2. INFINITE SERIES

2.1. A PRE-REQUISITE: SEQUENCES

We concluded the last section by asking what we would get if we considered the "Taylor polynomial of degree ∞ for the function e^x centered at 0",

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

As we said at the time, we have a lot of groundwork to consider first, such as the fundamental question of what it even means to add an infinite list of numbers together. As we will see in the next section, this is a delicate question. In order to put our explorations on solid ground, we begin by studying sequences.

A *sequence* is just an ordered list of objects. Our sequences are (almost) always lists of real numbers, so another definition for us would be that a sequence is a real-valued function whose domain is the positive integers. The sequence whose nth term is a_n is denoted $\{a_n\}$, or if there might be confusion otherwise, $\{a_n\}_{n=1}^{\infty}$, which indicates that the sequence starts when $n = 1$ and continues forever.

Sequences are specified in several different ways. Perhaps the simplest way is to specify the first few terms, for example

$$\{a_n\} = \{2, 4, 6, 8, 10, 12, 14, \dots\},$$

is a perfectly clear definition of the sequence of positive even integers. This method is slightly less clear when

$$\{a_n\} = \{2, 3, 5, 7, 11, 13, 17, \dots\},$$

although with a bit of imagination, one can deduce that a_n is the nth prime number (for technical reasons, 1 is *not* considered to be a prime number). Of course, this method completely breaks down when the sequence has no discernible pattern, such as

$$\{a_n\} = \{0, 4, 3, 2, 11, 29, 54, 59, 35, 41, 46, \dots\}.$$

To repeat: the above is *not* a good definition of a series. Indeed, this sequence is the number of home runs that Babe Ruth hit from 1914 onward; if the Red Sox had been able to predict the pattern, would they have sold his contract to the Yankees after the 1919 season?

Another method of specifying a sequence is by giving a formula for the *general (nth) term*. For example,

$$\{a_n\} = \{2n\}$$

is another definition of the positive even integers, while $\{a_n\} = \{n^2\}$ defines the sequence $\{a_n\} = \{1, 4, 9, 16, 25, \dots\}$ of squares.

Example 1. Guess a formula for the general term of the sequence

$$\left\{\frac{3}{1}, -\frac{5}{4}, \frac{7}{9}, -\frac{9}{16}, \frac{11}{25}, \dots\right\}.$$

Solution. It is good to tackle this problem one piece at a time. First, notice that the sequence alternates in sign. Since the sequence begins with a positive term, this shows that we should have a factor of $(-1)^{n-1}$ in the formula for a_n. Next, the numerators of these fractions list all the odd numbers starting with 3, so we guess $2n + 1$ for the numerators. The denominators seem to be the squares, so we guess n^2 for these. Putting this together we have $a_n = (-1)^{n-1}(2n + 1)/n^2$. ●

A sequence grows *geometrically* (or *exponentially*) if each term is obtained by multiplying the previous term by a fixed *common ratio*, typically denoted by r. Therefore, letting a denote the first term of a geometric sequence, the sequence can be defined as $\{ar^n\}_{n=0}^{\infty}$; note here that the subscripted $n = 0$ indicates that the sequence starts with $n = 0$. Geometric sequences are also called "geometric progressions".

There is a legendary (but probably fictitious) myth involving a geometric sequence. It is said that when the inventor of chess (an ancient Indian mathematician, in most accounts) showed his invention to his ruler, the ruler was so pleased that he gave the inventor the right to name his prize. The inventor asked for 1 grain of wheat for the first square of the board, 2 grains for the second square, 4 grains for the third square, 8 grains for the fourth square, and so on. The ruler, although initially offended that the inventor asked for so little, agreed to the offer. Days later, the ruler asked his treasurer why it was taking so long to pay the inventor, and the treasurer pointed out that to pay for the 64th square alone, it would take

$$2^{64} = 18,446,744,073,709,551,616$$

grains of wheat. To put this in perspective, a single grain of wheat contains about $1/5$ of a calorie, so 2^{64} grains of wheat contain approximately $3,719,465,121,917,178,880$ calories. Assuming a 2000 calorie per day diet, the amount of wheat *just for the 64th square of the chessboard* would feed 6 billion people for almost 850 years[†] (although they should probably supplement their diet with Vitamin C to prevent scurvy). The legend concludes with

[†] *Wolfram Alpha* (`http://www.wolframalpha.com/`) is an incredible resource for performing calculations such as this; simply type "2^64 grains of wheat" into its search box and wonder at the results.

the ruler insisting that the inventor participate in the grain counting, in order to make sure that it is "accurate," an offer which the inventor presumably refused.

While sequences are interesting in their own right, we are mostly interested in applying tools for sequences to our study of infinite sums. Therefore, the two most important questions about sequences for us are:

Does the sequence converge or diverge?
If the sequence converges, what does it converge to?

Intuitively, the notion of convergence is often quite clear. For example, the sequence

$$\{0.3, 0.33, 0.333, 0.3333, \dots\}$$

converges to $1/3$, while the sequence

$$\{1/n\} = \{1/1, 1/2, 1/3, 1/4, \dots,\}$$

converges to 0. Slightly more formally, the sequence $\{a_n\}$ converges to the number L if by taking n large enough, we can make the terms of the sequence as close to L as we like. By being a bit more specific in this description, we arrive at the formal definition of convergence below.

Converges to L. The sequence $\{a_n\}$ is said to *converge to L* if for every number $\epsilon > 0$, there is some number N so that $|a_n - L| < \epsilon$ for all $n \geqslant N$.

To indicate that the sequence $\{a_n\}$ converges to L, then we may write

$$\lim_{n\to\infty} a_n = L,$$

or simply

$$a_n \to L \text{ as } n \to \infty.$$

Before moving on, we check one of our previous observations using the formal definition of convergence.

Example 2. Show, using the definition, that the sequence $\{1/n\}$ converges to 0.

Solution. Let $\epsilon > 0$ be any positive number. We want to show that

$$|a_n - 0| = 1/n < \epsilon$$

for sufficiently large n. Solving the above inequality for n, we see that $|a_n - 0| < \epsilon$ for all $n > 1/\epsilon$, proving that $1/n \to 0$ as $n \to \infty$. ●

In practice, we rarely use the formal definition of convergence for examples such as this. After all, the numerator of $1/n$ is constant and the denominator increases without bound, so it is clear that the limit is 0. Many types of limits can be computed with this reasoning and a few techniques, as we show in the next two examples.

Example 3. Compute $\lim_{n\to\infty} \frac{7n+3}{5n+\sqrt{n}}$.

Solution. A common technique with limits is to divide both the numerator and denominator by the fastest growing function of n involved in the expression. In this case the fastest growing function of n involved is n (choosing $7n$ would work just as well), so we divide by n:

$$\lim_{n\to\infty} \frac{7n+3}{5n+\sqrt{n}} = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{7+3/n}{5+\sqrt{n}/n} = \lim_{n\to\infty} \frac{7+3/n}{5+1/\sqrt{n}}.$$

As $n \to \infty$, $3/n \to 0$ and $1/\sqrt{n} \to 0$, so the limit of this sequence is $7/5$. ●

Example 4. Determine the limit as $n \to \infty$ of the sequence $\{\sqrt{n+2} - \sqrt{n}\}$.

