_2} v) - \frac{\mathrm{d}}{\mathrm{d}x}(\overline{p_1} v) + \overline{p_0} v\right)} \\ &= \overline{v}(p_2 u'' + p_1 u' + p_0 u) - u\overline{(\overline{p_2} v'' + (2\overline{p_2}' - \overline{p_1})v' + (\overline{p_2}'' - \overline{p_1}' + \overline{p_0})v)} \\ &= u'' p_2 \overline{v} + u' p_1 \overline{v} + u\left[-p_2 \overline{v}'' + (-2p_2' + p_1)\overline{v}' + (-p_2'' + p_1')\overline{v}\right].\end{aligned}$$

We will not verify Lagrange's identity for the general case.

Integrating Lagrange's identity on its interval of validity gives us **Green's formula**.

$$\int_a^b \left(\overline{v} L[u] - u\overline{L^*[v]}\right) \mathrm{d}x = B[u, v]\big|_{x=b} - B[u, v]\big|_{x=a}$$

Result 16.7.1 The adjoint of the operator

$$L[y] = p_n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_0 y$$

is defined

$$L^*[y] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\overline{p_n} y) + (-1)^{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(\overline{p_{n-1}} y) + \cdots + \overline{p_0} y.$$

If each of the p_k is k times continuously differentiable and u and v are n times continuously differentiable, then Lagrange's identity states

$$\overline{v} L[y] - u \overline{L^*[v]} = \frac{\mathrm{d}}{\mathrm{d}x} B[u, v] = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{m=1}^{n} \sum_{\substack{j+k=m-1 \\ j \geq 0, k \geq 0}} (-1)^j u^{(k)} (p_m \overline{v})^{(j)}.$$

Integrating Lagrange's identity on it's domain of validity yields Green's formula,

$$\int_a^b \left(\overline{v} L[u] - u \overline{L^*[v]} \right) \mathrm{d}x = B[u, v]\big|_{x=b} - B[u, v]\big|_{x=a}.$$

16.8 Additional Exercises

Exact Equations
Nature of Solutions
Transformation to a First Order System
The Wronskian
Well-Posed Problems
The Fundamental Set of Solutions
Adjoint Equations

Exercise 16.7
Find the adjoint of the Bessel equation of order ν,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

and the Legendre equation of order α,

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

Hint, Solution

Exercise 16.8
Find the adjoint of

$$x^2y'' - xy' + 3y = 0.$$

Hint, Solution

16.9 Hints

Hint 16.1

Hint 16.2

Hint 16.3

Hint 16.4

Hint 16.5
The difference of any two of the u_i's is a homogeneous solution.

Hint 16.6

Exact Equations
Nature of Solutions
Transformation to a First Order System
The Wronskian
Well-Posed Problems
The Fundamental Set of Solutions
Adjoint Equations

Hint 16.7

Hint 16.8

16.10 Solutions

Solution 16.1

The second order, linear, homogeneous differential equation is

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{16.4}$$

An exact equation can be written in the form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[a(x)y' + b(x)y\right] = 0.$$

If Equation 16.4 is exact, then we can write it in the form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[P(x)y' + f(x)y\right] = 0$$

for some function $f(x)$. We carry out the differentiation to write the equation in standard form:

$$P(x)y'' + \left(P'(x) + f(x)\right)y' + f'(x)y = 0 \tag{16.5}$$

We equate the coefficients of Equations 16.4 and 16.5 to obtain a set of equations.

$$P'(x) + f(x) = Q(x), \quad f'(x) = R(x).$$

In order to eliminate $f(x)$, we differentiate the first equation and substitute in the expression for $f'(x)$ from the second equation. This gives us a *necessary* condition for Equation 16.4 to be exact:

$$\boxed{P''(x) - Q'(x) + R(x) = 0} \tag{16.6}$$

Now we demonstrate that Equation 16.6 is a *sufficient* condition for exactness. Suppose that Equation 16.6 holds. Then we can replace R by $Q' - P''$ in the differential equation.

$$Py'' + Qy' + (Q' - P'')y = 0$$

We recognize the right side as an exact differential.

$$(Py' + (Q - P')y)' = 0$$

Thus Equation 16.6 is a sufficient condition for exactness. We can integrate to reduce the problem to a first order differential equation.

$$Py' + (Q - P')y = c$$

Solution 16.2
Suppose that there is an integrating factor $\mu(x)$ that will make

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

exact. We multiply by this integrating factor.

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0. \tag{16.7}$$

We apply the exactness condition from Exercise 16.1 to obtain a differential equation for the integrating factor.

$$(\mu P)'' - (\mu Q)' + \mu R = 0$$
$$\mu'' P + 2\mu' P' + \mu P'' - \mu' Q - \mu Q' + \mu R = 0$$
$$\boxed{P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0}$$

Solution 16.3
We consider the differential equation,

$$y'' + xy' + y = 0.$$

Since

$$(1)'' - (x)' + 1 = 0$$

we see that this is an exact equation. We rearrange terms to form exact derivatives and then integrate.

$$(y')' + (xy)' = 0$$
$$y' + xy = c$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{x^2/2} y\right] = c\,\mathrm{e}^{x^2/2}$$
$$\boxed{y = c\,\mathrm{e}^{-x^2/2}\int \mathrm{e}^{x^2/2}\,\mathrm{d}x + d\,\mathrm{e}^{-x^2/2}}$$

Solution 16.4

Consider the initial value problem,

$$y'' + p(x)y' + q(x)y = f(x),$$
$$y(x_0) = y_0, \quad y'(x_0) = y_1.$$

If $p(x)$, $q(x)$ and $f(x)$ are continuous on an interval $(a \ldots b)$ with $x_0 \in (a \ldots b)$, then the problem has a unique solution on that interval.

1.

$$xy'' + 3y = x$$
$$y'' + \frac{3}{x}y = 1$$

Unique solutions exist on the intervals $(-\infty \ldots 0)$ and $(0 \ldots \infty)$.

2.

$$x(x-1)y'' + 3xy' + 4y = 2$$
$$y'' + \frac{3}{x-1}y' + \frac{4}{x(x-1)}y = \frac{2}{x(x-1)}$$

Unique solutions exist on the intervals $(-\infty \ldots 0)$, $(0 \ldots 1)$ and $(1 \ldots \infty)$.

3.

$$\mathrm{e}^x\, y'' + x^2 y' + y = \tan x$$
$$y'' + x^2\,\mathrm{e}^{-x}\, y' + \mathrm{e}^{-x}\, y = \mathrm{e}^{-x} \tan x$$

Unique solutions exist on the intervals $\left(\frac{(2n-1)\pi}{2} \dots \frac{(2n+1)\pi}{2}\right)$ for $n \in \mathbb{Z}$.

Solution 16.5
We know that the general solution is

$$y = y_p + c_1 y_1 + c_2 y_2,$$

where y_p is a particular solution and y_1 and y_2 are linearly independent homogeneous solutions. Since y_p can be any particular solution, we choose $y_p = u_1$. Now we need to find two homogeneous solutions. Since $L[u_i] = f(x)$, $L[u_1 - u_2] = L[u_2 - u_3] = 0$. Finally, we note that since the u_i's are linearly independent, $y_1 = u_1 - u_2$ and $y_2 = u_2 - u_3$ are linearly independent. Thus the general solution is

$$\boxed{y = u_1 + c_1(u_1 - u_2) + c_2(u_2 - u_3).}$$

Solution 16.6
The Wronskian of the solutions is

$$W(x) = \begin{vmatrix} \mathrm{e}^x & \mathrm{e}^{-x} \\ \mathrm{e}^x & -\mathrm{e}^{-x} \end{vmatrix} = -2.$$

Since the Wronskian is nonzero, the solutions are independent.

The fundamental set of solutions, $\{u_1, u_2\}$, is a linear combination of e^x and e^{-x}.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \mathrm{e}^x \\ \mathrm{e}^{-x} \end{pmatrix}$$

The coefficients are

$$\begin{aligned}
\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} &= \begin{pmatrix} \mathrm{e}^{0} & \mathrm{e}^{0} \\ \mathrm{e}^{-0} & -\mathrm{e}^{-0} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\
&= \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\end{aligned}$$

$$u_1 = \frac{1}{2}(\mathrm{e}^{x} + \mathrm{e}^{-x}), \qquad u_2 = \frac{1}{2}(\mathrm{e}^{x} - \mathrm{e}^{-x}).$$

The fundamental set of solutions at $x = 0$ is

$$\boxed{\{\cosh x, \sinh x\}.}$$

Exact Equations

Nature of Solutions

Transformation to a First Order System

The Wronskian

Well-Posed Problems

The Fundamental Set of Solutions

Adjoint Equations

Solution 16.7

1. The Bessel equation of order ν is

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0.$$

The adjoint equation is

$$x^2\mu'' + (4x - x)\mu' + (2 - 1 + x^2 - \nu^2)\mu = 0$$
$$\boxed{x^2\mu'' + 3x\mu' + (1 + x^2 - \nu^2)\mu = 0.}$$

2. The Legendre equation of order α is

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

The adjoint equation is

$$(1 - x^2)\mu'' + (-4x + 2x)\mu' + (-2 + 2 + \alpha(\alpha + 1))\mu = 0$$
$$\boxed{(1 - x^2)\mu'' - 2x\mu' + \alpha(\alpha + 1)\mu = 0}$$

Solution 16.8
The adjoint of

$$x^2y'' - xy' + 3y = 0$$

is

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2y) + \frac{\mathrm{d}}{\mathrm{d}x}(xy) + 3y = 0$$
$$(x^2y'' + 4xy' + 2y) + (xy' + y) + 3y = 0$$
$$\boxed{x^2y'' + 5xy' + 6y = 0.}$$

16.11 Quiz

Problem 16.1
What is the differential equation whose solution is the two parameter family of curves $y = c_1 \sin(2x + c_2)$?
Solution

16.12 Quiz Solutions

Solution 16.1

We take the first and second derivative of $y = c_1 \sin(2x + c_2)$.

$$y' = 2c_1 \cos(2x + c_2)$$
$$y'' = -4c_1 \sin(2x + c_2)$$

This gives us three equations involving x, y, y', y'' and the parameters c_1 and c_2. We eliminate the the parameters to obtain the differential equation. Clearly we have,

$$y'' + 4y = 0.$$

Chapter 17

Techniques for Linear Differential Equations

My new goal in life is to take the meaningless drivel out of human interaction.

-Dave Ozenne

The n^{th} order linear homogeneous differential equation can be written in the form:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0.$$

In general it is not possible to solve second order and higher linear differential equations. In this chapter we will examine equations that have special forms which allow us to either reduce the order of the equation or solve it.

17.1 Constant Coefficient Equations

The n^{th} order constant coefficient differential equation has the form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

We will find that solving a constant coefficient differential equation is no more difficult than finding the roots of a polynomial. For notational simplicity, we will first consider second order equations. Then we will apply the same techniques to higher order equations.

17.1.1 Second Order Equations

Factoring the Differential Equation. Consider the second order constant coefficient differential equation:

$$y'' + 2ay' + by = 0. \tag{17.1}$$

Just as we can factor a second degree polynomial:

$$\lambda^2 + 2a\lambda + b = (\lambda - \alpha)(\lambda - \beta), \alpha = -a + \sqrt{a^2 - b} \quad \text{and} \quad \beta = -a - \sqrt{a^2 - b},$$

we can factor Equation 17.1.

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2a\frac{\mathrm{d}}{\mathrm{d}x} + b\right) y = \left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \beta\right) y$$

Once we have factored the differential equation, we can solve it by solving a series of two first order differential equations. We set $u = \left(\frac{\mathrm{d}}{\mathrm{d}x} - \beta\right) y$ to obtain a first order equation:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - \alpha\right) u = 0,$$

which has the solution:

$$u = c_1 \mathrm{e}^{\alpha x}.$$

To find the solution of Equation 17.1, we solve

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - \beta\right) y = u = c_1 \mathrm{e}^{\alpha x}.$$

We multiply by the integrating factor and integrate.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\beta x} y\right) = c_1 \mathrm{e}^{(\alpha-\beta)x}$$
$$y = c_1 \mathrm{e}^{\beta x} \int \mathrm{e}^{(\alpha-\beta)x} \, \mathrm{d}x + c_2 \mathrm{e}^{\beta x}$$

We first consider the case that α and β are distinct.

$$y = c_1 \mathrm{e}^{\beta x} \frac{1}{\alpha-\beta} \mathrm{e}^{(\alpha-\beta)x} + c_2 \mathrm{e}^{\beta x}$$

We choose new constants to write the solution in a simpler form.

$$y = c_1 \mathrm{e}^{\alpha x} + c_2 \mathrm{e}^{\beta x}$$

Now we consider the case $\alpha = \beta$.

$$y = c_1 \mathrm{e}^{\alpha x} \int 1 \, \mathrm{d}x + c_2 \mathrm{e}^{\alpha x}$$
$$y = c_1 x \mathrm{e}^{\alpha x} + c_2 \mathrm{e}^{\alpha x}$$

The solution of Equation 17.1 is

$$y = \begin{cases} c_1 \mathrm{e}^{\alpha x} + c_2 \mathrm{e}^{\beta x}, & \alpha \neq \beta, \\ c_1 \mathrm{e}^{\alpha x} + c_2 x \mathrm{e}^{\alpha x}, & \alpha = \beta. \end{cases} \tag{17.2}$$

Example 17.1.1 *Consider the differential equation:* $y'' + y = 0$. *To obtain the general solution, we factor the equation and apply the result in Equation 17.2.*

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - \imath\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \imath\right) y = 0$$
$$y = c_1 \mathrm{e}^{\imath x} + c_2 \mathrm{e}^{-\imath x}.$$

Example 17.1.2 *Next we solve* $y'' = 0$.

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - 0\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - 0\right) y = 0$$
$$y = c_1 \mathrm{e}^{0x} + c_2 x \mathrm{e}^{0x}$$
$$y = c_1 + c_2 x$$

Substituting the Form of the Solution into the Differential Equation. Note that if we substitute $y = \mathrm{e}^{\lambda x}$ into the differential equation (17.1), we will obtain the quadratic polynomial (17.1.1) for λ.

$$y'' + 2ay' + by = 0$$
$$\lambda^2 \mathrm{e}^{\lambda x} + 2a\lambda \mathrm{e}^{\lambda x} + b \mathrm{e}^{\lambda x} = 0$$
$$\lambda^2 + 2a\lambda + b = 0$$

This gives us a superficially different method for solving constant coefficient equations. We substitute $y = \mathrm{e}^{\lambda x}$ into the differential equation. Let α and β be the roots of the quadratic in λ. If the roots are distinct, then the linearly independent solutions are $y_1 = \mathrm{e}^{\alpha x}$ and $y_2 = \mathrm{e}^{\beta x}$. If the quadratic has a double root at $\lambda = \alpha$, then the linearly independent solutions are $y_1 = \mathrm{e}^{\alpha x}$ and $y_2 = x \mathrm{e}^{\alpha x}$.

Example 17.1.3 *Consider the equation:*

$$y'' - 3y' + 2y = 0.$$

The substitution $y = \mathrm{e}^{\lambda x}$ *yields*

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0.$$

Thus the solutions are e^{x} *and* e^{2x}.

Example 17.1.4 *Next consider the equation:*

$$y'' - 2y' + 4y = 0.$$

The substitution $y = e^{\lambda x}$ *yields*

$$\lambda^2 - 2\lambda + 4 = (\lambda - 2)^2 = 0.$$

Because the polynomial has a double root, the solutions are e^{2x} *and* $x\, e^{2x}$.