Solution. Here we use another frequent technique: when dealing with square roots, it is often helpful to multiply by the "conjugate":

$$\sqrt{n+2} - \sqrt{n} = \left(\sqrt{n+2} - \sqrt{n}\right)\left(\frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}}\right) = \frac{2}{\sqrt{n+2}+\sqrt{n}}.$$

Now we can analyze this fraction instead. The numerator is constant and the denominator grows without bound, so the sequence converges to 0. ●

You should convince yourself that a given sequence can converge to at most one number. If the sequence $\{a_n\}$ does not converge to any number, then we say that it *diverges*. There are two different types of divergence, and it is important to distinguish them. First we have the type of divergence exhibited by the sequence $\{n!\}$:

> **Diverges to Infinity.** The sequence $\{a_n\}$ is said to *diverge to infinity* if for every number $\ell > 0$, there is some number N so that $a_n > \ell$ for all $n \geqslant N$.

If the sequence $\{a_n\}$ diverges to infinity, then we write $\lim_{n\to\infty} a_n = \infty$. Note that the sequence $\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}$ demonstrates a different type of divergence, which is sometimes referred to as "oscillatory divergence." Both types of divergence show up in our next example.

Example 5. Showing, using only the definition of convergence, that:

(a) $\lim_{n\to\infty} r^n = \infty$ if $r > 1$,

(b) $\lim_{n\to\infty} r^n = 1$ if $r = 1$,

(b) $\lim_{n\to\infty} r^n = 0$ if $-1 < r < 1$ (i.e., if $|r| < 1$),

(c) $\lim_{n\to\infty} r^n$ does not exist if $r \leqslant -1$.

Solution. Beginning with (a), assume that $r > 1$ and fix a number $M > 1$. By taking logarithms, we see that $r^n > M$ if and only if $\ln r^n > \ln M$, or equivalently, if and only if

$$n \ln r > \ln M.$$

Since both M and r are greater than 1, $\ln r, \ln M > 0$. This shows that $r^n > \ln M$ for all $n > \ln M/\ln r$, proving that $\ln r^n = \infty$ when $r > 1$.

Part (b) is obvious, as $1^n = 1$ for all n.

For part (c), let us assume that $|r| < 1$. We want to prove that $\lim_{n\to\infty} r^n = 0$, which means (according to our definition above) that for every $\epsilon > 0$, there is some number N so that

$$|r^n - 0| = |r^n| = |r|^n < \epsilon$$

for all $n \geqslant N$. Again taking logarithms, this holds if $n \ln |r| < \ln \epsilon$. Since $|r| < 1$, its logarithm is negative, so when we divide both sides of this inequality by $\ln |r|$ we flip the inequality. This $|r|^n < \epsilon$ for all $n > \ln \epsilon/\ln |r|$, proving that $\lim_{n\to\infty} r^n = 0$ when $|r| < 1$.

This leaves us with (d). We have already observed that the sequence $\{(-1)^n\}$ diverges, so suppose that $r < -1$. In this case the sequence $\{r^n\}$ can be viewed as a "shuffle" of two sequences, the negative sequence $\{r, r^3, r^5, \dots\}$ and the positive sequence $\{r^2, r^4, r^6, \dots\}$. By part (a), $r^{2n} = (r^2)^n \to \infty$ as $n \to \infty$, while $r^{2n+1} = r \cdot r^{2n} \to -\infty$ as $n \to \infty$, so in this case the sequence $\{r^n\}$ does not have a limit. ●

Another way to specify a sequence is with initial conditions and a recurrence. For example, the factorials† can be specified by the *recurrence*

$$a_n = n \cdot a_{n-1} \text{ for } n \geqslant 2,$$

and the *initial condition* $a_1 = 1$.

†If the inventor of chess had been extremely greedy, he would have asked for 1! grains of wheat for the first square, 2! grains of wheat for the second square, 3! grains of wheat for the third square, and so on, because

$$\begin{aligned} 64! \quad = \quad & 126,886,932,185,884,164,103,433,389,335,161,480,802,865,516, \\ & 174,545,192,198,801,894,375,214,704,230,400,000,000,000,000,000, \end{aligned}$$

almost 127 octovigintillion, and about a billion times the estimated number of atoms in the universe.

A more complicated recurrence relation is provided by the Fibonacci numbers $\{f_n\}_{n=0}^{\infty}$, defined by

$$f_n = f_{n-1} + f_{n-2} \text{ for } n \geqslant 2,$$

and the initial conditions $f_0 = f_1 = 1$. This sequence begins

$$\{f_n\}_{n=0}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots\}.$$

We now make an important definition. In this definition, note that we twist the notion of "increasing" a bit; what we call increasing sequences should really be called "nondecreasing sequences", but this awkward term is rarely used.

> **Monotone Sequences.** The sequence $\{a_n\}$ is *(weakly) increasing* if $a_n \leqslant a_{n+1}$ for all n, and *(weakly) decreasing* if $a_n \geqslant a_{n+1}$ for all n. A sequence is *monotone* if it is either increasing or decreasing.

It is good to practice identifying monotone sequences. Here are a few examples to practice on:

- $\{1/n\}$ is decreasing,
- $\{1 - 1/n\}$ is increasing,
- $\left\{\dfrac{n}{n^2+1}\right\}$ is decreasing.

You should also verify that the geometric sequence $\{ar^n\}$ is decreasing for $0 < r < 1$, increasing for $r > 1$, and not monotone if r is negative.

It is frequently helpful to know that a given sequence converges, even if we do not know its limit. We have a powerful tool to establish this for monotone sequences. First, we need another definition.

> **Bounded Sequences.** The sequence $\{a_n\}$ is *bounded* if there are numbers ℓ and u such that $\ell \leqslant a_n \leqslant u$ for all n.

For monotone sequences, boundedness implies convergence:

> **The Monotone Convergence Theorem.** Every bounded monotone sequence converges.

We delay the proof of this theorem to Exercise 92. Knowing that a limit exists can sometimes be enough to solve for its true value, as our next example demonstrates.

Example 6. Prove that the sequence defined recursively by $a_1 = 2$ and

$$a_{n+1} = \frac{2a_n}{1+a_n}$$

for $n \geqslant 1$ is decreasing, and use the Monotone Convergence Theorem to compute its limit.

Solution. First we ask: for which values of n is $a_n \geqslant a_{n+1}$? Substituting the definition of a_{n+1}, this inequality becomes

$$a_n \geqslant \frac{2a_n}{1+a_n},$$

which simplifies to $a_n^2 \geqslant a_n$. So we have our answer: $a_n \geqslant a_{n+1}$ whenever $a_n \geqslant 1$. We are given that $a_1 = 2$, so $a_1 \geqslant a_2$. For the other values of n, notice that if $a_n \geqslant 1$, then

$$a_{n+1} = \frac{2a_n}{1+a_n} = \frac{a_n + a_n}{1+a_n} \geqslant \frac{1+a_n}{1+a_n},$$

so since $a_1 \geqslant 1$, we see that $a_2 \geqslant 1$, which implies that $a_3 \geqslant 1$, and so on. In the end, we may conclude that $a_n \geqslant 1$ for *all* values of n. By our work above, this shows that $a_n \geqslant a_{n+1}$ for all values of n, so the sequence is decreasing.

When dealing with monotone sequences, a common technique is to first prove that the sequence has a limit, and then use this fact to find the limit. In order to prove that $\{a_n\}$ has a limit, we need to show that it is bounded. From our previous work, we know that $a_n \geqslant 1$ for all n, and an upper bound is also easy:

$$a_{n+1} = \frac{2a_n}{1+a_n} \leqslant \frac{2a_n}{a_n} = 2.$$

Therefore, $1 \leqslant a_n \leqslant 2$ for all n, so the sequence $\{a_n\}$ has a limit.

Let L denote this limit. Then we have

$$a_{n+1} = \frac{2a_n}{1+a_n} \to L$$

as $n \to \infty$. But as $n \to \infty$, $a_n \to \infty$ as well, so we must have

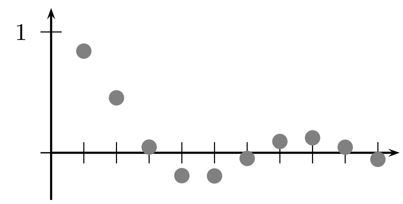
$$\frac{2L}{1+L} = L,$$

which simplifies to $L^2 = L$. There are two possible solutions, $L = 0$ and $L = 1$, but we can rule out $L = 0$ because it lies outside of our bounds, so it must be the case that this sequence converges to 1. ●

For the rest of the section, we study some other methods for computing limits. One technique to find the limit of the sequence $\{a_n\}$ is to "sandwich" it between a lower bound $\{\ell_n\}$ and an upper bound $\{u_n\}$.

> **The Sandwich Theorem.** Suppose the sequences $\{a_n\}$, $\{\ell_n\}$, and $\{u_n\}$ satisfy $\ell_n \leqslant a_n \leqslant u_n$ for all large n and $\ell_n \to L$ and $u_n \to L$ then $a_n \to L$ as well.

The Sandwich Theorem[†] hopefully seems intuitively obvious. We ask the reader to give a formal proof in Exercise 94. Examples 7 and 8 illustrate its use.

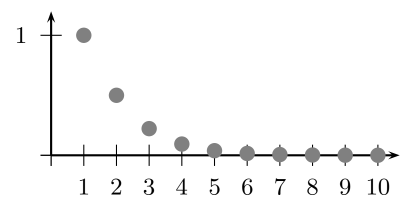


Example 7. Compute $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n}$.

Solution. For all n we have

$$-\frac{1}{n} \leqslant \frac{\sin n}{n} \leqslant \frac{1}{n},$$

so since $-1/n \to 0$ and $1/n \to 0$, $\dfrac{\sin n}{n} \to 0$ by the Sandwich Theorem. ●



Example 8. Show that $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Solution. Evaluating this limit merely requires finding the right bound:

$$\frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \dots \cdot \frac{2}{n} \cdot \frac{1}{n}, \leqslant \frac{1}{n}$$

so $n!/n^n$ is sandwiched between 0 and $1/n$. Since $1/n \to 0$ and 0 *is* 0, we see by the Sandwich Theorem that $n!/n^n \to 0$. ●

The following useful result allows us to switch from limits of sequences to limits of functions.

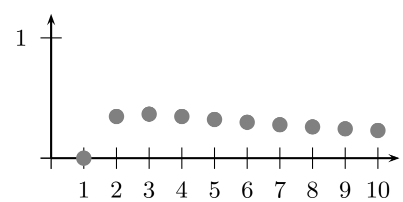
> **Theorem.** If $\{a_n\}$ is a function satisfying $a_n = f(n)$ and $\displaystyle\lim_{x\to\infty} f(x)$ exists, then $\displaystyle\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(n)$.

Proof. If $\displaystyle\lim_{n\to\infty} f(x) = L$ then for each $\epsilon > 0$ there is a number N such that $|f(x) - L| < \epsilon$ whenever $x > N$. Of course, this means that $|a_n - L| < \epsilon$ whenever $n > N$, proving the theorem. □

[†] In 1942, Arther Stone and John Tukey proved a theorem called the Ham Sandwich Theorem, which states that given any sandwich composed of bread, ham, and cheese, there is some plane (i.e., straight cut) that slices the sandwich into two pieces, each containing the same amount of bread, the same amount of ham, and the same amount of cheese.

Why might we want to switch from sequences to functions? Generally we do so in order to invoke l'Hôpital's Rule†:

> **l'Hôpital's Rule.** Suppose that c is a real number or $c = \pm\infty$, and that f and g are differentiable and $g'(x) \neq 0$ near c (except possibly at $x = c$). If either
>
> $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$$
>
> or
>
> $$\lim_{x\to c} f(x) = \pm \lim_{x\to c} g(x) = \pm\infty$$
>
> then
>
> $$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$
>
> provided that this limit exists.



Example 9. Compute $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n}$.

Solution. We know that

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x},$$

and we can use l'Hôpital's Rule to evaluate this second limit since both $\ln x$ and x tend to ∞ as $x \to \infty$:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

This shows that $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$. ●

Another very useful result allows us to "move" limits inside continuous functions:

> **The Continuous Function Limits Theorem.** Suppose the sequence $\{a_n\}$ converges to L and that f is continuous at L and defined for all values a_n. Then the sequence $\{f(a_n)\}$ converges to $f(L)$.

† l'Hôpital's Rule is one of the many misnomers in mathematics. It is named after Guillaume de l'Hôpital (1661–1704) because it appeared in a calculus book he authored (the first calculus book ever written, in fact), but l'Hôpital's Rule was actually discovered by his mathematics tutor, Johann Bernoulli (1667–1748). The two had a contract entitling l'Hôpital to use Bernoulli's discoveries however he wished.

We conclude the section with two examples using this theorem.

Example 10. Compute $\lim_{n\to\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right)$.

Solution. Since $1/n \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\pi}{2} - \frac{1}{n} = \frac{\pi}{2}.$$

Therefore,

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{2} - \frac{1}{n}\right) = \sin\left(\frac{\pi}{2}\right) = 1,$$

by the Continuous Function Limits Theorem. ●

Our final example is less straight-forward, but more important.

Example 11. Compute $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

Solution. As with most problems which have a variable in the exponent, it is a good idea to rewrite the limit using e and $\ln$:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} e^{n\ln\left(1+\frac{1}{n}\right)}$$

Consider only the exponent for now, $n\ln\left(1 + \frac{1}{n}\right)$. This is an $\infty \cdot 0$ indeterminate form, so we rewrite it to give a $\frac{0}{0}$ form:

$$n\ln\left(1 + \frac{1}{n}\right) = \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}}.$$

Applying l'Hôpital's Rule, we see that

$$\lim_{n\to\infty} \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{1}{n^2}}{-\frac{1}{n^2}} = 1.$$

Thus the exponents converge to 1. Because e^x is a continuous function, we can now apply the Continuous Function Limits Theorem to see that

$$\lim_{n\to\infty} e^{n\ln\left(1 + \frac{1}{n}\right)} = e^{\lim_{n\to\infty} n\ln\left(1 + \frac{1}{n}\right)} = e^1 = e,$$

so the solution to the example is e. ●

A wonderful resource for integer sequences is the *On-Line Encyclopedia of Integer Sequences*, maintained by Neil Sloane (1939–), available at

`http://www.research.att.com/~njas/sequences/`

The encyclopedia currently contains more than 165,000 sequences. Each of these sequences is numbered, and so the sequence also contains a sequence $\{a_n\}$ (number 91,967) in which a_n is the nth term of the nth sequence. What is a_{91967}?