Result 17.1.1 Consider the second order constant coefficient differential equation:

$$y'' + 2ay' + by = 0.$$

We can factor the differential equation into the form:

$$\left(\frac{d}{dx} - \alpha\right)\left(\frac{d}{dx} - \beta\right) y = 0,$$

which has the solution:

$$y = \begin{cases} c_1 e^{\alpha x} + c_2 e^{\beta x}, & \alpha \neq \beta, \\ c_1 e^{\alpha x} + c_2 x\, e^{\alpha x}, & \alpha = \beta. \end{cases}$$

We can also determine α and β by substituting $y = e^{\lambda x}$ into the differential equation and factoring the polynomial in λ.

Shift Invariance. Note that if $u(x)$ is a solution of a constant coefficient equation, then $u(x + c)$ is also a solution. This is useful in applying initial or boundary conditions.

Example 17.1.5 *Consider the problem*

$$y'' - 3y' + 2y = 0, \qquad y(0) = a, \quad y'(0) = b.$$

We know that the general solution is

$$y = c_1 e^{x} + c_2 e^{2x}.$$

Applying the initial conditions, we obtain the equations,

$$c_1 + c_2 = a, \quad c_1 + 2c_2 = b.$$

The solution is

$$y = (2a - b)\,\mathrm{e}^x + (b - a)\,\mathrm{e}^{2x}.$$

Now suppose we wish to solve the same differential equation with the boundary conditions $y(1) = a$ *and* $y'(1) = b$. *All we have to do is shift the solution to the right.*

$$y = (2a - b)\,\mathrm{e}^{x-1} + (b - a)\,\mathrm{e}^{2(x-1)}.$$

17.1.2 Real-Valued Solutions

If the coefficients of the differential equation are real, then the solution can be written in terms of real-valued functions (Result 16.2.2). For a real root $\lambda = \alpha$ of the polynomial in λ, the corresponding solution, $y = \mathrm{e}^{\alpha x}$, is real-valued.

Now recall that the complex roots of a polynomial with real coefficients occur in complex conjugate pairs. Assume that $\alpha \pm \imath\beta$ are roots of

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$$

The corresponding solutions of the differential equation are $\mathrm{e}^{(\alpha+\imath\beta)x}$ and $\mathrm{e}^{(\alpha-\imath\beta)x}$. Note that the linear combinations

$$\frac{\mathrm{e}^{(\alpha+\imath\beta)x} + \mathrm{e}^{(\alpha-\imath\beta)x}}{2} = \mathrm{e}^{\alpha x}\cos(\beta x), \qquad \frac{\mathrm{e}^{(\alpha+\imath\beta)x} - \mathrm{e}^{(\alpha-\imath\beta)x}}{\imath 2} = \mathrm{e}^{\alpha x}\sin(\beta x),$$

are real-valued solutions of the differential equation. We could also obtain real-valued solution by taking the real and imaginary parts of either $\mathrm{e}^{(\alpha+\imath\beta)x}$ or $\mathrm{e}^{(\alpha-\imath\beta)x}$.

$$\Re\left(\mathrm{e}^{(\alpha+\imath\beta)x}\right) = \mathrm{e}^{\alpha x}\cos(\beta x), \qquad \Im\left(\mathrm{e}^{(\alpha+\imath\beta)x}\right) = \mathrm{e}^{\alpha x}\sin(\beta x)$$

Example 17.1.6 *Consider the equation*

$$y'' - 2y' + 2y = 0.$$

The substitution $y = \mathrm{e}^{\lambda x}$ *yields*

$$\lambda^2 - 2\lambda + 2 = (\lambda - 1 - \imath)(\lambda - 1 + \imath) = 0.$$

The linearly independent solutions are

$$\mathrm{e}^{(1+\imath)x}, \quad \textit{and} \quad \mathrm{e}^{(1-\imath)x}.$$

We can write the general solution in terms of real functions.

$$\boxed{y = c_1 \mathrm{e}^x \cos x + c_2 \mathrm{e}^x \sin x}$$

Exercise 17.1
Find the general solution of

$$y'' + 2ay' + by = 0$$

for $a, b \in \mathbb{R}$. There are three distinct forms of the solution depending on the sign of $a^2 - b$.
Hint, Solution

Exercise 17.2
Find the fundamental set of solutions of

$$y'' + 2ay' + by = 0$$

at the point $x = 0$, for $a, b \in \mathbb{R}$. Use the general solutions obtained in Exercise 17.1.
Hint, Solution

Result 17.1.2 . Consider the second order constant coefficient equation

$$y'' + 2ay' + by = 0.$$

The general solution of this differential equation is

$$y = \begin{cases} e^{-ax}\left(c_1 e^{\sqrt{a^2-b}\,x} + c_2 e^{-\sqrt{a^2-b}\,x}\right) & \text{if } a^2 > b, \\ e^{-ax}\left(c_1 \cos(\sqrt{b-a^2}\,x) + c_2 \sin(\sqrt{b-a^2}\,x)\right) & \text{if } a^2 < b, \\ e^{-ax}(c_1 + c_2 x) & \text{if } a^2 = b. \end{cases}$$

The fundamental set of solutions at $x = 0$ is

$$\begin{cases} \left\{e^{-ax}\left(\cosh(\sqrt{a^2-b}\,x) + \frac{a}{\sqrt{a^2-b}}\sinh(\sqrt{a^2-b}\,x)\right), e^{-ax}\frac{1}{\sqrt{a^2-b}}\sinh(\sqrt{a^2-b}\,x)\right\} & \text{if } a^2 > b, \\ \left\{e^{-ax}\left(\cos(\sqrt{b-a^2}\,x) + \frac{a}{\sqrt{b-a^2}}\sin(\sqrt{b-a^2}\,x)\right), e^{-ax}\frac{1}{\sqrt{b-a^2}}\sin(\sqrt{b-a^2}\,x)\right\} & \text{if } a^2 < b, \\ \{(1+ax)\,e^{-ax}, x\,e^{-ax}\} & \text{if } a^2 = b. \end{cases}$$

To obtain the fundamental set of solutions at the point $x = \xi$, substitute $(x - \xi)$ for x in the above solutions.

17.1.3 Higher Order Equations

The constant coefficient equation of order n has the form

$$L[y] = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{17.3}$$

The substitution $y = \mathrm{e}^{\lambda x}$ will transform this differential equation into an algebraic equation.

$$L[\mathrm{e}^{\lambda x}] = \lambda^n \mathrm{e}^{\lambda x} + a_{n-1}\lambda^{n-1} \mathrm{e}^{\lambda x} + \cdots + a_1\lambda \mathrm{e}^{\lambda x} + a_0 \mathrm{e}^{\lambda x} = 0$$
$$\left(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0\right) \mathrm{e}^{\lambda x} = 0$$
$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

Assume that the roots of this equation, $\lambda_1, \ldots, \lambda_n$, are distinct. Then the n linearly independent solutions of Equation 17.3 are

$$\mathrm{e}^{\lambda_1 x}, \ldots, \mathrm{e}^{\lambda_n x}.$$

If the roots of the algebraic equation are not distinct then we will not obtain all the solutions of the differential equation. Suppose that $\lambda_1 = \alpha$ is a double root. We substitute $y = \mathrm{e}^{\lambda x}$ into the differential equation.

$$L[\mathrm{e}^{\lambda x}] = [(\lambda - \alpha)^2(\lambda - \lambda_3)\cdots(\lambda - \lambda_n)] \mathrm{e}^{\lambda x} = 0$$

Setting $\lambda = \alpha$ will make the left side of the equation zero. Thus $y = \mathrm{e}^{\alpha x}$ is a solution. Now we differentiate both sides of the equation with respect to λ and interchange the order of differentiation.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} L[\mathrm{e}^{\lambda x}] = L\left[\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathrm{e}^{\lambda x}\right] = L\left[x \mathrm{e}^{\lambda x}\right]$$

Let $p(\lambda) = (\lambda - \lambda_3)\cdots(\lambda - \lambda_n)$. We calculate $L\left[x \mathrm{e}^{\lambda x}\right]$ by applying L and then differentiating with respect to λ.

$$\begin{aligned}
L\left[x \mathrm{e}^{\lambda x}\right] &= \frac{\mathrm{d}}{\mathrm{d}\lambda} L[\mathrm{e}^{\lambda x}] \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}[(\lambda - \alpha)^2(\lambda - \lambda_3)\cdots(\lambda - \lambda_n)] \mathrm{e}^{\lambda x} \\
&= \frac{\mathrm{d}}{\mathrm{d}\lambda}[(\lambda - \alpha)^2 p(\lambda)] \mathrm{e}^{\lambda x} \\
&= \left[2(\lambda - \alpha)p(\lambda) + (\lambda - \alpha)^2 p'(\lambda) + (\lambda - \alpha)^2 p(\lambda) x\right] \mathrm{e}^{\lambda x} \\
&= (\lambda - \alpha)\left[2p(\lambda) + (\lambda - \alpha)p'(\lambda) + (\lambda - \alpha)p(\lambda)x\right] \mathrm{e}^{\lambda x}
\end{aligned}$$

Since setting $\lambda = \alpha$ will make this expression zero, $L[x \mathrm{e}^{\alpha x}] = 0$, $x \mathrm{e}^{\alpha x}$ is a solution of Equation 17.3. You can verify that $\mathrm{e}^{\alpha x}$ and $x \mathrm{e}^{\alpha x}$ are linearly independent. Now we have generated all of the solutions for the differential equation.

If $\lambda = \alpha$ is a root of multiplicity m then by repeatedly differentiating with respect to λ you can show that the corresponding solutions are

$$\mathrm{e}^{\alpha x}, x \mathrm{e}^{\alpha x}, x^2 \mathrm{e}^{\alpha x}, \ldots, x^{m-1} \mathrm{e}^{\alpha x}.$$

Example 17.1.7 *Consider the equation*

$$y''' - 3y' + 2y = 0.$$

The substitution $y = \mathrm{e}^{\lambda x}$ *yields*

$$\lambda^3 - 3\lambda + 2 = (\lambda - 1)^2(\lambda + 2) = 0.$$

Thus the general solution is

$$\boxed{y = c_1 \mathrm{e}^x + c_2 x \mathrm{e}^x + c_3 \mathrm{e}^{-2x}.}$$

Result 17.1.3 Consider the n^{th} order constant coefficient equation

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0.$$

Let the factorization of the algebraic equation obtained with the substitution $y = \mathrm{e}^{\lambda x}$ be

$$(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p} = 0.$$

A set of linearly independent solutions is given by

$$\{\mathrm{e}^{\lambda_1 x}, x \mathrm{e}^{\lambda_1 x}, \ldots, x^{m_1-1} \mathrm{e}^{\lambda_1 x}, \ldots, \mathrm{e}^{\lambda_p x}, x \mathrm{e}^{\lambda_p x}, \ldots, x^{m_p-1} \mathrm{e}^{\lambda_p x}\}.$$

If the coefficients of the differential equation are real, then we can find a real-valued set of solutions.

Example 17.1.8 *Consider the equation*

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + 2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0.$$

The substitution $y = \mathrm{e}^{\lambda x}$ *yields*

$$\lambda^4 + 2\lambda^2 + 1 = (\lambda - i)^2(\lambda + i)^2 = 0.$$

Thus the linearly independent solutions are

$$\mathrm{e}^{\imath x}, x\,\mathrm{e}^{\imath x}, \mathrm{e}^{-\imath x} \text{ and } x\,\mathrm{e}^{-\imath x}.$$

Noting that

$$\mathrm{e}^{\imath x} = \cos(x) + \imath \sin(x),$$

we can write the general solution in terms of sines and cosines.

$$\boxed{y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x}$$

17.2 Euler Equations

Consider the equation

$$L[y] = x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + ax\frac{\mathrm{d}y}{\mathrm{d}x} + by = 0, \quad x > 0.$$

Let's say, for example, that y has units of distance and x has units of time. Note that each term in the differential equation has the same dimension.

$$(\text{time})^2 \frac{(\text{distance})}{(\text{time})^2} = (\text{time})\frac{(\text{distance})}{(\text{time})} = (\text{distance})$$

Thus this is a second order Euler, or equidimensional equation. We know that the first order Euler equation, $xy' + ay = 0$, has the solution $y = cx^a$. Thus for the second order equation we will try a solution of the form $y = x^\lambda$. The substitution

$y = x^\lambda$ will transform the differential equation into an algebraic equation.

$$\begin{aligned} L[x^\lambda] = x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}[x^\lambda] + ax\frac{\mathrm{d}}{\mathrm{d}x}[x^\lambda] + bx^\lambda = 0 \\ \lambda(\lambda-1)x^\lambda + a\lambda x^\lambda + bx^\lambda = 0 \\ \lambda(\lambda-1) + a\lambda + b = 0 \end{aligned}$$

Factoring yields

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0.$$

If the two roots, λ_1 and λ_2, are distinct then the general solution is

$$y = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}.$$

If the roots are not distinct, $\lambda_1 = \lambda_2 = \lambda$, then we only have the one solution, $y = x^\lambda$. To generate the other solution we use the same approach as for the constant coefficient equation. We substitute $y = x^\lambda$ into the differential equation and differentiate with respect to λ.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda} L[x^\lambda] &= L[\frac{\mathrm{d}}{\mathrm{d}\lambda} x^\lambda] \\ &= L[\ln x \ x^\lambda] \end{aligned}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} x^\lambda = \frac{\mathrm{d}}{\mathrm{d}\lambda} \mathrm{e}^{\lambda \ln x} = \ln x \ \mathrm{e}^{\lambda \ln x} = \ln x \ x^\lambda.$$

Now we apply L and then differentiate with respect to λ.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda} L[x^\lambda] &= \frac{\mathrm{d}}{\mathrm{d}\lambda} (\lambda - \alpha)^2 x^\lambda \\ &= 2(\lambda - \alpha) x^\lambda + (\lambda - \alpha)^2 \ln x \ x^\lambda \end{aligned}$$

Equating these two results,

$$L[\ln x \; x^\lambda] = 2(\lambda - \alpha)x^\lambda + (\lambda - \alpha)^2 \ln x \; x^\lambda.$$

Setting $\lambda = \alpha$ will make the right hand side zero. Thus $y = \ln x \; x^\alpha$ is a solution.