EXERCISES FOR SECTION 2.1

Write down a formula for the general term, a_n, of each of the sequences in Exercises 1–8.

1. $3, 5, 7, 9, 11, \ldots$

2. $4, 16, 64, 256, 1024, \ldots$

3. $2, -4, 8, -16, 32, \ldots$

4. $5/2, 7/5, 9/10, 11/17, 13/26, \ldots$

5. $3, 1, 3, 1, 3, \ldots$

6. $-4/3, 7/9, -10/27, 13/81, -16/243, \ldots$

7. $1/2, 2/4, 6/8, 24/16, 120/32, \ldots$

8. $2, 4 \cdot 2, 6 \cdot 4 \cdot 2, 8 \cdot 6 \cdot 4 \cdot 2, 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2, \ldots$ (Try to find something simpler than $2n \cdot (2n-2) \cdots 4 \cdot 2$).

Exercises 9–12 give initial terms and recurrence relations for sequences. Use these to compute the first 5 terms and try to write a formula for the general term, a_n.

9. $a_1 = 2$, $a_n = a_{n-1} + 3$

10. $a_1 = 2$, $a_n = na_{n-1}$

11. $a_1 = 1$, $a_n = a_{n-1} + n$

12. $a_1 = 1$, $a_n = n^3 a_{n-1}$

13. Suppose that $\{a_n\}$ is a geometric sequence. If $a_2 = 6$ and $a_5 = 162$, what are the possibilities for a_1?

14. Suppose that $\{a_n\}$ is a geometric sequence. If $a_2 = 2$ and $a_4 = 6$, what are the possibilities for a_1?

Determine if the sequences in Exercises 15–22 converge or diverge. If they converge, find their limits.

15. $a_n = \dfrac{7n+5}{4n+2}$

16. $a_n = \dfrac{\sqrt{7n+5}}{\sqrt{4n+2}}$

17. $a_n = \dfrac{7n^3+5}{4n^3+2}$

18. $a_n = \dfrac{7\sqrt[4]{n}+5}{4\sqrt[4]{n}+2}$

19. $a_n = \sqrt{2n+2} - \sqrt{2n}$

20. $a_n = \sqrt{2n+2} + \sqrt{2n}$

21. $a_n = 2\arctan(3n^2)$

22. $a_n = 2\arctan(\sqrt{n}+10)$

Compute the limits in Exercises 23–30.

23. $\displaystyle\lim_{n\to\infty} \sqrt{\frac{7n+5}{4n+2}}$

24. $\displaystyle\lim_{n\to\infty} \frac{(n+2)!}{n!(3+5n)^2}$

25. $\displaystyle\lim_{n\to\infty} \frac{5n}{\ln(2+3e^n)}$

26. $\displaystyle\lim_{n\to\infty} \frac{(n+2)!}{n^2 n!}$

27. $\displaystyle\lim_{n\to\infty} \frac{\ln n}{\ln 3n}$

28. $\displaystyle\lim_{n\to\infty} e^{1/n}$

29. $\lim\limits_{n\to\infty} \sqrt[n]{2n}$

30. $\lim\limits_{n\to\infty} (7n+3)^{5/n}$

Prove the limit laws states in Exercises 31–34.

31. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences,

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

32. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences,

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n.$$

33. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences,

$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$

34. If $\{a_n\}$ is a convergent sequence, then

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$

for any number c.

Compute the limits in Exercises 35–42. All of these limits are either 0 or infinity, explanation is required, but you need not apply l'Hôpital's Rule.

35. $\lim\limits_{n\to\infty} \dfrac{n^n}{n!}$

36. $\lim\limits_{n\to\infty} \dfrac{n!}{4^n}$

37. $\lim\limits_{n\to\infty} \dfrac{n^{1/4}}{(\ln n)^4}$

38. $\lim\limits_{n\to\infty} \dfrac{(2n+1)^{10}}{e^n}$

39. $\lim\limits_{n\to\infty} \dfrac{n}{(n+3)\ln n}$

40. $\lim\limits_{n\to\infty} \dfrac{n^3+2n}{\sqrt{n^7+2n^6}}$

41. $\lim\limits_{n\to\infty} \dfrac{n-2}{(\ln n)^{10}}$

42. $\lim\limits_{n\to\infty} \left(\dfrac{3n^2-n}{4n^2+1}\right)^n$

43. Arrange the functions

$$n, \quad n^n, \quad \ln n, \quad 3^n, \quad n\ln n, \quad 2^{n^2}, \quad \sqrt{n^6+1}$$

in increasing order, so that (for large n) each function is very much larger than the one that it follows.

44. Where does $n!$ fit in the list you made for Exercise 43?

45. Define the sequence $\{a_n\}$ recursively by

$$a_{n+1} = \frac{2a_n}{1+a_n}.$$

Show that if $a_1 = 1/2$ then $\{a_n\}$ is increasing. (Compare this to the result of Example 6).

46. Define the sequence $\{a_n\}$ recursively by

$$a_{n+1} = \frac{3+3a_n}{3+a_n}.$$

Show that if $a_1 = 1$ then $\{a_n\}$ is increasing, while if $a_1 = 2$ then $\{a_n\}$ is decreasing.

In Exercises 47–50, compute the integral to give a simplified formula for a_n and then determine $\lim\limits_{n\to\infty} a_n$.

47. $a_n = \displaystyle\int_1^n \frac{1}{x}\,dx$

48. $a_n = \displaystyle\int_1^n \frac{1}{x^2}\,dx$

49. $a_n = \displaystyle\int_1^n \frac{1}{x\ln x}\,dx$

50. $a_n = \displaystyle\int_1^n \frac{1}{x(\ln x)^2}\,dx$

51. What is $\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n}\right)^n$? More generally, what is $\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n$?

52. Let $\{s_n\}$ denote the sequence given by $1/2, 1/2 + 1/4, 1/2 + 1/4 + 1/8, 1/2 + 1/4 + 1/8 + 1/16, \ldots$. Conjecture a formula for s_n. What does this mean for $\lim\limits_{n\to\infty} s_n$?

Recall that the function $f(x)$ is *periodic* with period p if $f(x+p) = f(x)$ for all values of x. Similarly, the sequence $\{a_n\}_{n=1}^{\infty}$ is periodic with period p if $a_{n+p} = a_n$ for all $n \geqslant 1$ and p is the least number with this property. Determine the periods of the sequences in Exercises 53–56.

53. $\{\sin n\pi\}_{n=1}^{\infty}$

54. $\left\{\cos \dfrac{n\pi}{m}\right\}_{n=1}^{\infty}$

55. $\left\{\cos \dfrac{n\pi}{m} \sin \dfrac{n\pi}{m}\right\}_{n=1}^{\infty}$

56. $\left\{\sin^2 \dfrac{n\pi}{m}\right\}_{n=1}^{m}$

Compute the sequence $f(0), f'(0), f''(0), f^{(3)}(0), \ldots$ of derivatives for the functions $f(x)$ listed in Exercises 57–63.

57. $f(x) = e^x$

58. $f(x) = \sin x$

59. $f(x) = \cos x$

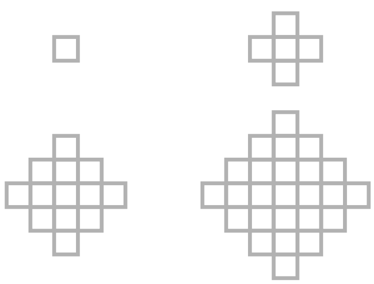
60. $f(x) = \sin 2x$

61. $f(x) = xe^x$

62. $f(x) = (1+x)^3$

63. $f(x) = \sqrt{1+x}$

64. Consider the sequence of figures below.

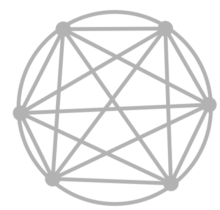


Let a_n denote the number of non-overlapping small squares in these figures, so $a_1 = 1$, $a_2 = 5$, $a_3 = 13$, and $a_4 = 25$. Write a formula for a_n. *Hint:* it may be helpful to consider the squares that are missing from the figures.

Exercises 65 and 66 concern the following sequence. Choose n points on a circle, and join each point to all the others. This divides the circle into a number, say a_n, of regions. For example, $a_1 = 1$ (by definition), $a_2 = 2$, and $a_3 = 4$:

65. Compute a_4 and a_5. Based on the first 5 terms of the sequence, conjecture a formula for the general term a_n.

66. Compute a_6. Does this match your conjecture?



67. (Due to Solomon Golomb (1932–)) There is a unique sequence $\{a_n\}$ of positive integers which is nondecreasing and contains exactly a_n occurrences of the number n for each n. Compute the first 10 terms of this sequence.†

68. Define the sequence $\{a_n\}$ recursively by

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even,} \\ 3a_n + 1 & \text{if } a_n \text{ is odd.} \end{cases}$$

Compute this sequence when $a_1 = 13$ and $a_1 = 24$. The $3n+1$ Conjecture, first posed by Lothar Collatz (1910–1990) in 1937 but still unproved, states that if you start with any positive integer as a_1 then this sequence will eventually reach 1, where it will end in the infinite periodic sequence $1, 4, 2, 1, 4, 2, \ldots$. About this conjecture, the prolific Hungarian mathematician Paul Erdős (1913–1996) said "mathematics is not yet ready for such problems."

69. Compute the first 10 terms of the sequence $a_n = n^2 + n + 41$. What do these numbers have in common?

◆ **70.** Do all terms in the sequence of Exercise 69 share this property?

†Amazingly, a_n is the nearest integer to $\varphi^{2-\varphi} n^{\varphi-1}$ where $\varphi = \dfrac{1+\sqrt{5}}{2}$ denotes the golden ratio.

◆ **71.** It has been known since Euclid (see also Exercises 32–34 in Section 2.3) that there are infinitely many primes‡, but how far apart can they be? Prove that for any positive integer n, the sequence

$$n! + 2, n! + 3, \ldots, n! + n$$

contains *no* prime numbers.

72. Let a_n denote the sum of the integers 1 up to n, so $a_4 = 1 + 2 + 3 + 4 = 10$. Compute the first 6 terms of a_n. Can you give a formula for a_n?

73. Say that a number is *polite* if it can be written as the sum of two or more consecutive positive integers. For example, 14 is polite because $14 = 2 + 3 + 4 + 5$. Let a_n denote the nth polite number. Compute the first 6 terms of a_n. Do you spot a pattern?

74. A *composition* of the integer n is a way of writing n as a sum of positive integers, in which the order of the integers *does* matter. For example, there are eight partitions of 4: 4, $3 + 1$, $1 + 3$, $2 + 2$, $2 + 1 + 1$, $1 + 2 + 1$, $1 + 1 + 2$, and $1 + 1 + 1 + 1$. Let a_n denote the number of compositions of n. Compute the first 6 terms of a_n. Can you conjecture a formula?

75. Let a_n denote the number of compositions of n into 1s and 2s. Relate $\{a_n\}$ to a sequence from this section.

76. For $n \geqslant 2$, let a_n denote the number of compositions of n into parts greater than 1. Relate $\{a_n\}$ to a sequence from this section.

◆ **77.** A *partition* of the integer n is a way of writing n as a sum of positive integers, in which the order of the integers does not matter. For example, there are five partitions of 4: 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$, and $1 + 1 + 1 + 1$. Let a_n denote the number of partitions of n. Compute the first 6 terms of a_n. Can you conjecture a formula?

◆ **78.** Prove that every infinite sequence $\{a_n\}$ has an infinite monotone subsequence $a_{i_1}, a_{i_2}, a_{i_3}, \ldots$ (with $i_1 < i_2 < \cdots$). *Hint:* a sequence without a greatest element must clearly have an infinite increasing subsequence. So suppose that $\{a_n\}$ has a greatest element, a_{i_1}. Now consider the sequence $a_{i_1+1}, a_{i_1+2}, \ldots$.

◆ **79.** Prove that every sequence $a_1, a_2, \ldots, a_{n^2+1}$ of length $n^2 + 1$ has a monotone subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}$ (with $i_1 < i_2 < \cdots < i_{n+1}$) of length at least $n + 1$. *Hint:* let d_i denote the longest weakly increasing subsequence beginning with a_i, i.e., the largest m so that you can find $a_i = a_{i_1} \leqslant a_{i_2} \leqslant \cdots \leqslant a_{i_m}$ for $i = i_1 < i_2 < \cdots < i_m$. If $d_i \leqslant n$ for all $i = 1, 2, \ldots, n^2 + 1$, how many terms of the sequence must share the same d_i? What does this mean for that subsequence? This result is known as the Erdős-Szekeres Theorem, after Paul Erdős and George Szekeres (1911–2005), who proved it in 1935.

◆ **80.** For each positive integer n, let a_n denote the greatest number which can be expressed as the product of positive integers with sum n. For example, $a_6 = 9$ because $3 \cdot 3$ is the greatest product of positive integers with sum 6. Find a formula for a_n.

Describe the sequences in Exercises 81–84. *Warning:* some of these have very creative definitions. You should use all tools at your disposal, including the Internet.

◆ **81.** $1, 2, 4, 6, 10, 12, 16, 18, \ldots$

◆ **82.** $1896, 1900, 1904, 1906, 1908, 1912, 1920, 1924, \ldots$

◆ **83.** $3, 3, 5, 4, 4, 3, 5, 5, \ldots$

◆ **84.** $1, 2, 3, 2, 1, 2, 3, 4, \ldots$

◆ **85.** How many 0s are there in the decimal expansion of 10!? What about 50! and 100!? *Hint:* the number of 0s is equal to the number of 10s which divide these numbers. These 10s can come by multiplying a number divisible by 10 itself, or by multiplying a number divisible by 5 by a number divisible by 2.

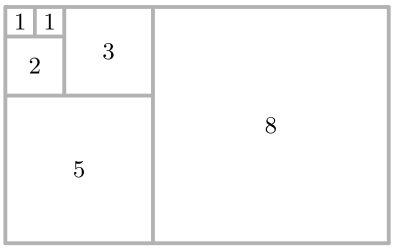
◆ **86.** How many ways are there to order a deck of cards so that each of the suits is together?

‡The distinguished Israeli mathematician Noga Alon recounts:

> "I was interviewed in the Israeli Radio for five minutes and I said that more than 2000 years ago, Euclid proved that there are infinitely many primes. Immediately the host interuppted me and asked: 'Are there still infinitely many primes?"'

◆ **87.** How many ways are there to order a deck of cards so that all of the spades are together? (But the cards of the other suits may be in any order.)