If you are in the mood for a little algebra you can show by repeatedly differentiating with respect to λ that if $\lambda = \alpha$ is a root of multiplicity m in an n^{th} order Euler equation then the associated solutions are

$$x^\alpha, \ln x \; x^\alpha, (\ln x)^2 x^\alpha, \ldots, (\ln x)^{m-1} x^\alpha.$$

Example 17.2.1 *Consider the Euler equation*

$$xy'' - y' + \frac{y}{x} = 0.$$

The substitution $y = x^\lambda$ *yields the algebraic equation*

$$\lambda(\lambda - 1) - \lambda + 1 = (\lambda - 1)^2 = 0.$$

Thus the general solution is

$$\boxed{y = c_1 x + c_2 x \ln x.}$$

17.2.1 Real-Valued Solutions

If the coefficients of the Euler equation are real, then the solution can be written in terms of functions that are real-valued when x is real and positive, (Result 16.2.2). If $\alpha \pm \imath\beta$ are the roots of

$$\lambda(\lambda - 1) + a\lambda + b = 0$$

then the corresponding solutions of the Euler equation are

$$x^{\alpha + \imath\beta} \quad \text{and} \quad x^{\alpha - \imath\beta}.$$

We can rewrite these as

$$x^\alpha \, \mathrm{e}^{\imath\beta \ln x} \quad \text{and} \quad x^\alpha \, \mathrm{e}^{-\imath\beta \ln x}.$$

Note that the linear combinations

$$\frac{x^\alpha \mathrm{e}^{\imath\beta\ln x} + x^\alpha \mathrm{e}^{-\imath\beta\ln x}}{2} = x^\alpha \cos(\beta \ln x), \quad \text{and} \quad \frac{x^\alpha \mathrm{e}^{\imath\beta\ln x} - x^\alpha \mathrm{e}^{-\imath\beta\ln x}}{\imath 2} = x^\alpha \sin(\beta \ln x),$$

are real-valued solutions when x is real and positive. Equivalently, we could take the real and imaginary parts of either $x^{\alpha+\imath\beta}$ or $x^{\alpha-\imath\beta}$.

$$\Re\left(x^\alpha \mathrm{e}^{\imath\beta\ln x}\right) = x^\alpha \cos(\beta \ln x), \qquad \Im\left(x^\alpha \mathrm{e}^{\imath\beta\ln x}\right) = x^\alpha \sin(\beta \ln x)$$

Result 17.2.1 Consider the second order Euler equation

$$x^2 y'' + (2a+1)xy' + by = 0.$$

The general solution of this differential equation is

$$y = \begin{cases} x^{-a}\left(c_1 x^{\sqrt{a^2-b}} + c_2 x^{-\sqrt{a^2-b}}\right) & \text{if } a^2 > b, \\ x^{-a}\left(c_1 \cos\left(\sqrt{b-a^2}\,\ln x\right) + c_2 \sin\left(\sqrt{b-a^2}\,\ln x\right)\right) & \text{if } a^2 < b, \\ x^{-a}\left(c_1 + c_2 \ln x\right) & \text{if } a^2 = b. \end{cases}$$

The fundamental set of solutions at $x = \xi$ is

$$y = \begin{cases} \left\{ \left(\frac{x}{\xi}\right)^{-a}\left(\cosh\left(\sqrt{a^2-b}\,\ln\frac{x}{\xi}\right) + \frac{a}{\sqrt{a^2-b}}\sinh\left(\sqrt{a^2-b}\,\ln\frac{x}{\xi}\right)\right), \right. & \\ \qquad \left. \left(\frac{x}{\xi}\right)^{-a}\frac{\xi}{\sqrt{a^2-b}}\sinh\left(\sqrt{a^2-b}\,\ln\frac{x}{\xi}\right) \right\} & \text{if } a^2 > b, \\ \left\{ \left(\frac{x}{\xi}\right)^{-a}\left(\cos\left(\sqrt{b-a^2}\,\ln\frac{x}{\xi}\right) + \frac{a}{\sqrt{b-a^2}}\sin\left(\sqrt{b-a^2}\,\ln\frac{x}{\xi}\right)\right), \right. & \\ \qquad \left. \left(\frac{x}{\xi}\right)^{-a}\frac{\xi}{\sqrt{b-a^2}}\sin\left(\sqrt{b-a^2}\,\ln\frac{x}{\xi}\right) \right\} & \text{if } a^2 < b, \\ \left\{ \left(\frac{x}{\xi}\right)^{-a}\left(1 + a\ln\frac{x}{\xi}\right), \left(\frac{x}{\xi}\right)^{-a}\xi\ln\frac{x}{\xi} \right\} & \text{if } a^2 = b. \end{cases}$$

Example 17.2.2 *Consider the Euler equation*

$$x^2y'' - 3xy' + 13y = 0.$$

The substitution $y = x^\lambda$ *yields*

$$\lambda(\lambda - 1) - 3\lambda + 13 = (\lambda - 2 - \imath 3)(\lambda - 2 + \imath 3) = 0.$$

The linearly independent solutions are

$$\left\{x^{2+\imath 3}, x^{2-\imath 3}\right\}.$$

We can put this in a more understandable form.

$$x^{2+\imath 3} = x^2\,\mathrm{e}^{\imath 3 \ln x} = x^2 \cos(3 \ln x) + x^2 \sin(3 \ln x)$$

We can write the general solution in terms of real-valued functions.

$$\boxed{y = c_1 x^2 \cos(3 \ln x) + c_2 x^2 \sin(3 \ln x)}$$

Result 17.2.2 Consider the n^{th} order Euler equation

$$x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1} x^{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1 x \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0.$$

Let the factorization of the algebraic equation obtained with the substitution $y = x^\lambda$ be

$$(\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_p)^{m_p} = 0.$$

A set of linearly independent solutions is given by

$$\{x^{\lambda_1}, \ln x\ x^{\lambda_1}, \ldots, (\ln x)^{m_1 - 1} x^{\lambda_1}, \ldots, x^{\lambda_p}, \ln x\ x^{\lambda_p}, \ldots, (\ln x)^{m_p - 1} x^{\lambda_p}\}.$$

If the coefficients of the differential equation are real, then we can find a set of solutions that are real valued when x is real and positive.

17.3 Exact Equations

Exact equations have the form

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x, y, y', y'', \ldots) = f(x).$$

If you can write an equation in the form of an exact equation, you can integrate to reduce the order by one, (or solve the equation for first order). We will consider a few examples to illustrate the method.

Example 17.3.1 *Consider the equation*

$$y'' + x^2 y' + 2xy = 0.$$

We can rewrite this as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y' + x^2 y\right] = 0.$$

Integrating yields a first order inhomogeneous equation.

$$y' + x^2 y = c_1$$

We multiply by the integrating factor $I(x) = \exp(\int x^2 \, \mathrm{d}x)$ *to make this an exact equation.*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^3/3} y\right) = c_1 \, \mathrm{e}^{x^3/3}$$

$$\mathrm{e}^{x^3/3} y = c_1 \int \mathrm{e}^{x^3/3} \, \mathrm{d}x + c_2$$

$$\boxed{y = c_1 \, \mathrm{e}^{-x^3/3} \int \mathrm{e}^{x^3/3} \, \mathrm{d}x + c_2 \, \mathrm{e}^{-x^3/3}}$$

Result 17.3.1 If you can write a differential equation in the form

$$\frac{\mathrm{d}}{\mathrm{d}x} F(x, y, y', y'', \ldots) = f(x),$$

then you can integrate to reduce the order of the equation.

$$F(x, y, y', y'', \ldots) = \int f(x) \, \mathrm{d}x + c$$

17.4 Equations Without Explicit Dependence on y

Example 17.4.1 *Consider the equation*

$$y'' + \sqrt{x} y' = 0.$$

This is a second order equation for y, but note that it is a first order equation for y'. We can solve directly for y'.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(\frac{2}{3}x^{3/2}\right) y'\right) = 0$$

$$y' = c_1 \exp\left(-\frac{2}{3}x^{3/2}\right)$$

Now we just integrate to get the solution for y.

$$\boxed{y = c_1 \int \exp\left(-\frac{2}{3}x^{3/2}\right) \mathrm{d}x + c_2}$$

Result 17.4.1 If an n^{th} order equation does not explicitly depend on y then you can consider it as an equation of order $n-1$ for y'.

17.5 Reduction of Order

Consider the second order linear equation

$$L[y] \equiv y'' + p(x)y' + q(x)y = f(x).$$

Suppose that we know one homogeneous solution y_1. We make the substitution $y = uy_1$ and use that $L[y_1] = 0$.

$$\begin{gathered} L[uy_1] = 0u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0 \\ u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0 \\ u''y_1 + u'(2y_1' + py_1) = 0 \end{gathered}$$

Thus we have reduced the problem to a first order equation for u'. An analogous result holds for higher order equations.

Result 17.5.1 Consider the n^{th} order linear differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x).$$

Let y_1 be a solution of the homogeneous equation. The substitution $y = uy_1$ will transform the problem into an $(n-1)^{th}$ order equation for u'. For the second order problem

$$y'' + p(x)y' + q(x)y = f(x)$$

this reduced equation is

$$u''y_1 + u'(2y_1' + py_1) = f(x).$$

Example 17.5.1 *Consider the equation*

$$y'' + xy' - y = 0.$$

By inspection we see that $y_1 = x$ is a solution. We would like to find another linearly independent solution. The substitution $y = xu$ yields

$$xu'' + (2 + x^2)u' = 0$$
$$u'' + \left(\frac{2}{x} + x\right) u' = 0$$

The integrating factor is $I(x) = \exp(2 \ln x + x^2/2) = x^2 \exp(x^2/2)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\, \mathrm{e}^{x^2/2}\, u'\right) = 0$$
$$u' = c_1 x^{-2}\, \mathrm{e}^{-x^2/2}$$
$$u = c_1 \int x^{-2}\, \mathrm{e}^{-x^2/2}\, \mathrm{d}x + c_2$$
$$y = c_1 x \int x^{-2}\, \mathrm{e}^{-x^2/2}\, \mathrm{d}x + c_2 x$$

Thus we see that a second solution is

$$\boxed{y_2 = x \int x^{-2}\, \mathrm{e}^{-x^2/2}\, \mathrm{d}x.}$$

17.6 *Reduction of Order and the Adjoint Equation

Let L be the linear differential operator

$$L[y] = p_n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1} \frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_0 y,$$

where each p_j is a j times continuously differentiable complex valued function. Recall that the adjoint of L is

$$L^*[y] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\overline{p_n} y) + (-1)^{n-1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(\overline{p_{n-1}} y) + \cdots + \overline{p_0} y.$$

If u and v are n times continuously differentiable, then Lagrange's identity states

$$\overline{v}L[u] - u\overline{L^*[v]} = \frac{\mathrm{d}}{\mathrm{d}x}B[u,v],$$

where

$$B[u,v] = \sum_{m=1}^{n} \sum_{\substack{j+k=m-1\\ j\geq 0, k\geq 0}} (-1)^j u^{(k)} (p_m\overline{v})^{(j)}.$$

For second order equations,

$$B[u,v] = up_1\overline{v} + u'p_2\overline{v} - u(p_2\overline{v})'.$$

(See Section 16.7.)

If we can find a solution to the homogeneous adjoint equation, $L^*[y] = 0$, then we can reduce the order of the equation $L[y] = f(x)$. Let ψ satisfy $L^*[\psi] = 0$. Substituting $u = y$, $v = \psi$ into Lagrange's identity yields

$$\overline{\psi}L[y] - y\overline{L^*[\psi]} = \frac{\mathrm{d}}{\mathrm{d}x}B[y,\psi]$$
$$\overline{\psi}L[y] = \frac{\mathrm{d}}{\mathrm{d}x}B[y,\psi].$$

The equation $L[y] = f(x)$ is equivalent to the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}B[y,\psi] = \overline{\psi}f$$
$$B[y,\psi] = \int \overline{\psi(x)} f(x)\,\mathrm{d}x,$$

which is a linear equation in y of order $n-1$.

Example 17.6.1 *Consider the equation*

$$L[y] = y'' - x^2y' - 2xy = 0.$$

Method 1. *Note that this is an exact equation.*

$$\frac{\mathrm{d}}{\mathrm{d}x}(y' - x^2 y) = 0$$
$$y' - x^2 y = c_1$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x^3/3}\, y\right) = c_1\,\mathrm{e}^{-x^3/3}$$
$$\boxed{y = c_1\,\mathrm{e}^{x^3/3} \int \mathrm{e}^{-x^3/3}\,\mathrm{d}x + c_2\,\mathrm{e}^{x^3/3}}$$

Method 2. *The adjoint equation is*

$$L^*[y] = y'' + x^2 y' = 0.$$

By inspection we see that $\psi = (\text{constant})$ is a solution of the adjoint equation. To simplify the algebra we will choose $\psi = 1$. Thus the equation $L[y] = 0$ is equivalent to

$$B[y, 1] = c_1$$
$$y(-x^2) + \frac{\mathrm{d}}{\mathrm{d}x}[y](1) - y\frac{\mathrm{d}}{\mathrm{d}x}[1] = c_1$$
$$y' - x^2 y = c_1.$$

By using the adjoint equation to reduce the order we obtain the same solution as with Method 1.

17.7 Additional Exercises

Constant Coefficient Equations

Exercise 17.3 (mathematica/ode/techniques_linear/constant.nb)
Find the solution of each one of the following initial value problems. Sketch the graph of the solution and describe its behavior as t increases.

1. $6y'' - 5y' + y = 0$, $y(0) = 4$, $y'(0) = 0$
2. $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 2$
3. $y'' + 4y' + 4y = 0$, $y(-1) = 2$, $y'(-1) = 1$

Hint, Solution

Exercise 17.4 (mathematica/ode/techniques_linear/constant.nb)
Substitute $y = \mathrm{e}^{\lambda x}$ to find two linearly independent solutions to

$$y'' - 4y' + 13y = 0.$$

that are real-valued when x is real-valued.
Hint, Solution

Exercise 17.5 (mathematica/ode/techniques_linear/constant.nb)
Find the general solution to

$$y''' - y'' + y' - y = 0.$$

Write the solution in terms of functions that are real-valued when x is real-valued.
Hint, Solution

Exercise 17.6
Substitute $y = \mathrm{e}^{\lambda x}$ to find the fundamental set of solutions at $x = 0$ for each of the equations:

1. $y'' + y = 0$,

2. $y'' - y = 0$,

3. $y'' = 0$.

What are the fundamental set of solutions at $x = 1$ for each of these equations.
Hint, Solution

Exercise 17.7
Consider a ball of mass m hanging by an ideal spring of spring constant k. The ball is suspended in a fluid which damps the motion. This resistance has a coefficient of friction, μ. Find the differential equation for the displacement of the mass from its equilibrium position by balancing forces. Denote this displacement by $y(t)$. If the damping force is weak, the mass will have a decaying, oscillatory motion. If the damping force is strong, the mass will not oscillate. The displacement will decay to zero. The value of the damping which separates these two behaviors is called critical damping.

Find the solution which satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$. Use the solutions obtained in Exercise 17.2 or refer to Result 17.1.2.

Consider the case $m = k = 1$. Find the coefficient of friction for which the displacement of the mass decays most rapidly. Plot the displacement for strong, weak and critical damping.
Hint, Solution

Exercise 17.8
Show that $y = c\cos(x - \phi)$ is the general solution of $y'' + y = 0$ where c and ϕ are constants of integration. (It is not sufficient to show that $y = c\cos(x - \phi)$ satisfies the differential equation. $y = 0$ satisfies the differential equation, but is is certainly not the general solution.) Find constants c and ϕ such that $y = \sin(x)$.

Is $y = c\cosh(x - \phi)$ the general solution of $y'' - y = 0$? Are there constants c and ϕ such that $y = \sinh(x)$?
Hint, Solution

Exercise 17.9 (mathematica/ode/techniques_linear/constant.nb)
Let $y(t)$ be the solution of the initial-value problem

$$y'' + 5y' + 6y = 0; \qquad y(0) = 1, \quad y'(0) = V.$$

For what values of V does $y(t)$ remain nonnegative for all $t > 0$?