◆ **88.** Use the figure below to conjecture and prove a simplified formula for the sum $\sum_{k=0}^{n} f_k^2$, where f_k denotes the kth Fibonacci number.



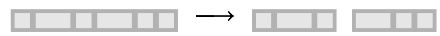
In addition to their recurrence, Fibonacci numbers can also be described in a more concrete way: the nth Fibonacci number f_n counts the number of different ways to *tile* a board of size $n \times 1$ using "squares" of size 1×1 and "dominos" of size 2×1. For example, $f_4 = 5$ because there are four ways to tile a 4×1 board with these pieces:

◆ **89.** Verify that the nth Fibonacci number counts the different ways to tile an $n \times 1$ board using 1×1 squares and 2×1 dominos.

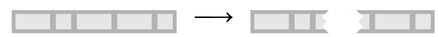
This interpretation can be quite useful in proving identities involving the Fibonacci numbers. Consider, for example, the identity

$$f_{2n} = f_n^2 + f_{n-1}^2.$$

In order to show that this holds for all $n \geqslant 1$, since f_{2n} counts tilings of a $2n \times 1$ board, we need only show that there are $f_n^2 + f_{n-1}^2$ such tilings. Take a particular tiling of a $2n \times 1$ board and chop it into two $n \times 1$ boards. There are two possibilities. First, we might chop the board into two tilings of an $n \times 1$ board, as shown below with an 8×1 board:



Secondly, we might chop through a 2×1 domino, thereby getting two $(n-1) \times 1$ tilings:



The number of $n \times 1$ tilings is f_n, so there are f_n^2 ways in which our chop could break up the $2n \times 1$ tiling in the first manner. Similarly, the number of $(n-1) \times 1$ tilings is f_{n-1}, so there are f_{n-1}^2 in which our chop could break up the $2n \times 1$ tiling in the second manner. Therefore, since we have accounted for all $2n \times 1$ tilings, we see that $f_{2n} = f_n^2 + f_{n-1}^2$, as desired.

Give similar arguments to verify the identities in Exercises 90 and 91.

◆ **90.** Prove that $\sum_{k=0}^{n} f_k = f_{n+2} - 1$.

◆ **91.** Prove that $f_{3n+2} = f_{n+1}^3 + 3f_{n+1}^2 f_n + f_n^3$.

◆ **92.** Prove the Monotone Convergence Theorem in the case where the sequence is monotonically increasing.

◆ **93.** Prove that the converse to the Monotone Convergence Theorem is also true, i.e., that every monotone sequence that is not bounded diverges.

◆ **94.** Prove the Sandwich Theorem.

ANSWERS TO SELECTED EXERCISES, SECTION 2.1

1. $a_n = 2n + 1$
3. $a_n = (-1)^{n-1}2^n$
5. $a_n = \begin{cases} 3 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$
7. $a_n = \dfrac{n!}{2^n}$
9. $\{2, 5, 8, 11, 14, \ldots\}$, $a_n = 3n - 1$
11. $\{1, 3, 6, 10, 15, \ldots\}$, $a_n = {}^{n(n+1)}\!/_2$
13. The common ratio must be 3, so $a_1 = 2$.
15. Converges to $7/4$.
17. Converges to $7/4$.
19. Converges to 0.
21. Converges to π.
23. Converges to $\sqrt{7/4}$.

2.2. AN INTRODUCTION TO SERIES

In everyday language, the words series and sequence mean the same thing. However, in mathematics, it is vital to recognize the difference. A *series* is the result of adding a sequence of numbers together. While you may never have thought of it this way, we deal with series all the time when we write expressions like

$$\frac{1}{3} = 0.333\ldots,$$

since this means that

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots$$

In general we are concerned with infinite series such as

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

for various sequences $\{a_n\}$. First though, we need to decide what it means to add an infinite sequence of numbers together. Clearly we can't just add the numbers together until we reach the end (like we do with finite sums), because we won't ever get to the end. Instead, we adopt the following limit-based definition.

Convergence and Divergence of Series. If the sequence $\{s_n\}$ of *partial sums* defined by

$$s_n = a_1 + a_2 + \cdots + a_n$$

has a limit as $n \to \infty$ then we say that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n,$$

and in this case we say that $\sum_{n=1}^{\infty} a_n$ *converges*. Otherwise, $\sum_{n=1}^{\infty} a_n$ *diverges*.

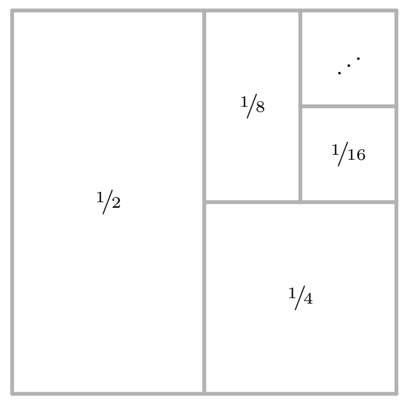
We begin with a particularly simple example.

Example 1 (Powers of 2). The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1.

Solution. We begin by computing a few partial sums:

$$\begin{array}{lllllll} s_1 & = & 1/2 & = & 1/2 & = & 1 - 1/2 \\ s_2 & = & 1/2 + 1/4 & = & 3/4 & = & 1 - 1/4 \\ s_3 & = & 1/2 + 1/4 + 1/8 & = & 7/8 & = & 1 - 1/8 \\ s_4 & = & 1/2 + 1/4 + 1/8 + 1/16 & = & 15/16 & = & 1 - 1/16 \end{array}$$

These partial sums suggest that $s_n = 1 - 1/2^n$. Once we have guessed this pattern, it is easy to prove. If $s_n = 1 - 1/2^n$, then $s_{n+1} = 1 - 1/2^n + 1/2^{n+1} = 1 - 1/2^{n+1}$, so the formula is correct for all values of n (this technique of proof is known as mathematical induction). With this formula, we see that $\lim_{n\to\infty} s_n = 1$, so $\sum_{n=1}^{\infty} 1/2^n = 1$. ●



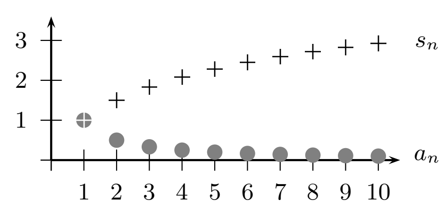
There is an alternative, more geometrical, way to see that this series converges to 1. Divide the unit square in half, giving two squares of area 1/2. Now divide one of these squares in half, giving two squares of area 1/4. Now divide one of these in half, giving two squares of area 1/16. If we continue forever, we will subdivide the unit square (which has area 1) into squares of area 1/2, 1/4, 1/8, . . . , verifying that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Note that although we were able to find an explicit formula for the partial sums s_n in Example 1, this is *not* possible in general.

Our next example shows a series $\sum a_n$ which diverges even though the sequence $\{a_n\}$ gets arbitrarily small, thereby demonstrating that the difference between convergent and divergent series is quite subtle.

Example 2 (The Harmonic Series). The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.