Hint, Solution

Exercise 17.10 (mathematica/ode/techniques_linear/constant.nb)
Find two linearly independent solutions of

$$y'' + \operatorname{sign}(x) y = 0, \quad -\infty < x < \infty.$$

where $\operatorname{sign}(x) = \pm 1$ according as x is positive or negative. (The solution should be continuous and have a continuous first derivative.)
Hint, Solution

Euler Equations

Exercise 17.11
Find the general solution of

$$x^2 y'' + x y' + y = 0, \qquad x > 0.$$

Hint, Solution

Exercise 17.12
Substitute $y = x^\lambda$ to find the general solution of

$$x^2 y'' - 2xy + 2y = 0.$$

Hint, Solution

Exercise 17.13 (mathematica/ode/techniques_linear/constant.nb)
Substitute $y = x^\lambda$ to find the general solution of

$$x y''' + y'' + \frac{1}{x} y' = 0.$$

Write the solution in terms of functions that are real-valued when x is real-valued and positive.
Hint, Solution

Exercise 17.14
Find the general solution of

$$x^2y'' + (2a+1)xy' + by = 0.$$

Hint, Solution

Exercise 17.15
Show that

$$y_1 = \mathrm{e}^{ax}, \quad y_2 = \lim_{\alpha \to a} \frac{\mathrm{e}^{\alpha x} - \mathrm{e}^{-\alpha x}}{\alpha}$$

are linearly indepedent solutions of

$$y'' - a^2 y = 0$$

for all values of a. It is common to abuse notation and write the second solution as

$$y_2 = \frac{\mathrm{e}^{ax} - \mathrm{e}^{-ax}}{a}$$

where the limit is taken if $a = 0$. Likewise show that

$$y_1 = x^a, \quad y_2 = \frac{x^a - x^{-a}}{a}$$

are linearly indepedent solutions of

$$x^2y'' + xy' - a^2 y = 0$$

for all values of a.
Hint, Solution

Exercise 17.16 (mathematica/ode/techniques_linear/constant.nb)
Find two linearly independent solutions (i.e., the general solution) of

$$\text{(a) } x^2y'' - 2xy' + 2y = 0, \quad \text{(b) } x^2y'' - 2y = 0, \quad \text{(c) } x^2y'' - xy' + y = 0.$$

Hint, Solution

Exact Equations

Exercise 17.17
Solve the differential equation

$$y'' + y' \sin x + y \cos x = 0.$$

Hint, Solution

Equations Without Explicit Dependence on y
Reduction of Order

Exercise 17.18
Consider

$$(1 - x^2)y'' - 2xy' + 2y = 0, \quad -1 < x < 1.$$

Verify that $y = x$ is a solution. Find the general solution.
Hint, Solution

Exercise 17.19
Consider the differential equation

$$y'' - \frac{x+1}{x}y' + \frac{1}{x}y = 0.$$

Since the coefficients sum to zero, $(1 - \frac{x+1}{x} + \frac{1}{x} = 0)$, $y = \mathrm{e}^x$ is a solution. Find another linearly independent solution.
Hint, Solution

Exercise 17.20
One solution of

$$(1 - 2x)y'' + 4xy' - 4y = 0$$

is $y = x$. Find the general solution.
Hint, Solution

Exercise 17.21
Find the general solution of

$$(x-1)y'' - xy' + y = 0,$$

given that one solution is $y = e^x$. (you may assume $x > 1$)
Hint, Solution

*Reduction of Order and the Adjoint Equation

17.8 Hints

Hint 17.1
Substitute $y = \mathrm{e}^{\lambda x}$ into the differential equation.

Hint 17.2
The fundamental set of solutions is a linear combination of the homogeneous solutions.

Constant Coefficient Equations

Hint 17.3

Hint 17.4

Hint 17.5
It is a constant coefficient equation.

Hint 17.6
Use the fact that if $u(x)$ is a solution of a constant coefficient equation, then $u(x + c)$ is also a solution.

Hint 17.7
The force on the mass due to the spring is $-ky(t)$. The frictional force is $-\mu y'(t)$.

Note that the initial conditions describe the second fundamental solution at $t = 0$.

Note that for large t, $t\,\mathrm{e}^{\alpha t}$ is much small than $\mathrm{e}^{\beta t}$ if $\alpha < \beta$. (Prove this.)

Hint 17.8
By definition, the general solution of a second order differential equation is a two parameter family of functions that satisfies the differential equation. The trigonometric identities in Appendix M may be useful.

Hint 17.9

Hint 17.10

Euler Equations

Hint 17.11

Hint 17.12

Hint 17.13

Hint 17.14
Substitute $y = x^\lambda$ into the differential equation. Consider the three cases: $a^2 > b$, $a^2 < b$ and $a^2 = b$.

Hint 17.15

Hint 17.16

Exact Equations

Hint 17.17
It is an exact equation.

Equations Without Explicit Dependence on $\mathbf{y}$

Reduction of Order

Hint 17.18

Hint 17.19
Use reduction of order to find the other solution.

Hint 17.20
Use reduction of order to find the other solution.

Hint 17.21

*Reduction of Order and the Adjoint Equation

17.9 Solutions

Solution 17.1

We substitute $y = e^{\lambda x}$ into the differential equation.

$$y'' + 2ay' + by = 0$$
$$\lambda^2 + 2a\lambda + b = 0$$
$$\lambda = -a \pm \sqrt{a^2 - b}$$

If $a^2 > b$ then the two roots are distinct and real. The general solution is

$$y = c_1 e^{\left(-a+\sqrt{a^2-b}\right)x} + c_2 e^{\left(-a-\sqrt{a^2-b}\right)x}.$$

If $a^2 < b$ then the two roots are distinct and complex-valued. We can write them as

$$\lambda = -a \pm \imath\sqrt{b - a^2}.$$

The general solution is

$$y = c_1 e^{\left(-a+\imath\sqrt{b-a^2}\right)x} + c_2 e^{\left(-a-\imath\sqrt{b-a^2}\right)x}.$$

By taking the sum and difference of the two linearly independent solutions above, we can write the general solution as

$$y = c_1 e^{-ax} \cos\left(\sqrt{b - a^2}\, x\right) + c_2 e^{-ax} \sin\left(\sqrt{b - a^2}\, x\right).$$

If $a^2 = b$ then the only root is $\lambda = -a$. The general solution in this case is then

$$y = c_1 e^{-ax} + c_2 x e^{-ax}.$$

In summary, the general solution is

$$\boxed{y = \begin{cases} e^{-ax}\left(c_1 e^{\sqrt{a^2-b}\,x} + c_2 e^{-\sqrt{a^2-b}\,x}\right) & \text{if } a^2 > b, \\ e^{-ax}\left(c_1 \cos\left(\sqrt{b-a^2}\,x\right) + c_2 \sin\left(\sqrt{b-a^2}\,x\right)\right) & \text{if } a^2 < b, \\ e^{-ax}(c_1 + c_2 x) & \text{if } a^2 = b. \end{cases}}$$

Solution 17.2
First we note that the general solution can be written,

$$y = \begin{cases} e^{-ax}\left(c_1 \cosh\left(\sqrt{a^2-b}\,x\right) + c_2 \sinh\left(\sqrt{a^2-b}\,x\right)\right) & \text{if } a^2 > b, \\ e^{-ax}\left(c_1 \cos\left(\sqrt{b-a^2}\,x\right) + c_2 \sin\left(\sqrt{b-a^2}\,x\right)\right) & \text{if } a^2 < b, \\ e^{-ax}(c_1 + c_2 x) & \text{if } a^2 = b. \end{cases}$$

We first consider the case $a^2 > b$. The derivative is

$$y' = e^{-ax}\left(\left(-ac_1 + \sqrt{a^2-b}\,c_2\right)\cosh\left(\sqrt{a^2-b}\,x\right) + \left(-ac_2 + \sqrt{a^2-b}\,c_1\right)\sinh\left(\sqrt{a^2-b}\,x\right)\right).$$

The conditions, $y_1(0) = 1$ and $y_1'(0) = 0$, for the first solution become,

$$c_1 = 1, \qquad -ac_1 + \sqrt{a^2-b}\,c_2 = 0,$$
$$c_1 = 1, \qquad c_2 = \frac{a}{\sqrt{a^2-b}}.$$

The conditions, $y_2(0) = 0$ and $y_2'(0) = 1$, for the second solution become,

$$c_1 = 0, \qquad -ac_1 + \sqrt{a^2-b}\,c_2 = 1,$$
$$c_1 = 0, \qquad c_2 = \frac{1}{\sqrt{a^2-b}}.$$

The fundamental set of solutions is

$$\left\{ e^{-ax}\left(\cosh\left(\sqrt{a^2-b}\,x\right) + \frac{a}{\sqrt{a^2-b}}\sinh\left(\sqrt{a^2-b}\,x\right)\right), e^{-ax}\frac{1}{\sqrt{a^2-b}}\sinh\left(\sqrt{a^2-b}\,x\right)\right\}.$$

Now consider the case $a^2 < b$. The derivative is

$$y' = e^{-ax}\left(\left(-ac_1 + \sqrt{b-a^2}\,c_2\right)\cos\left(\sqrt{b-a^2}\,x\right) + \left(-ac_2 - \sqrt{b-a^2}\,c_1\right)\sin\left(\sqrt{b-a^2}\,x\right)\right).$$

Clearly, the fundamental set of solutions is

$$\left\{ e^{-ax} \left(\cos\left(\sqrt{b-a^2}\,x\right) + \frac{a}{\sqrt{b-a^2}} \sin\left(\sqrt{b-a^2}\,x\right) \right), e^{-ax} \frac{1}{\sqrt{b-a^2}} \sin\left(\sqrt{b-a^2}\,x\right) \right\}.$$

Finally we consider the case $a^2 = b$. The derivative is

$$y' = e^{-ax}(-ac_1 + c_2 + -ac_2 x).$$

The conditions, $y_1(0) = 1$ and $y_1'(0) = 0$, for the first solution become,

$$\begin{aligned} c_1 = 1, \qquad -ac_1 + c_2 = 0, \\ c_1 = 1, \qquad c_2 = a. \end{aligned}$$

The conditions, $y_2(0) = 0$ and $y_2'(0) = 1$, for the second solution become,

$$\begin{aligned} c_1 = 0, \qquad -ac_1 + c_2 = 1, \\ c_1 = 0, \qquad c_2 = 1. \end{aligned}$$

The fundamental set of solutions is

$$\left\{ (1 + ax)\, e^{-ax}, x\, e^{-ax} \right\}.$$

In summary, the fundamental set of solutions at $x = 0$ is

$$\boxed{\begin{cases} \left\{ e^{-ax} \left(\cosh\left(\sqrt{a^2-b}\,x\right) + \frac{a}{\sqrt{a^2-b}} \sinh\left(\sqrt{a^2-b}\,x\right) \right), e^{-ax} \frac{1}{\sqrt{a^2-b}} \sinh\left(\sqrt{a^2-b}\,x\right) \right\} & \text{if } a^2 > b, \\ \left\{ e^{-ax} \left(\cos\left(\sqrt{b-a^2}\,x\right) + \frac{a}{\sqrt{b-a^2}} \sin\left(\sqrt{b-a^2}\,x\right) \right), e^{-ax} \frac{1}{\sqrt{b-a^2}} \sin\left(\sqrt{b-a^2}\,x\right) \right\} & \text{if } a^2 < b, \\ \{(1 + ax)\, e^{-ax}, x\, e^{-ax}\} & \text{if } a^2 = b. \end{cases}}$$

Constant Coefficient Equations

Solution 17.3

1. We consider the problem
$$6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0.$$
We make the substitution $y = \mathrm{e}^{\lambda x}$ in the differential equation.
$$6\lambda^2 - 5\lambda + 1 = 0$$
$$(2\lambda - 1)(3\lambda - 1) = 0$$
$$\lambda = \left\{\frac{1}{3}, \frac{1}{2}\right\}$$
The general solution of the differential equation is
$$y = c_1 \mathrm{e}^{t/3} + c_2 \mathrm{e}^{t/2}.$$
We apply the initial conditions to determine the constants.
$$c_1 + c_2 = 4, \quad \frac{c_1}{3} + \frac{c_2}{2} = 0$$
$$c_1 = 12, \quad c_2 = -8$$
The solution subject to the initial conditions is
$$\boxed{y = 12\,\mathrm{e}^{t/3} - 8\,\mathrm{e}^{t/2}.}$$
The solution is plotted in Figure 17.1. The solution tends to $-\infty$ as $t \to \infty$.

2. We consider the problem
$$y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2.$$
We make the substitution $y = \mathrm{e}^{\lambda x}$ in the differential equation.
$$\lambda^2 - 2\lambda + 5 = 0$$
$$\lambda = 1 \pm \sqrt{1 - 5}$$
$$\lambda = \{1 + \imath 2, 1 - \imath 2\}$$

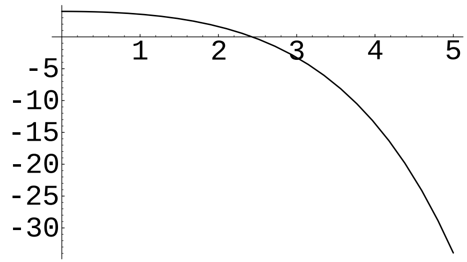

Figure 17.1: The solution of $6y'' - 5y' + y = 0$, $y(0) = 4$, $y'(0) = 0$.

The general solution of the differential equation is

$$y = c_1 \, \mathrm{e}^t \cos(2t) + c_2 \, \mathrm{e}^t \sin(2t).$$

We apply the initial conditions to determine the constants.

$$y(\pi/2) = 0 \quad \Rightarrow \quad -c_1 \, \mathrm{e}^{\pi/2} = 0 \quad \Rightarrow \quad c_1 = 0$$
$$y'(\pi/2) = 2 \quad \Rightarrow \quad -2c_2 \, \mathrm{e}^{\pi/2} = 2 \quad \Rightarrow \quad c_2 = -\mathrm{e}^{-\pi/2}$$

The solution subject to the initial conditions is

$$\boxed{y = -\mathrm{e}^{t-\pi/2} \sin(2t).}$$

The solution is plotted in Figure 17.2. The solution oscillates with an amplitude that tends to ∞ as $t \to \infty$.

3. We consider the problem

$$y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1.$$

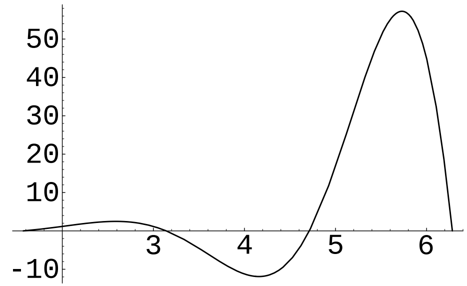

Figure 17.2: The solution of $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 2$.

We make the substitution $y = e^{\lambda x}$ in the differential equation.

$$\lambda^2 + 4\lambda + 4 = 0$$
$$(\lambda + 2)^2 = 0$$
$$\lambda = -2$$

The general solution of the differential equation is

$$y = c_1 e^{-2t} + c_2 t e^{-2t}.$$

We apply the initial conditions to determine the constants.

$$c_1 e^2 - c_2 e^2 = 2, \quad -2c_1 e^2 + 3c_2 e^2 = 1$$
$$c_1 = 7 e^{-2}, \quad c_2 = 5 e^{-2}$$

The solution subject to the initial conditions is

$$\boxed{y = (7 + 5t) e^{-2(t+1)}}$$

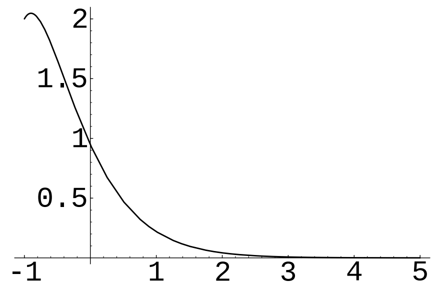

Figure 17.3: The solution of $y'' + 4y' + 4y = 0$, $y(-1) = 2$, $y'(-1) = 1$.

The solution is plotted in Figure 17.3. The solution vanishes as $t \to \infty$.

$$\lim_{t\to\infty} (7 + 5t)\, \mathrm{e}^{-2(t+1)} = \lim_{t\to\infty} \frac{7 + 5t}{\mathrm{e}^{2(t+1)}} = \lim_{t\to\infty} \frac{5}{2\, \mathrm{e}^{2(t+1)}} = 0$$

Solution 17.4

$$y'' - 4y' + 13y = 0.$$

With the substitution $y = \mathrm{e}^{\lambda x}$ we obtain

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda \mathrm{e}^{\lambda x} + 13 \mathrm{e}^{\lambda x} = 0$$
$$\lambda^2 - 4\lambda + 13 = 0$$
$$\lambda = 2 \pm 3i.$$

Thus two linearly independent solutions are

$$\mathrm{e}^{(2+3i)x}, \quad \text{and} \quad \mathrm{e}^{(2-3i)x}.$$

Noting that

$$\mathrm{e}^{(2+3i)x} = \mathrm{e}^{2x}[\cos(3x) + \imath\sin(3x)]$$
$$\mathrm{e}^{(2-3i)x} = \mathrm{e}^{2x}[\cos(3x) - \imath\sin(3x)],$$

we can write the two linearly independent solutions

$$\boxed{y_1 = \mathrm{e}^{2x}\cos(3x), \qquad y_2 = \mathrm{e}^{2x}\sin(3x).}$$

Solution 17.5

We note that

$$y''' - y'' + y' - y = 0$$

is a constant coefficient equation. The substitution, $y = \mathrm{e}^{\lambda x}$, yields

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$
$$(\lambda - 1)(\lambda - i)(\lambda + i) = 0.$$

The corresponding solutions are e^x, $\mathrm{e}^{\imath x}$, and $\mathrm{e}^{-\imath x}$. We can write the general solution as

$$\boxed{y = c_1\,\mathrm{e}^x + c_2\cos x + c_3\sin x.}$$

Solution 17.6

We start with the equation $y'' + y = 0$. We substitute $y = \mathrm{e}^{\lambda x}$ into the differential equation to obtain

$$\lambda^2 + 1 = 0, \qquad \lambda = \pm i.$$

A linearly independent set of solutions is

$$\{\mathrm{e}^{\imath x}, \mathrm{e}^{-\imath x}\}.$$

The fundamental set of solutions has the form

$$y_1 = c_1\,\mathrm{e}^{\imath x} + c_2\,\mathrm{e}^{-\imath x},$$
$$y_2 = c_3\,\mathrm{e}^{\imath x} + c_4\,\mathrm{e}^{-\imath x}.$$

By applying the constraints

$$y_1(0) = 1, \quad y_1'(0) = 0,$$
$$y_2(0) = 0, \quad y_2'(0) = 1,$$

we obtain

$$y_1 = \frac{\mathrm{e}^{\imath x} + \mathrm{e}^{-\imath x}}{2} = \cos x,$$
$$y_2 = \frac{\mathrm{e}^{\imath x} + \mathrm{e}^{-\imath x}}{\imath 2} = \sin x.$$

Now consider the equation $y'' - y = 0$. By substituting $y = \mathrm{e}^{\lambda x}$ we find that a set of solutions is

$$\{\mathrm{e}^x, \mathrm{e}^{-x}\}.$$

By taking linear combinations of these we see that another set of solutions is

$$\{\cosh x, \sinh x\}.$$

Note that this is the fundamental set of solutions.

Next consider $y'' = 0$. We can find the solutions by substituting $y = \mathrm{e}^{\lambda x}$ or by integrating the equation twice. The fundamental set of solutions as $x = 0$ is

$$\{1, x\}.$$

Note that if $u(x)$ is a solution of a constant coefficient differential equation, then $u(x + c)$ is also a solution. Also note that if $u(x)$ satisfies $y(0) = a$, $y'(0) = b$, then $u(x - x_0)$ satisfies $y(x_0) = a$, $y'(x_0) = b$. Thus the fundamental sets of solutions at $x = 1$ are

1. $\{\cos(x - 1), \sin(x - 1)\}$,
2. $\{\cosh(x - 1), \sinh(x - 1)\}$,
3. $\{1, x - 1\}$.

Solution 17.7

Let $y(t)$ denote the displacement of the mass from equilibrium. The forces on the mass are $-ky(t)$ due to the spring and $-\mu y'(t)$ due to friction. We equate the external forces to $my''(t)$ to find the differential equation of the motion.

$$my'' = -ky - \mu y'$$

$$\boxed{y'' + \frac{\mu}{m} y' + \frac{k}{m} y = 0}$$

The solution which satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$ is

$$\boxed{y(t) = \begin{cases} e^{-\mu t/(2m)} \frac{2m}{\sqrt{\mu^2 - 4km}} \sinh\left(\sqrt{\mu^2 - 4km}\, t/(2m)\right) & \text{if } \mu^2 > km, \\ e^{-\mu t/(2m)} \frac{2m}{\sqrt{4km - \mu^2}} \sin\left(\sqrt{4km - \mu^2}\, t/(2m)\right) & \text{if } \mu^2 < km, \\ t\, e^{-\mu t/(2m)} & \text{if } \mu^2 = km. \end{cases}}$$

We respectively call these cases: strongly damped, weakly damped and critically damped. In the case that $m = k = 1$ the solution is

$$y(t) = \begin{cases} e^{-\mu t/2} \frac{2}{\sqrt{\mu^2 - 4}} \sinh\left(\sqrt{\mu^2 - 4}\, t/2\right) & \text{if } \mu > 2, \\ e^{-\mu t/2} \frac{2}{\sqrt{4 - \mu^2}} \sin\left(\sqrt{4 - \mu^2}\, t/2\right) & \text{if } \mu < 2, \\ t\, e^{-t} & \text{if } \mu = 2. \end{cases}$$

Note that when t is large, $t\, e^{-t}$ is much smaller than $e^{-\mu t/2}$ for $\mu < 2$. To prove this we examine the ratio of these functions as $t \to \infty$.

$$\begin{aligned} \lim_{t \to \infty} \frac{t\, e^{-t}}{e^{-\mu t/2}} &= \lim_{t \to \infty} \frac{t}{e^{(1-\mu/2)t}} \\ &= \lim_{t \to \infty} \frac{1}{(1 - \mu/2)\, e^{(1-\mu)t}} \\ &= 0 \end{aligned}$$

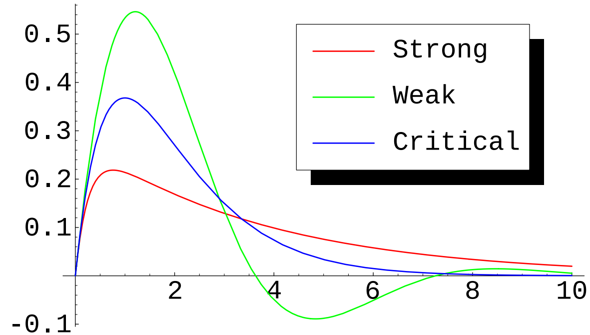

Figure 17.4: Strongly, weakly and critically damped solutions.

Using this result, we see that the critically damped solution decays faster than the weakly damped solution.

We can write the strongly damped solution as

$$\mathrm{e}^{-\mu t/2} \frac{2}{\sqrt{\mu^2-4}} \left(\mathrm{e}^{\sqrt{\mu^2-4}\, t/2} - \mathrm{e}^{-\sqrt{\mu^2-4}\, t/2} \right).$$

For large t, the dominant factor is $\mathrm{e}^{\left(\sqrt{\mu^2-4}-\mu\right)t/2}$. Note that for $\mu > 2$,

$$\sqrt{\mu^2-4} = \sqrt{(\mu+2)(\mu-2)} > \mu - 2.$$

Therefore we have the bounds

$$-2 < \sqrt{\mu^2-4} - \mu < 0.$$

This shows that the critically damped solution decays faster than the strongly damped solution. $\mu = 2$ gives the fastest decaying solution. Figure 17.4 shows the solution for $\mu = 4$, $\mu = 1$ and $\mu = 2$.

Solution 17.8

Clearly $y = c\cos(x - \phi)$ satisfies the differential equation $y'' + y = 0$. Since it is a two-parameter family of functions, it must be the general solution.

Using a trigonometric identity we can rewrite the solution as

$$y = c\cos\phi\cos x + c\sin\phi\sin x.$$

Setting this equal to $\sin x$ gives us the two equations

$$\begin{aligned} c\cos\phi &= 0, \\ c\sin\phi &= 1, \end{aligned}$$

which has the solutions $c = 1$, $\phi = (2n + 1/2)\pi$, and $c = -1$, $\phi = (2n - 1/2)\pi$, for $n \in \mathbb{Z}$.

Clearly $y = c\cosh(x - \phi)$ satisfies the differential equation $y'' - y = 0$. Since it is a two-parameter family of functions, it must be the general solution.

Using a trigonometric identity we can rewrite the solution as

$$y = c\cosh\phi\cosh x + c\sinh\phi\sinh x.$$

Setting this equal to $\sinh x$ gives us the two equations

$$\begin{aligned} c\cosh\phi &= 0, \\ c\sinh\phi &= 1, \end{aligned}$$

which has the solutions $c = -i$, $\phi = \imath(2n + 1/2)\pi$, and $c = i$, $\phi = \imath(2n - 1/2)\pi$, for $n \in \mathbb{Z}$.

Solution 17.9

We substitute $y = \mathrm{e}^{\lambda t}$ into the differential equation.

$$\begin{aligned} \lambda^2 \mathrm{e}^{\lambda t} + 5\lambda \mathrm{e}^{\lambda t} + 6 \mathrm{e}^{\lambda t} &= 0 \\ \lambda^2 + 5\lambda + 6 &= 0 \\ (\lambda + 2)(\lambda + 3) &= 0 \end{aligned}$$

The general solution of the differential equation is

$$y = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}.$$

The initial conditions give us the constraints:

$$c_1 + c_2 = 1,$$
$$-2c_1 - 3c_2 = V.$$

The solution subject to the initial conditions is

$$\boxed{y = (3 + V) \mathrm{e}^{-2t} - (2 + V) \mathrm{e}^{-3t}.}$$

This solution will be non-negative for $t > 0$ if $V \geq -3$.

Solution 17.10

For negative x, the differential equation is

$$y'' - y = 0.$$

We substitute $y = \mathrm{e}^{\lambda x}$ into the differential equation to find the solutions.

$$\lambda^2 - 1 = 0$$
$$\lambda = \pm 1$$
$$y = \left\{\mathrm{e}^{x}, \mathrm{e}^{-x}\right\}$$

We can take linear combinations to write the solutions in terms of the hyperbolic sine and cosine.

$$y = \{\cosh(x), \sinh(x)\}$$

For positive x, the differential equation is

$$y'' + y = 0.$$

We substitute $y = e^{\lambda x}$ into the differential equation to find the solutions.

$$\lambda^2 + 1 = 0$$
$$\lambda = \pm \imath$$
$$y = \left\{ e^{\imath x}, e^{-\imath x} \right\}$$

We can take linear combinations to write the solutions in terms of the sine and cosine.

$$y = \{\cos(x), \sin(x)\}$$

We will find the fundamental set of solutions at $x = 0$. That is, we will find a set of solutions, $\{y_1, y_2\}$ that satisfy the conditions:

$$y_1(0) = 1 \quad y_1'(0) = 0$$
$$y_2(0) = 0 \quad y_2'(0) = 1$$

Clearly, these solutions are

$$\boxed{y_1 = \begin{cases} \cosh(x) & x < 0 \\ \cos(x) & x \geq 0 \end{cases} \qquad y_2 = \begin{cases} \sinh(x) & x < 0 \\ \sin(x) & x \geq 0 \end{cases}}$$

Euler Equations

Solution 17.11
We consider an Euler equation,

$$x^2 y'' + x y' + y = 0, \quad x > 0.$$

We make the change of independent variable $\xi = \ln x$, $u(\xi) = y(x)$ to obtain

$$u'' + u = 0.$$

We make the substitution $u(\xi) = \mathrm{e}^{\lambda\xi}$.

$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

A set of linearly independent solutions for $u(\xi)$ is

$$\{\mathrm{e}^{\imath\xi}, \mathrm{e}^{-\imath\xi}\}.$$

Since

$$\cos\xi = \frac{\mathrm{e}^{\imath\xi} + \mathrm{e}^{-\imath\xi}}{2} \quad \text{and} \quad \sin\xi = \frac{\mathrm{e}^{\imath\xi} - \mathrm{e}^{-\imath\xi}}{\imath 2},$$

another linearly independent set of solutions is

$$\{\cos\xi, \sin\xi\}.$$

The general solution for $y(x)$ is

$$\boxed{y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x).}$$

Solution 17.12

Consider the differential equation

$$x^2 y'' - 2xy + 2y = 0.$$

With the substitution $y = x^\lambda$ this equation becomes

$$\lambda(\lambda - 1) - 2\lambda + 2 = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$\lambda = 1, 2.$$

The general solution is then

$$\boxed{y = c_1 x + c_2 x^2.}$$

Solution 17.13
We note that

$$xy''' + y'' + \frac{1}{x}y' = 0$$

is an Euler equation. The substitution $y = x^\lambda$ yields

$$\lambda^3 - 3\lambda^2 + 2\lambda + \lambda^2 - \lambda + \lambda = 0$$
$$\lambda^3 - 2\lambda^2 + 2\lambda = 0.$$

The three roots of this algebraic equation are

$$\lambda = 0, \quad \lambda = 1 + i, \quad \lambda = 1 - \imath$$

The corresponding solutions to the differential equation are

$$\begin{array}{lll} y = x^0 & y = x^{1+\imath} & y = x^{1-\imath} \\ y = 1 & y = x\,\mathrm{e}^{\imath \ln x} & y = x\,\mathrm{e}^{-\imath \ln x}. \end{array}$$

We can write the general solution as

$$\boxed{y = c_1 + c_2 x \cos(\ln x) + c_3 \sin(\ln x).}$$

Solution 17.14
We substitute $y = x^\lambda$ into the differential equation.

$$x^2 y'' + (2a + 1)xy' + by = 0$$
$$\lambda(\lambda - 1) + (2a + 1)\lambda + b = 0$$
$$\lambda^2 + 2a\lambda + b = 0$$
$$\lambda = -a \pm \sqrt{a^2 - b}$$

For $a^2 > b$ then the general solution is

$$y = c_1 x^{-a+\sqrt{a^2-b}} + c_2 x^{-a-\sqrt{a^2-b}}.$$

For $a^2 < b$, then the general solution is

$$y = c_1 x^{-a+\imath\sqrt{b-a^2}} + c_2 x^{-a-\imath\sqrt{b-a^2}}.$$

By taking the sum and difference of these solutions, we can write the general solution as

$$y = c_1 x^{-a} \cos\left(\sqrt{b-a^2}\,\ln x\right) + c_2 x^{-a} \sin\left(\sqrt{b-a^2}\,\ln x\right).$$

For $a^2 = b$, the quadratic in lambda has a double root at $\lambda = a$. The general solution of the differential equation is

$$y = c_1 x^{-a} + c_2 x^{-a} \ln x.$$

In summary, the general solution is:

$$\boxed{y = \begin{cases} x^{-a}\left(c_1 x^{\sqrt{a^2-b}} + c_2 x^{-\sqrt{a^2-b}}\right) & \text{if } a^2 > b, \\ x^{-a}\left(c_1 \cos\left(\sqrt{b-a^2}\,\ln x\right) + c_2 \sin\left(\sqrt{b-a^2}\,\ln x\right)\right) & \text{if } a^2 < b, \\ x^{-a}\left(c_1 + c_2 \ln x\right) & \text{if } a^2 = b. \end{cases}}$$

Solution 17.15

For $a \neq 0$, two linearly independent solutions of

$$y'' - a^2 y = 0$$

are

$$y_1 = \mathrm{e}^{ax}, \quad y_2 = \mathrm{e}^{-ax}.$$

For $a = 0$, we have

$$y_1 = \mathrm{e}^{0x} = 1, \quad y_2 = x\,\mathrm{e}^{0x} = x.$$

In this case the solution are defined by

$$y_1 = [\mathrm{e}^{ax}]_{a=0}\,, \quad y_2 = \left[\frac{\mathrm{d}}{\mathrm{d}a}\,\mathrm{e}^{ax}\right]_{a=0}.$$

By the definition of differentiation, $f'(0)$ is

$$f'(0) = \lim_{a\to 0} \frac{f(a) - f(-a)}{2a}.$$

Thus the second solution in the case $a = 0$ is

$$y_2 = \lim_{a\to 0} \frac{\mathrm{e}^{ax} - \mathrm{e}^{-ax}}{a}$$

Consider the solutions

$$y_1 = \mathrm{e}^{ax}, \quad y_2 = \lim_{\alpha\to a} \frac{\mathrm{e}^{\alpha x} - \mathrm{e}^{-\alpha x}}{\alpha}.$$

Clearly y_1 is a solution for all a. For $a \neq 0$, y_2 is a linear combination of e^{ax} and e^{-ax} and is thus a solution. Since the coefficient of e^{-ax} in this linear combination is non-zero, it is linearly independent to y_1. For $a = 0$, y_2 is one half the derivative of e^{ax} evaluated at $a = 0$. Thus it is a solution.