Solution. The proof we give is due to the French philosopher Nicolas Oresme (1323–1382), and stands as one of the pinacles of medieval mathematical achievement. We simply group the terms together so that each group sums to at least 1/2:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\underbrace{\frac{1}{3}+\frac{1}{4}}_{\geqslant 2\cdot\frac{1}{4}=\frac{1}{2}}\\ &\quad+\underbrace{\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}}_{\geqslant 4\cdot\frac{1}{8}=\frac{1}{2}}\\ &\quad+\underbrace{\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}}_{\geqslant 8\cdot\frac{1}{16}=\frac{1}{2}}\\ &\quad+\cdots\\ &\geqslant 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots,\end{aligned}$$

and therefore the series diverges. ●

The name of this series is due to Pythagoras, and his first experiments with music. Pythagoras noticed that striking a glass half-full of water would produce a note one octave higher than striking a glass full of water. A glass one-third full of water similarly produces a note at a "perfect fifth" of a whole glass, while a glass one-quarter full produces a note two octaves higher, and a glass one-fifth full produces a "major third." These higher frequencies are referred to as *harmonics*, and all musical instruments produce harmonics in addition to the *fundamental frequency* which they are playing (the instrument's "timbre" describes the amounts in which these different harmonics occur). This is what led Pythagoras to call the series $1 + 1/2 + 1/3 + \cdots$ the harmonic series.

Example 3. Suppose that scientists measure the total yearly precipitation at a certain point for 100 years. On average, how many of those years will have record high precipitation?

Solution. Suppose that the data is uncorrelated from year to year (in particular, that the amount of precipitation one year has no effect on the precipitation the next), and that the data shows no long-term trends (such as might be suggested if the climate were changing). In other words, suppose that the precipitations by year are *independent identically distributed random variables*.

Letting a_n denote the amount of precipitation in the nth year, the question asks: for how many values n can we expect a_n to be the maximum of $a_1, \ldots, a_n$? By definition, a_1 is a maximum. The second year precipitation, a_2, then has a $1/2$ of being the maximum of a_1, a_2, while in general there is a $1/n$ chance that a_n is the maximum of $a_1, \ldots, a_n$. This shows that the expected number of record high years of precipitation is

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{100}.$$

The value of this sum is approximately 5.18738, so this is the expected number of record high years of precipitation. ●

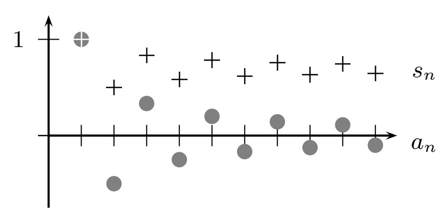
Example 2 provides a bit of intuition as to why the harmonic series diverges. Suppose that the precipitation data is collected *forever*. Then the expected number of record years is $\sum 1/n$. On the other hand, it seems natural that we should expect to see new record years no matter how long the data has been collected, although the record years will occur increasingly rarely. Therefore we should expect the harmonic series to diverge.

By making half the terms of the harmonic series negative, we obtain a convergent series:

Example 4 (The Alternating Harmonic Series). The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

converges.



Solution. Let s_n denote the nth partial sum of this series. If we group the first $2n$ terms in pairs, we have

$$s_{2n} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right).$$

Since $1 > 1/2$, $1/3 > 1/4$, and so on, each of these groups is positive. Therefore $s_{2n+2} > s_{2n}$, so the even partial sums are monotonically increasing.

Moreover, by grouping the terms in a different way,

$$s_{2n} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2n-2} - \frac{1}{2n-1}\right) - \frac{1}{2n},$$

we see that $s_{2n} < 1$ for all n. This shows that the sequence $\{s_{2n}\}$ is bounded and monotone. The Monotone Convergence Theorem (from the previous section) therefore implies that the sequence $\{s_{2n}\}$ has a limit; suppose that $\lim_{n\to\infty} s_{2n} = L$.

Now we consider the odd partial sums: $s_{2n+1} = s_{2n} + 1/2n+1$, so

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} \frac{1}{2n+1} = L.$$

Since both the even and odd partial sums converge to the same value, the sum of the series exists and is at most 1. ●

This leaves open a natural question: what is the sum of the alternating harmonic series? Our proof shows that the sum of this series is sandwiched between its even partial sums (which are under-estimates) and its odd partial sums (which are over-estimates), so

$$\frac{1}{2} = 1 - \frac{1}{2} \leqslant \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \leqslant 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6},$$

and we could get better estimates by including more terms. Back in Euler's time it would be difficult to guess what this series converges to, but with computers and the web, it is quite easy. In only a few seconds, a computer can compute that the 5 millionth partial sum of the alternating harmonic series is approximately 0.693147, and the *Inverse Symbolic Calculator* at

`http://oldweb.cecm.sfu.ca/projects/ISC/`

(which attempts to find "nice" expressions for decimal numbers) lists $\ln 2$ as its best guess for 0.693147. Exercises 46 and 47 in Section 2.4 and Exercise 24 in Section 3.1 prove that this is indeed the sum.

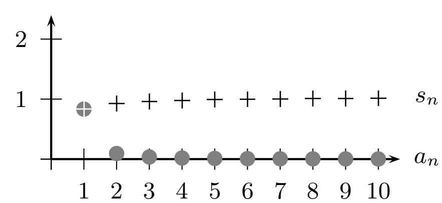
We next consider the result of adding the same terms in a different order.

Example 5 (A Troublesome Inequality). The series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right)$$

does not equal $\sum (-1)^{n+1}/n$, despite having the same terms.

Solution. Simplifying the inside of this series,



$$\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} = \frac{8n-3}{(4n-3)(4n-1)(2n)}$$

shows that it contains only positive terms. Therefore its value is at least its first two terms added together,

$$\sum_{n=1}^{\infty} \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right) \geqslant \left(1 + \frac{1}{3} - \frac{1}{2} \right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4} \right) = \frac{389}{420} > \frac{9}{10}.$$

However, if we remove the parentheses then it is clear that the fractions we are adding in this series are precisely the terms of the alternating harmonic series, whose value is strictly less than $9/10$. To repeat,

$$\sum_{n=1}^{\infty} \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right) \neq \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

even though both sides contain the same fractions with the same signs! ●

We will explore this phenomenon more in Section 2.8. Until then, just remember that *the order in which you sum a series which has both negative and positive terms might affect the answer.* In other words, when you are adding infinitely many numbers, some of which are positive and some of which are negative, addition is *not necessarily* commutative.

It is because of examples such as this that we need to be extremely careful when dealing with series. This is why we will make sure to prove every tool we use, even when those tools are "obvious."

Next we consider a series which is sometimes referred to as Grandi's series, after the Italian mathematician, philosopher, and priest Guido Grandi (1671–1742), who studied the series in a 1703 work.

Example 6 (An Oscillating Sum). The series $\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges.

Solution. While the harmonic series diverges to infinity, the series $1-1+1-1+\cdots$ diverges because its partial sums oscillate between 0 and 1:

$$s_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

so $\lim_{n\to\infty} s_n$ does not exist. ●

This is an example of a series which can be shown to diverge by the first general test in our toolbox:

The Test for Divergence. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.

Proof. It is easier to prove the contrapositive: if $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$. Since we are assuming that $\sum a_n$ converges, $\lim_{n\to\infty} s_n$ exists. Suppose that $\lim_{n\to\infty} s_n = L$. Then

$$a_n = s_n - s_{n-1} \to L - L = 0$$

as $n \to 0$, proving the theorem. □

It is important to remember that the converse to the Test for Divergence is false, i.e., even if the terms of a series tend to 0, the series may still diverge. Indeed, the harmonic series is just such a series: $1/n \to 0$ as $n \to \infty$, but $\sum 1/n$ diverges.