For $a \neq 0$, two linearly independent solutions of

$$x^2 y'' + x y' - a^2 y = 0$$

are

$$y_1 = x^a, \quad y_2 = x^{-a}.$$

For $a = 0$, we have

$$y_1 = [x^a]_{a=0} = 1, \quad y_2 = \left[\frac{\mathrm{d}}{\mathrm{d}a} x^a\right]_{a=0} = \ln x.$$

Consider the solutions

$$y_1 = x^a, \quad y_2 = \frac{x^a - x^{-a}}{a}$$

Clearly y_1 is a solution for all a. For $a \neq 0$, y_2 is a linear combination of x^a and x^{-a} and is thus a solution. For $a = 0$, y_2 is one half the derivative of x^a evaluated at $a = 0$. Thus it is a solution.

Solution 17.16

1.

$$x^2y'' - 2xy' + 2y = 0$$

We substitute $y = x^\lambda$ into the differential equation.

$$\lambda(\lambda - 1) - 2\lambda + 2 = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda - 1)(\lambda - 2) = 0$$
$$\boxed{y = c_1x + c_2x^2}$$

2.

$$x^2y'' - 2y = 0$$

We substitute $y = x^\lambda$ into the differential equation.

$$\lambda(\lambda - 1) - 2 = 0$$
$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda + 1)(\lambda - 2) = 0$$
$$\boxed{y = \frac{c_1}{x} + c_2x^2}$$

3.

$$x^2y'' - xy' + y = 0$$

We substitute $y = x^\lambda$ into the differential equation.

$$\lambda(\lambda - 1) - \lambda + 1 = 0$$
$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)^2 = 0$$

Since there is a double root, the solution is:

$$\boxed{y = c_1 x + c_2 x \ln x.}$$

Exact Equations

Solution 17.17
We note that

$$y'' + y' \sin x + y \cos x = 0$$

is an exact equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}[y' + y \sin x] = 0$$
$$y' + y \sin x = c_1$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[y\,\mathrm{e}^{-\cos x}\right] = c_1\,\mathrm{e}^{-\cos x}$$
$$\boxed{y = c_1\,\mathrm{e}^{\cos x} \int \mathrm{e}^{-\cos x}\,\mathrm{d}x + c_2\,\mathrm{e}^{\cos x}}$$

Equations Without Explicit Dependence on y
Reduction of Order

Solution 17.18

$$(1 - x^2)y'' - 2xy' + 2y = 0, \quad -1 < x < 1$$

We substitute $y = x$ into the differential equation to check that it is a solution.

$$(1 - x^2)(0) - 2x(1) + 2x = 0$$

We look for a second solution of the form $y = xu$. We substitute this into the differential equation and use the fact that x is a solution.

$$
\begin{gathered}
(1-x^2)(xu''+2u') - 2x(xu'+u) + 2xu = 0 \\
(1-x^2)(xu''+2u') - 2x(xu') = 0 \\
(1-x^2)xu'' + (2-4x^2)u' = 0 \\
\frac{u''}{u'} = \frac{2-4x^2}{x(x^2-1)} \\
\frac{u''}{u'} = -\frac{2}{x} + \frac{1}{1-x} - \frac{1}{1+x} \\
\ln(u') = -2\ln(x) - \ln(1-x) - \ln(1+x) + \text{const} \\
\ln(u') = \ln\left(\frac{c}{x^2(1-x)(1+x)}\right) \\
u' = \frac{c}{x^2(1-x)(1+x)} \\
u' = c\left(\frac{1}{x^2} + \frac{1}{2(1-x)} + \frac{1}{2(1+x)}\right) \\
u = c\left(-\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x)\right) + \text{const} \\
u = c\left(-\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right) + \text{const}
\end{gathered}
$$

A second linearly independent solution is

$$
\boxed{y = -1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right).}
$$

Solution 17.19

We are given that $y = \mathrm{e}^x$ is a solution of

$$y'' - \frac{x+1}{x} y' + \frac{1}{x} y = 0.$$

To find another linearly independent solution, we will use reduction of order. Substituting

$$\begin{aligned} y &= u\,\mathrm{e}^x \\ y' &= (u' + u)\,\mathrm{e}^x \\ y'' &= (u'' + 2u' + u)\,\mathrm{e}^x \end{aligned}$$

into the differential equation yields

$$\begin{gathered} u'' + 2u' + u - \frac{x+1}{x}(u' + u) + \frac{1}{x} u = 0. \\ u'' + \frac{x-1}{x} u' = 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[u' \exp\left(\int \left(1 - \frac{1}{x} \right) \mathrm{d}x \right) \right] = 0 \\ u'\,\mathrm{e}^{x - \ln x} = c_1 \\ u' = c_1 x\,\mathrm{e}^{-x} \\ u = c_1 \int x\,\mathrm{e}^{-x}\,\mathrm{d}x + c_2 \\ u = c_1 (x\,\mathrm{e}^{-x} + \mathrm{e}^{-x}) + c_2 \\ y = c_1 (x + 1) + c_2\,\mathrm{e}^x \end{gathered}$$

Thus a second linearly independent solution is

$$\boxed{y = x + 1.}$$

Solution 17.20
We are given that $y = x$ is a solution of

$$(1 - 2x)y'' + 4xy' - 4y = 0.$$

To find another linearly independent solution, we will use reduction of order. Substituting

$$\begin{aligned} y &= xu \\ y' &= xu' + u \\ y'' &= xu'' + 2u' \end{aligned}$$

into the differential equation yields

$$\begin{gathered} (1 - 2x)(xu'' + 2u') + 4x(xu' + u) - 4xu = 0, \\ (1 - 2x)xu'' + (4x^2 - 4x + 2)u' = 0, \\ \frac{u''}{u'} = \frac{4x^2 - 4x + 2}{x(2x - 1)}, \\ \frac{u''}{u'} = 2 - \frac{2}{x} + \frac{2}{2x - 1}, \\ \ln(u') = 2x - 2\ln x + \ln(2x - 1) + \mathsf{const}, \\ u' = c_1 \left(\frac{2}{x} - \frac{1}{x^2} \right) e^{2x}, \\ u = c_1 \frac{1}{x} e^{2x} + c_2, \\ \boxed{y = c_1 e^{2x} + c_2 x.} \end{gathered}$$

Solution 17.21
One solution of

$$(x - 1)y'' - xy' + y = 0,$$

is $y_1 = \mathrm{e}^x$. We find a second solution with reduction of order. We make the substitution $y_2 = u\,\mathrm{e}^x$ in the differential equation. We determine u up to an additive constant.

$$
\begin{gathered}
(x-1)(u'' + 2u' + u)\,\mathrm{e}^x - x(u' + u)\,\mathrm{e}^x + u\,\mathrm{e}^x = 0 \\
(x-1)u'' + (x-2)u' = 0 \\
\frac{u''}{u'} = -\frac{x-2}{x-1} = -1 + \frac{1}{x-1} \\
\ln|u'| = -x + \ln|x-1| + c \\
u' = c(x-1)\,\mathrm{e}^{-x} \\
u = -cx\,\mathrm{e}^{-x}
\end{gathered}
$$

The second solution of the differential equation is $y_2 = x$.

*Reduction of Order and the Adjoint Equation

Chapter 18

Techniques for Nonlinear Differential Equations

In mathematics you don't understand things. You just get used to them.

- Johann von Neumann

18.1 Bernoulli Equations

Sometimes it is possible to solve a nonlinear equation by making a change of the dependent variable that converts it into a linear equation. One of the most important such equations is the *Bernoulli equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)y^\alpha, \quad \alpha \neq 1.$$

The change of dependent variable $u = y^{1-\alpha}$ will yield a first order linear equation for u which when solved will give us an implicit solution for y. (See Exercise 18.4.)

Result 18.1.1 The Bernoulli equation $y' + p(t)y = q(t)y^\alpha, \ \alpha \neq 1$ can be transformed to the first order linear equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + (1-\alpha)p(t)u = (1-\alpha)q(t)$$

with the change of variables $u = y^{1-\alpha}$.

Example 18.1.1 *Consider the Bernoulli equation*

$$y' = \frac{2}{x}y + y^2.$$

First we divide by y^2.

$$y^{-2}y' = \frac{2}{x}y^{-1} + 1$$

We make the change of variable $u = y^{-1}$.

$$-u' = \frac{2}{x}u + 1$$
$$u' + \frac{2}{x}u = -1$$

The integrating factor is $I(x) = \exp(\int \frac{2}{x}\,dx) = x^2$.

$$\begin{aligned}
\frac{d}{dx}(x^2 u) &= -x^2 \\
x^2 u &= -\frac{1}{3}x^3 + c \\
u &= -\frac{1}{3}x + \frac{c}{x^2} \\
y &= \left(-\frac{1}{3}x + \frac{c}{x^2}\right)^{-1}
\end{aligned}$$

Thus the solution for y *is*

$$\boxed{y = \frac{3x^2}{c - x^2}.}$$

18.2 Riccati Equations

Factoring Second Order Operators. Consider the second order linear equation

$$L[y] = \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)\right] y = y'' + p(x)y' + q(x)y = f(x).$$

If we were able to factor the linear operator L into the form

$$L = \left[\frac{\mathrm{d}}{\mathrm{d}x} + a(x)\right]\left[\frac{\mathrm{d}}{\mathrm{d}x} + b(x)\right], \tag{18.1}$$

then we would be able to solve the differential equation. Factoring reduces the problem to a system of first order equations. We start with the factored equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + a(x)\right]\left[\frac{\mathrm{d}}{\mathrm{d}x} + b(x)\right] y = f(x).$$

We set $u = \left[\frac{\mathrm{d}}{\mathrm{d}x} + b(x)\right] y$ and solve the problem

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + a(x)\right] u = f(x).$$

Then to obtain the solution we solve

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + b(x)\right] y = u.$$

Example 18.2.1 *Consider the equation*

$$y'' + \left(x - \frac{1}{x}\right) y' + \left(\frac{1}{x^2} - 1\right) y = 0.$$

Let's say by some insight or just random luck we are able to see that this equation can be factored into

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + x\right]\left[\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x}\right] y = 0.$$

We first solve the equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + x\right] u = 0.$$
$$u' + xu = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2/2} u\right) = 0$$
$$u = c_1 \,\mathrm{e}^{-x^2/2}$$

Then we solve for y with the equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{x}\right] y = u = c_1 \,\mathrm{e}^{-x^2/2}.$$

$$y' - \frac{1}{x} y = c_1 \,\mathrm{e}^{-x^2/2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1} y\right) = c_1 x^{-1} \,\mathrm{e}^{-x^2/2}$$

$$\boxed{y = c_1 x \int x^{-1} \,\mathrm{e}^{-x^2/2}\, dx + c_2 x}$$

If we were able to solve for a and b in Equation 18.1 in terms of p and q then we would be able to solve any second order differential equation. Equating the two operators,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p\frac{\mathrm{d}}{\mathrm{d}x} + q &= \left[\frac{\mathrm{d}}{\mathrm{d}x} + a\right]\left[\frac{\mathrm{d}}{\mathrm{d}x} + b\right] \\ &= \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (a+b)\frac{\mathrm{d}}{\mathrm{d}x} + (b' + ab).\end{aligned}$$

Thus we have the two equations

$$a + b = p, \quad \text{and} \quad b' + ab = q.$$

Eliminating a,

$$\begin{aligned} b' + (p - b)b &= q \\ b' &= b^2 - pb + q \end{aligned}$$

Now we have a nonlinear equation for b that is no easier to solve than the original second order linear equation.

Riccati Equations. Equations of the form

$$y' = a(x)y^2 + b(x)y + c(x)$$

are called Riccati equations. From the above derivation we see that for every second order differential equation there is a corresponding Riccati equation. Now we will show that the converse is true.

We make the substitution

$$y = -\frac{u'}{au}, \qquad y' = -\frac{u''}{au} + \frac{(u')^2}{au^2} + \frac{a'u'}{a^2u},$$

in the Riccati equation.

$$\begin{gathered}
y' = ay^2 + by + c \\
-\frac{u''}{au} + \frac{(u')^2}{au^2} + \frac{a'u'}{a^2u} = a\frac{(u')^2}{a^2u^2} - b\frac{u'}{au} + c \\
-\frac{u''}{au} + \frac{a'u'}{a^2u} + b\frac{u'}{au} - c = 0 \\
u'' - \left(\frac{a'}{a} + b\right) u' + acu = 0
\end{gathered}$$

Now we have a second order linear equation for u.

Result 18.2.1 The substitution $y = -\frac{u'}{au}$ transforms the Riccati equation

$$y' = a(x)y^2 + b(x)y + c(x)$$

into the second order linear equation

$$u'' - \left(\frac{a'}{a} + b\right) u' + acu = 0.$$

Example 18.2.2 *Consider the Riccati equation*

$$y' = y^2 + \frac{1}{x}y + \frac{1}{x^2}.$$

With the substitution $y = -\frac{u'}{u}$ *we obtain*

$$u'' - \frac{1}{x}u' + \frac{1}{x^2}u = 0.$$

This is an Euler equation. The substitution $u = x^\lambda$ *yields*

$$\lambda(\lambda - 1) - \lambda + 1 = (\lambda - 1)^2 = 0.$$

Thus the general solution for u *is*

$$u = c_1 x + c_2 x \log x.$$

Since $y = -\frac{u'}{u}$*,*

$$y = -\frac{c_1 + c_2(1 + \log x)}{c_1 x + c_2 x \log x}$$

$$\boxed{y = -\frac{1 + c(1 + \log x)}{x + cx \log x}}$$

18.3 Exchanging the Dependent and Independent Variables

Some differential equations can be put in a more elementary form by exchanging the dependent and independent variables. If the new equation can be solved, you will have an implicit solution for the initial equation. We will consider a few examples to illustrate the method.

Example 18.3.1 *Consider the equation*

$$y' = \frac{1}{y^3 - xy^2}.$$

Instead of considering y to be a function of x, consider x to be a function of y. That is, $x = x(y)$, $x' = \frac{\mathrm{d}x}{\mathrm{d}y}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{y^3 - xy^2}$$
$$\frac{\mathrm{d}x}{\mathrm{d}y} = y^3 - xy^2$$
$$x' + y^2 x = y^3$$

Now we have a first order equation for x.

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\mathrm{e}^{y^3/3}\, x\right) = y^3\, \mathrm{e}^{y^3/3}$$
$$\boxed{x = \mathrm{e}^{-y^3/3} \int y^3\, \mathrm{e}^{y^3/3}\; dy + c\, \mathrm{e}^{-y^3/3}}$$

Example 18.3.2 *Consider the equation*

$$y' = \frac{y}{y^2 + 2x}.$$

Interchanging the dependent and independent variables yields

$$\frac{1}{x'} = \frac{y}{y^2 + 2x}$$
$$x' = y + 2\frac{x}{y}$$
$$x' - 2\frac{x}{y} = y$$
$$\frac{\mathrm{d}}{\mathrm{d}y}(y^{-2}x) = y^{-1}$$
$$y^{-2}x = \log y + c$$
$$\boxed{x = y^2 \log y + cy^2}$$

Result 18.3.1 Some differential equations can be put in a simpler form by exchanging the dependent and independent variables. Thus a differential equation for $y(x)$ can be written as an equation for $x(y)$. Solving the equation for $x(y)$ will give an implicit solution for $y(x)$.

18.4 Autonomous Equations

Autonomous equations have no explicit dependence on x. The following are examples.

- $y'' + 3y' - 2y = 0$
- $y'' = y + (y')^2$
- $y''' + y''y = 0$

The change of variables $u(y) = y'$ reduces an n^{th} order autonomous equation in y to a non-autonomous equation of order $n - 1$ in $u(y)$. Writing the derivatives of y in terms of u,

$$\begin{aligned}
y' &= u(y) \\
y'' &= \frac{\mathrm{d}}{\mathrm{d}x} u(y) \\
&= \frac{\mathrm{d}y}{\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}y} u(y) \\
&= y'u' \\
&= u'u \\
y''' &= (u''u + (u')^2)u.
\end{aligned}$$

Thus we see that the equation for $u(y)$ will have an order of one less than the original equation.

Result 18.4.1 Consider an autonomous differential equation for $y(x)$, (autonomous equations have no explicit dependence on x.) The change of variables $u(y) = y'$ reduces an n^{th} order autonomous equation in y to a non-autonomous equation of order $n - 1$ in $u(y)$.

Example 18.4.1 *Consider the equation*

$$y'' = y + (y')^2.$$

With the substitution $u(y) = y'$*, the equation becomes*

$$u'u = y + u^2$$
$$u' = u + yu^{-1}.$$

We recognize this as a Bernoulli equation. The substitution $v = u^2$ *yields*

$$\frac{1}{2}v' = v + y$$
$$v' - 2v = 2y$$
$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\mathrm{e}^{-2y} v\right) = 2y\,\mathrm{e}^{-2y}$$
$$v(y) = c_1\,\mathrm{e}^{2y} + \mathrm{e}^{2y}\int 2y\,\mathrm{e}^{-2y}\;dy$$
$$v(y) = c_1\,\mathrm{e}^{2y} + \mathrm{e}^{2y}\left(-y\,\mathrm{e}^{-2y} + \int \mathrm{e}^{-2y}\;dy\right)$$
$$v(y) = c_1\,\mathrm{e}^{2y} + \mathrm{e}^{2y}\left(-y\,\mathrm{e}^{-2y} - \frac{1}{2}\,\mathrm{e}^{-2y}\right)$$
$$v(y) = c_1\,\mathrm{e}^{2y} - y - \frac{1}{2}.$$

Now we solve for u*.*

$$u(y) = \left(c_1\,\mathrm{e}^{2y} - y - \frac{1}{2}\right)^{1/2}.$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(c_1\,\mathrm{e}^{2y} - y - \frac{1}{2}\right)^{1/2}$$

This equation is separable.

$$dx = \frac{dy}{\left(c_1 \,\mathrm{e}^{2y} - y - \frac{1}{2}\right)^{1/2}}$$

$$\boxed{x + c_2 = \int \frac{1}{\left(c_1 \,\mathrm{e}^{2y} - y - \frac{1}{2}\right)^{1/2}} \, dy}$$

Thus we finally have arrived at an implicit solution for $y(x)$.

Example 18.4.2 *Consider the equation*

$$y'' + y^3 = 0.$$

With the change of variables, $u(y) = y'$, *the equation becomes*

$$u'u + y^3 = 0.$$

This equation is separable.

$$u \, du = -y^3 \, dy$$

$$\frac{1}{2}u^2 = -\frac{1}{4}y^4 + c_1$$

$$u = \left(2c_1 - \frac{1}{2}y^4\right)^{1/2}$$

$$y' = \left(2c_1 - \frac{1}{2}y^4\right)^{1/2}$$

$$\frac{dy}{(2c_1 - \frac{1}{2}y^4)^{1/2}} = dx$$

Integrating gives us the implicit solution

$$\boxed{\int \frac{1}{(2c_1 - \frac{1}{2}y^4)^{1/2}} \, dy = x + c_2.}$$

18.5 *Equidimensional-in-x Equations

Differential equations that are invariant under the change of variables $x = c\,\xi$ are said to be equidimensional-in-x. For a familiar example from linear equations, we note that the Euler equation is equidimensional-in-x. Writing the new derivatives under the change of variables,

$$x = c\,\xi, \qquad \frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}\xi}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{1}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}, \qquad \ldots.$$

Example 18.5.1 *Consider the Euler equation*

$$y'' + \frac{2}{x}y' + \frac{3}{x^2}y = 0.$$

Under the change of variables, $x = c\,\xi$, $y(x) = u(\xi)$, this equation becomes

$$\frac{1}{c^2}u'' + \frac{2}{c\,\xi}\frac{1}{c}u' + \frac{3}{c^2\,\xi^2}u = 0$$
$$u'' + \frac{2}{\xi}u' + \frac{3}{\xi^2}u = 0.$$

Thus this equation is invariant under the change of variables $x = c\,\xi$.

Example 18.5.2 *For a nonlinear example, consider the equation*

$$y''\,y' + \frac{y''}{x\,y} + \frac{y'}{x^2} = 0.$$

With the change of variables $x = c\,\xi$, $y(x) = u(\xi)$ the equation becomes

$$\frac{u''}{c^2}\frac{u'}{c} + \frac{u''}{c^3\,\xi\,u} + \frac{u'}{c^3\,\xi^2} = 0$$
$$u''\,u' + \frac{u''}{\xi\,u} + \frac{u'}{\xi^2} = 0.$$

We see that this equation is also equidimensional-in-x.

You may recall that the change of variables $x = \mathrm{e}^t$ reduces an Euler equation to a constant coefficient equation. To generalize this result to nonlinear equations we will see that the same change of variables reduces an equidimensional-in-x equation to an autonomous equation.

Writing the derivatives with respect to x in terms of t,

$$x = \mathrm{e}^t, \qquad \frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t} = \mathrm{e}^{-t}\,\frac{\mathrm{d}}{\mathrm{d}t}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}t}$$

$$x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) - x\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{\mathrm{d}}{\mathrm{d}t}.$$

Example 18.5.3 *Consider the equation in Example 18.5.2*

$$y''\,y' + \frac{y''}{x\,y} + \frac{y'}{x^2} = 0.$$

Applying the change of variables $x = \mathrm{e}^t$, $y(x) = u(t)$ *yields an autonomous equation for* $u(t)$.

$$x^2\,y''\,x\,y' + \frac{x^2\,y''}{y} + x\,y' = 0$$

$$\boxed{(u'' - u')u' + \frac{u'' - u'}{u} + u' = 0}$$

Result 18.5.1 A differential equation that is invariant under the change of variables $x = c\,\xi$ is equidimensional-in-x. Such an equation can be reduced to autonomous equation of the same order with the change of variables, $x = \mathrm{e}^t$.

18.6 *Equidimensional-in-y Equations

A differential equation is said to be equidimensional-in-y if it is invariant under the change of variables $y(x) = c\,v(x)$. Note that all linear homogeneous equations are equidimensional-in-y.

Example 18.6.1 *Consider the linear equation*

$$y'' + p(x)y' + q(x)y = 0.$$

With the change of variables $y(x) = cv(x)$ the equation becomes

$$cv'' + p(x)cv' + q(x)cv = 0$$
$$v'' + p(x)v' + q(x)v = 0$$

Thus we see that the equation is invariant under the change of variables.

Example 18.6.2 *For a nonlinear example, consider the equation*

$$y''y + (y')^2 - y^2 = 0.$$

Under the change of variables $y(x) = cv(x)$ the equation becomes.

$$cv''cv + (cv')^2 - (cv)^2 = 0$$
$$v''v + (v')^2 - v^2 = 0.$$

Thus we see that this equation is also equidimensional-in-y.

The change of variables $y(x) = \mathrm{e}^{u(x)}$ reduces an n^{th} order equidimensional-in-y equation to an equation of order $n-1$ for u'. Writing the derivatives of $\mathrm{e}^{u(x)}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{e}^u = u'\,\mathrm{e}^u$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,\mathrm{e}^u = (u'' + (u')^2)\,\mathrm{e}^u$$
$$\frac{d^3}{dx^3}\,\mathrm{e}^u = (u''' + 3u''u'' + (u')^3)\,\mathrm{e}^u\,.$$

Example 18.6.3 *Consider the linear equation in Example 18.6.1*

$$y'' + p(x)y' + q(x)y = 0.$$

Under the change of variables $y(x) = \mathrm{e}^{u(x)}$ *the equation becomes*

$$(u'' + (u')^2)\,\mathrm{e}^u + p(x)u'\,\mathrm{e}^u + q(x)\,\mathrm{e}^u = 0$$
$$\boxed{u'' + (u')^2 + p(x)u' + q(x) = 0.}$$

Thus we have a Riccati equation for u'. *This transformation might seem rather useless since linear equations are usually easier to work with than nonlinear equations, but it is often useful in determining the asymptotic behavior of the equation.*

Example 18.6.4 *From Example 18.6.2 we have the equation*

$$y''y + (y')^2 - y^2 = 0.$$

The change of variables $y(x) = \mathrm{e}^{u(x)}$ *yields*

$$(u'' + (u')^2)\,\mathrm{e}^u\,\mathrm{e}^u + (u'\,\mathrm{e}^u)^2 - (\mathrm{e}^u)^2 = 0$$
$$u'' + 2(u')^2 - 1 = 0$$
$$u'' = -2(u')^2 + 1$$

Now we have a Riccati equation for u'. We make the substitution $u' = \frac{v'}{2v}$.

$$\frac{v''}{2v} - \frac{(v')^2}{2v^2} = -2\frac{(v')^2}{4v^2} + 1$$
$$v'' - 2v = 0$$
$$v = c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}$$
$$u' = 2\sqrt{2}\frac{c_1 \mathrm{e}^{\sqrt{2}x} - c_2 \mathrm{e}^{-\sqrt{2}x}}{c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}}$$
$$u = 2\int \frac{c_1\sqrt{2}\,\mathrm{e}^{\sqrt{2}x} - c_2\sqrt{2}\,\mathrm{e}^{-\sqrt{2}x}}{c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}}\,dx + c_3$$
$$u = 2\log\left(c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}\right) + c_3$$
$$y = \left(c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}\right)^2 \mathrm{e}^{c_3}$$

The constants are redundant, the general solution is

$$\boxed{y = \left(c_1 \mathrm{e}^{\sqrt{2}x} + c_2 \mathrm{e}^{-\sqrt{2}x}\right)^2}$$

Result 18.6.1 A differential equation is equidimensional-in-y if it is invariant under the change of variables $y(x) = cv(x)$. An n^{th} order equidimensional-in-y equation can be reduced to an equation of order $n-1$ in u' with the change of variables $y(x) = \mathrm{e}^{u(x)}$.

18.7 *Scale-Invariant Equations

Result 18.7.1 An equation is scale invariant if it is invariant under the change of variables, $x = c\xi$, $y(x) = c^\alpha v(\xi)$, for some value of α. A scale-invariant equation can be transformed to an equidimensional-in-x equation with the change of variables, $y(x) = x^\alpha u(x)$.

Example 18.7.1 *Consider the equation*

$$y'' + x^2 y^2 = 0.$$

Under the change of variables $x = c\xi$, $y(x) = c^\alpha v(\xi)$ this equation becomes

$$\frac{c^\alpha}{c^2} v''(\xi) + c^2 x^2 c^{2\alpha} v^2(\xi) = 0.$$

Equating powers of c in the two terms yields $\alpha = -4$.

Introducing the change of variables $y(x) = x^{-4} u(x)$ yields

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[x^{-4} u(x)\right] + x^2 (x^{-4} u(x))^2 = 0$$

$$x^{-4} u'' - 8x^{-5} u' + 20x^{-6} u + x^{-6} u^2 = 0$$

$$\boxed{x^2 u'' - 8xu' + 20u + u^2 = 0.}$$

We see that the equation for u is equidimensional-in-x.

18.8 Exercises

Exercise 18.1

1. Find the general solution and the singular solution of the Clairaut equation,

$$y = xp + p^2.$$

2. Show that the singular solution is the envelope of the general solution.

Hint, Solution

Bernoulli Equations

Exercise 18.2 (mathematica/ode/techniques_nonlinear/bernoulli.nb)

Consider the Bernoulli equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)y^{\alpha}.$$

1. Solve the Bernoulli equation for $\alpha = 1$.

2. Show that for $\alpha \neq 1$ the substitution $u = y^{1-\alpha}$ reduces Bernoulli's equation to a linear equation.

3. Find the general solution to the following equations.

   (a)
$$t^2 \frac{\mathrm{d}y}{\mathrm{d}t} + 2ty - y^3 = 0, \ t > 0$$

   (b)
$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + y^2 = 0$$

Hint, Solution

Exercise 18.3
Consider a population, y. Let the birth rate of the population be proportional to y with constant of proportionality 1. Let the death rate of the population be proportional to y^2 with constant of proportionality $1/1000$. Assume that the population is large enough so that you can consider y to be continuous. What is the population as a function of time if the initial population is y_0?
Hint, Solution

Exercise 18.4
Show that the transformation $u = y^{1-n}$ reduces the equation to a linear first order equation. Solve the equations

1. $t^2 \frac{\mathrm{d}y}{\mathrm{d}t} + 2ty - y^3 = 0 \quad t > 0$

2. $\frac{\mathrm{d}y}{\mathrm{d}t} = (\Gamma \cos t + T)\, y - y^3$, Γ and T are real constants. (From a fluid flow stability problem.)

Hint, Solution

Riccati Equations

Exercise 18.5

1. Consider the Ricatti equation,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)y^2 + b(x)y + c(x).$$
Substitute
$$y = y_p(x) + \frac{1}{u(x)}$$
into the Ricatti equation, where y_p is some particular solution to obtain a first order linear differential equation for u.

2. Consider a Ricatti equation,
$$y' = 1 + x^2 - 2xy + y^2.$$

Verify that $y_p(x) = x$ is a particular solution. Make the substitution $y = y_p + 1/u$ to find the general solution.

What would happen if you continued this method, taking the general solution for y_p? Would you be able to find a more general solution?

3. The substitution

$$y = -\frac{u'}{au}$$

gives us the second order, linear, homogeneous differential equation,

$$u'' - \left(\frac{a'}{a} + b\right) u' + acu = 0.$$

The general solution for u has two constants of integration. However, the solution for y should only have one constant of integration as it satisfies a first order equation. Write y in terms of the solution for u and verify tha y has only one constant of integration.

Hint, Solution

Exchanging the Dependent and Independent Variables

Exercise 18.6
Solve the differential equation

$$y' = \frac{\sqrt{y}}{xy + y}.$$

Hint, Solution

Autonomous Equations
*Equidimensional-in-x Equations
*Equidimensional-in-y Equations
*Scale-Invariant Equations

18.9 Hints

Hint 18.1

Bernoulli Equations

Hint 18.2

Hint 18.3
The differential equation governing the population is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y - \frac{y^2}{1000}, \quad y(0) = y_0.$$

This is a Bernoulli equation.

Hint 18.4

Riccati Equations

Hint 18.5

Exchanging the Dependent and Independent Variables

Hint 18.6
Exchange the dependent and independent variables.

Autonomous Equations
*Equidimensional-in-x Equations

*Equidimensional-in-y Equations
*Scale-Invariant Equations

18.10 Solutions

Solution 18.1

We consider the Clairaut equation,

$$y = xp + p^2. \tag{18.2}$$

1. We differentiate Equation 18.2 with respect to x to obtain a second order differential equation.

$$y' = y' + xy'' + 2y'y''$$
$$y''(2y' + x) = 0$$

Equating the first or second factor to zero will lead us to two distinct solutions.

$$y'' = 0 \quad \text{or} \quad y' = -\frac{x}{2}$$

If $y'' = 0$ then $y' \equiv p$ is a constant, (say $y' = c$). From Equation 18.2 we see that the general solution is,

$$\boxed{y(x) = cx + c^2.} \tag{18.3}$$

Recall that the general solution of a first order differential equation has one constant of integration.

If $y' = -x/2$ then $y = -x^2/4 + \text{const}$. We determine the constant by substituting the expression into Equation 18.2.

$$-\frac{x^2}{4} + c = x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2$$

Thus we see that a singular solution of the Clairaut equation is

$$\boxed{y(x) = -\frac{1}{4}x^2.} \tag{18.4}$$

Recall that a singular solution of a first order nonlinear differential equation has no constant of integration.

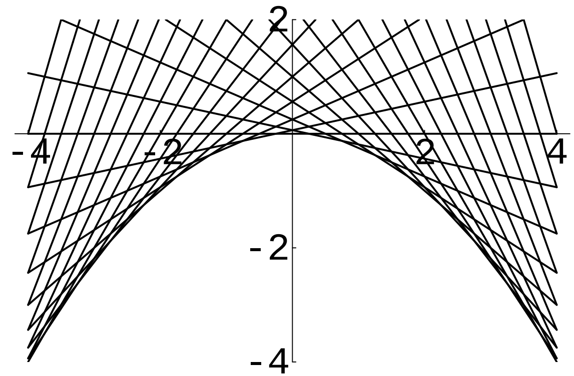

Figure 18.1: The Envelope of $y = cx + c^2$.

2. Equating the general and singular solutions, $y(x)$, and their derivatives, $y'(x)$, gives us the system of equations,

$$cx + c^2 = -\frac{1}{4}x^2, \qquad c = -\frac{1}{2}x.$$

Since the first equation is satisfied for $c = -x/2$, we see that the solution $y = cx + c^2$ is tangent to the solution $y = -x^2/4$ at the point $(-2c, -|c|)$. The solution $y = cx + c^2$ is plotted for $c = \ldots, -1/4, 0, 1/4, \ldots$ in Figure 18.1.

The envelope of a one-parameter family $F(x, y, c) = 0$ is given by the system of equations,

$$F(x, y, c) = 0, \qquad F_c(x, y, c) = 0.$$

For the family of solutions $y = cx + c^2$ these equations are

$$y = cx + c^2, \qquad 0 = x + 2c.$$

Substituting the solution of the second equation, $c = -x/2$, into the first equation gives the envelope,

$$y = \left(-\frac{1}{2}x\right)x + \left(-\frac{1}{2}x\right)^2 = -\frac{1}{4}x^2.$$

Thus we see that the singular solution is the envelope of the general solution.

Bernoulli Equations

Solution 18.2

1.

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)y$$
$$\frac{\mathrm{d}y}{y} = (q - p)\,\mathrm{d}t$$
$$\ln y = \int (q - p)\,\mathrm{d}t + c$$
$$\boxed{y = c\,\mathrm{e}^{\int (q-p)\,\mathrm{d}t}}$$

2. We consider the Bernoulli equation,

$$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t)y^{\alpha}, \quad \alpha \neq 1.$$

We divide by y^{α}.

$$y^{-\alpha}y' + p(t)y^{1-\alpha} = q(t)$$

This suggests the change of dependent variable $u = y^{1-\alpha}$, $u' = (1-\alpha)y^{-\alpha}y'$.

$$\frac{1}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}y^{1-\alpha} + p(t)y^{1-\alpha} = q(t)$$
$$\frac{\mathrm{d}u}{\mathrm{d}t} + (1-\alpha)p(t)u = (1-\alpha)q(t)$$

Thus we obtain a linear equation for u which when solved will give us an implicit solution for y.

3. (a)

$$t^2 \frac{\mathrm{d}y}{\mathrm{d}t} + 2ty - y^3 = 0, \quad t > 0$$
$$t^2 \frac{y'}{y^3} + 2t \frac{1}{y^2} = 1$$

We make the change of variables $u = y^{-2}$.

$$-\frac{1}{2} t^2 u' + 2tu = 1$$
$$u' - \frac{4}{t} u = -\frac{2}{t^2}$$

The integrating factor is

$$\mu = \mathrm{e}^{\int (-4/t) \, \mathrm{d}t} = \mathrm{e}^{-4 \ln t} = t^{-4}.$$

We multiply by the integrating factor and integrate to obtain the solution.

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(t^{-4} u \right) = -2t^{-6}$$
$$u = \frac{2}{5} t^{-1} + ct^4$$
$$y^{-2} = \frac{2}{5} t^{-1} + ct^4$$
$$y = \pm \frac{1}{\sqrt{\frac{2}{5} t^{-1} + ct^4}} \quad \boxed{y = \pm \frac{\sqrt{5t}}{\sqrt{2 + ct^5}}}$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + y^2 = 0$$
$$\frac{y'}{y^2} + \frac{2x}{y} = -1$$

We make the change of variables $u = y^{-1}$.

$$u' - 2xu = 1$$

The integrating factor is

$$\mu = \mathrm{e}^{\int (-2x)\,\mathrm{d}x} = \mathrm{e}^{-x^2}.$$

We multiply by the integrating factor and integrate to obtain the solution.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x^2} u\right) = \mathrm{e}^{-x^2}$$

$$u = \mathrm{e}^{x^2} \int \mathrm{e}^{-x^2}\,\mathrm{d}x + c\,\mathrm{e}^{x^2}$$

$$\boxed{y = \frac{\mathrm{e}^{-x^2}}{\int \mathrm{e}^{-x^2}\,\mathrm{d}x + c}}$$

Solution 18.3

The differential equation governing the population is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y - \frac{y^2}{1000}, \qquad y(0) = y_0.$$

We recognize this as a Bernoulli equation. The substitution $u(t) = 1/y(t)$ yields

$$-\frac{\mathrm{d}u}{\mathrm{d}t} = u - \frac{1}{1000}, \quad u(0) = \frac{1}{y_0}.$$

$$u' + u = \frac{1}{1000}$$

$$u = \frac{1}{y_0}\,\mathrm{e}^{-t} + \frac{\mathrm{e}^{-t}}{1000} \int_0^t \mathrm{e}^{\tau}\, d\tau$$

$$u = \frac{1}{1000} + \left(\frac{1}{y_0} - \frac{1}{1000}\right)\mathrm{e}^{-t}$$

Solving for $y(t)$,

$$\boxed{y(t) = \left(\frac{1}{1000} + \left(\frac{1}{y_0} - \frac{1}{1000}\right)\mathrm{e}^{-t}\right)^{-1}.}$$

As a check, we see that as $t \to \infty$, $y(t) \to 1000$, which is an equilibrium solution of the differential equation.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0 = y - \frac{y^2}{1000} \quad \to \quad y = 1000.$$

Solution 18.4

1.

$$t^2 \frac{\mathrm{d}y}{\mathrm{d}t} + 2ty - y^3 = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} + 2t^{-1}y = t^{-2}y^3$$

We make the change of variables $u(t) = y^{-2}(t)$.

$$u' - 4t^{-1}u = -2t^{-2}$$

This gives us a first order, linear equation. The integrating factor is

$$I(t) = \mathrm{e}^{\int -4t^{-1}\,dt} = \mathrm{e}^{-4\log t} = t^{-4}.$$

We multiply by the integrating factor and integrate.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{-4}u\right) = -2t^{-6}$$
$$t^{-4}u = \frac{2}{5}t^{-5} + c$$
$$u = \frac{2}{5}t^{-1} + ct^4$$

Finally we write the solution in terms of $y(t)$.

$$y(t) = \pm \frac{1}{\sqrt{\frac{2}{5} t^{-1} + c t^4}}$$

$$\boxed{y(t) = \pm \frac{\sqrt{5t}}{\sqrt{2 + c t^5}}}$$

2.

$$\frac{\mathrm{d}y}{\mathrm{d}t} - (\Gamma \cos t + T)\, y = -y^3$$

We make the change of variables $u(t) = y^{-2}(t)$.

$$u' + 2\,(\Gamma \cos t + T)\, u = 2$$

This gives us a first order, linear equation. The integrating factor is

$$I(t) = \mathrm{e}^{\int 2(\Gamma \cos t + T)\, dt} = \mathrm{e}^{2(\Gamma \sin t + Tt)}$$

We multiply by the integrating factor and integrate.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2(\Gamma \sin t + Tt)}\, u\right) = 2\, \mathrm{e}^{2(\Gamma \sin t + Tt)}$$

$$u = 2\, \mathrm{e}^{-2(\Gamma \sin t + Tt)} \left(\int \mathrm{e}^{2(\Gamma \sin t + Tt)}\, dt + c \right)$$

Finally we write the solution in terms of $y(t)$.

$$\boxed{y = \pm \frac{\mathrm{e}^{\Gamma \sin t + Tt}}{\sqrt{2\left(\int \mathrm{e}^{2(\Gamma \sin t + Tt)}\, dt + c\right)}}}$$

Riccati Equations

Solution 18.5

We consider the Ricatti equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)y^2 + b(x)y + c(x). \tag{18.5}$$

1. We substitute

   $$y = y_p(x) + \frac{1}{u(x)}$$

   into the Ricatti equation, where y_p is some particular solution.

   $$y_p' - \frac{u'}{u^2} = +a(x)\left(y_p^2 + 2\frac{y_p}{u} + \frac{1}{u^2}\right) + b(x)\left(y_p + \frac{1}{u}\right) + c(x)$$
   $$-\frac{u'}{u^2} = b(x)\frac{1}{u} + a(x)\left(2\frac{y_p}{u} + \frac{1}{u^2}\right)$$
   $$\boxed{u' = -\left(b + 2ay_p\right)u - a}$$

   We obtain a first order linear differential equation for u whose solution will contain one constant of integration.

2. We consider a Ricatti equation,

   $$y' = 1 + x^2 - 2xy + y^2. \tag{18.6}$$

   We verify that $y_p(x) = x$ is a solution.

   $$1 = 1 + x^2 - 2xx + x^2$$

   Substituting $y = y_p + 1/u$ into Equation 18.6 yields,

   $$u' = -\left(-2x + 2x\right)u - 1$$
   $$u = -x + c$$
   $$\boxed{y = x + \frac{1}{c - x}}$$

What would happen if we continued this method? Since $y = x + \frac{1}{c-x}$ is a solution of the Ricatti equation we can make the substitution,

$$y = x + \frac{1}{c-x} + \frac{1}{u(x)}, \tag{18.7}$$

which will lead to a solution for y which has two constants of integration. Then we could repeat the process, substituting the sum of that solution and $1/u(x)$ into the Ricatti equation to find a solution with three constants of integration. We know that the general solution of a first order, ordinary differential equation has only one constant of integration. Does this method for Ricatti equations violate this theorem? There's only one way to find out. We substitute Equation 18.7 into the Ricatti equation.

$$u' = -\left(-2x + 2\left(x + \frac{1}{c-x}\right)\right)u - 1$$
$$u' = -\frac{2}{c-x}u - 1$$
$$u' + \frac{2}{c-x}u = -1$$

The integrating factor is

$$I(x) = \mathrm{e}^{2/(c-x)} = \mathrm{e}^{-2\log(c-x)} = \frac{1}{(c-x)^2}.$$

Upon multiplying by the integrating factor, the equation becomes exact.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(c-x)^2}u\right) = -\frac{1}{(c-x)^2}$$
$$u = (c-x)^2\frac{-1}{c-x} + b(c-x)^2$$
$$u = x - c + b(c-x)^2$$

Thus the Ricatti equation has the solution,

$$y = x + \frac{1}{c-x} + \frac{1}{x - c + b(c-x)^2}.$$

It appears that we we have found a solution that has two constants of integration, but appearances can be deceptive. We do a little algebraic simplification of the solution.

$$y = x + \frac{1}{c - x} + \frac{1}{(b(c - x) - 1)(c - x)}$$
$$y = x + \frac{(b(c - x) - 1) + 1}{(b(c - x) - 1)(c - x)}$$
$$y = x + \frac{b}{b(c - x) - 1}$$
$$y = x + \frac{1}{(c - 1/b) - x}$$

This is actually a solution, (namely the solution we had before), with one constant of integration, (namely $c - 1/b$). Thus we see that repeated applications of the procedure will not produce more general solutions.

3. The substitution

$$y = -\frac{u'}{au}$$

gives us the second order, linear, homogeneous differential equation,

$$u'' - \left(\frac{a'}{a} + b\right) u' + acu = 0.$$

The solution to this linear equation is a linear combination of two homogeneous solutions, u_1 and u_2.

$$u = c_1 u_1(x) + c_2 u_2(x)$$

The solution of the Ricatti equation is then

$$y = -\frac{c_1 u_1'(x) + c_2 u_2'(x)}{a(x)(c_1 u_1(x) + c_2 u_2(x))}.$$

Since we can divide the numerator and denominator by either c_1 or c_2, this answer has only one constant of integration, (namely c_1/c_2 or c_2/c_1).

Exchanging the Dependent and Independent Variables

Solution 18.6
Exchanging the dependent and independent variables in the differential equation,

$$y' = \frac{\sqrt{y}}{xy + y},$$

yields

$$x'(y) = y^{1/2}x + y^{1/2}.$$

This is a first order differential equation for $x(y)$.

$$x' - y^{1/2} x = y^{1/2}$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[x \exp\left(-\frac{2y^{3/2}}{3}\right)\right] = y^{1/2} \exp\left(-\frac{2y^{3/2}}{3}\right)$$

$$x \exp\left(-\frac{2y^{3/2}}{3}\right) = -\exp\left(-\frac{2y^{3/2}}{3}\right) + c_1$$

$$x = -1 + c_1 \exp\left(\frac{2y^{3/2}}{3}\right)$$

$$\frac{x+1}{c_1} = \exp\left(\frac{2y^{3/2}}{3}\right)$$

$$\log\left(\frac{x+1}{c_1}\right) = \frac{2}{3} y^{3/2}$$

$$y = \left(\frac{3}{2}\log\left(\frac{x+1}{c_1}\right)\right)^{2/3}$$

$$\boxed{y = \left(c + \frac{3}{2}\log(x+1)\right)^{2/3}}$$

Autonomous Equations
*Equidimensional-in-x Equations
*Equidimensional-in-y Equations
*Scale-Invariant Equations