Before concluding the section, we make one more general observation. The convergence of a series depends only on how small its "tail" is. Thus it does not matter from the point of view of convergence/divergence if we ignore the first 10 (or the first 10 octovigintillion) terms of a series:

Tail Observation. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if its "tail"

$$\sum_{n=N}^{\infty} a_n$$

converges for some value of N.

While our techniques in this section have mostly been ad hoc, our goal in this chapter is to develop several tests which we can apply to a wide range of series. Our list of tests will grow to include†:

- The Test for Divergence
- The Integral Test
- The p-Series Test
- The Comparison Test
- The Ratio Test
- The Absolute Convergence Theorem
- The Alternating Series Test

It is important to realize that each test has distinct strengths and weaknesses, so if one test is inconclusive, you need to push onward and try more tests until you find one that can handle the series in question.

†This is in addition to the numerous more specialized tests developed in the exercises.

EXERCISES FOR SECTION 2.2

In all of the following problems, s_n denotes the nth partial sum of $\sum a_n$, that is,

$$s_n = a_1 + a_2 + \cdots + a_n.$$

1. If $\sum_{n=1}^{\infty} a_n = 3$ then what are $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} s_n$?

2. If $\lim_{n\to\infty} s_n = 4$ then what are $\lim_{n\to\infty} a_n$ and $\sum_{n=1}^{\infty} a_n$?

3. Compute the 4th partial sum of $\sum_{n=1}^{\infty} \frac{2}{n+2}$.

4. Compute the 4th partial sum of $\sum_{n=1}^{\infty} \frac{2}{n^2+2}$.

In Exercises 5–10, write down a formula for a_n and sum the series if it converges.

5. $s_n = \frac{3n+2}{n-4}$

6. $s_n = \frac{3}{n-4}$

7. $s_n = (-1)^n$

8. $s_n = \frac{n(n+1)}{2}$

9. $s_n = \sin n$

10. $s_n = \arctan n$

Determine if the series in Exercises 11–17 diverge by the Test for Divergence. (Note that if they do not diverge by the Test for Divergence, then we don't yet know if they converge or not.)

11. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3n^2+2n+1}$

12. $\sum_{n=1}^{\infty} \frac{\sin n}{n}$

13. $\sum_{n=1}^{\infty} \cos {}^1\!/_{n^2}$

14. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

15. $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$

16. $3+5+7+11+13+17+\cdots$, the sum of the primes.

17. ${}^1\!/_3 + {}^1\!/_5 + {}^1\!/_7 + {}^1\!/_{11} + {}^1\!/_{13} + {}^1\!/_{17} + \cdots$, the sum of the reciprocals of the primes.

In Exercises 18–21, reindex the series so that they begin at $n = 1$.

18. $\sum_{n=2}^{\infty} \frac{n^2}{2^n}$

19. $\sum_{n=4}^{\infty} \frac{n^2-n}{(n+5)^3}$

20. $\sum_{n=-4}^{\infty} \frac{n^2-n}{(n+5)^3}$

21. $\sum_{n=0}^{\infty} \frac{n\sin n}{(n+2)^3}$

Determine if the series in Exercises 22–25 converge at $x = -2$ and at $x = 2$.

22. $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$

23. $\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$

24. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n}$

25. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n2^n}$

Use the rules of limits described in Exercises 31–34 of Section 2.1 to prove the statements in Exercises 26–29.

26. If $\sum a_n$ and $\sum b_n$ are both convergent series, prove that

$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n.$$

27. If $\sum a_n$ and $\sum b_n$ are both convergent series, prove that

$$\sum_{n=1}^{\infty}(a_n-b_n)=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}b_n.$$

28. If $\sum a_n$ is a convergent series, prove that

$$\sum_{n=1}^{\infty}ca_n=c\sum_{n=1}^{\infty}a_n$$

for any number c.

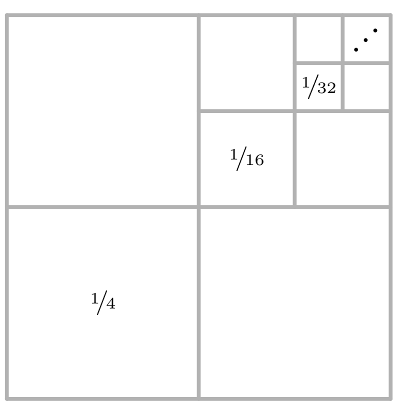
29. If $\sum a_n$ is a divergent series, prove that

$$\sum_{n=1}^{\infty}ca_n$$

diverges for any number c.

30. Archimedes (circa 287 BC–212 BC) was one of the first mathematicians to consider infinite series. In his treatise *The Quadrature of the Parabola*, he uses the figure shown below to prove that

$$1/4+1/4^2+1/4^3+\cdots=1/3.$$



Explain his proof in words.

31. Prove that if $\sum a_n$ converges, then its partial sums s_n are bounded.

32. Give an example showing that the converse to Exercise 31 is false, i.e., give a sequence $\{a_n\}$ whose partial sums are bounded but such that $\sum a_n$ does not converge.

◆ **33.** Suppose that $a_n \to 0$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty}(a_{2n}+a_{2n+1})$ converges.

◆ **34.** Give an example showing that the hypothesis $a_n \to 0$ in Exercise 33 is necessary.

Sometimes a series can be rewritten in a such a way that nearly every term cancels with a succeeding or preceding term, for example,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n(n+1)} &= \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots\\ &= 1.\end{aligned}$$

These series are called *telescoping*. In Exercises 35–36, express the series as telescoping series to compute their sums.

35. $\displaystyle\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$.

36. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{f_{n-1}f_{n+1}}$, where f_n is the nth Fibonacci number.

37. Explain why the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 3}+\frac{1}{4\cdot 4}+\cdots$$

is smaller than

$$1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots,$$

and then show that this second series $\sum 1/n(n+1)$ is a convergent telescoping series.

Exercises 38–41 concern the situation described in Example 2.

38. How many record years should we expect in 10 years of observations?

39. What is the probability that in 100 years of observations, the first year is the only record?

40. What is the probability that in 100 years of observations, there are only two record years?

41. What is the probability that in 100 years of observations, there are at most three record years?

When a musical instrument produces a sound, it also produces harmonics of that sound to some degree. For example, if you play a middle A on an instrument, you are playing a note at 440 Hz, but the instrument also produces sounds at 880 Hz, 1320 Hz, 1760 Hz, and so on. In a remarkable effect known as *restoration of the fundamental*, if the sound at 440 Hz is artificially removed, most listeners' brains will "fill it in," and perceive the collection of sounds as a middle A nonetheless. Indeed, your brain will perform the same feat even if several of the first few frequencies are removed. In Exercises 42–45, determine what fundamental frequency your brain will perceive the given collection of frequencies as.

42. 646 Hz, 969 Hz, 1292 Hz, 1615 Hz, ...

43. 784 Hz, 1176 Hz, 1568 Hz, 1960 Hz, ...

44. 789 Hz, 1052 Hz, 1315 Hz, 1578 Hz, ...

45. Discuss how restoration of the fundamental can be used to play music on a speaker which can't produce low notes.

Exercises 46–48 concern the following procedure, which was brought to my attention by Professor Pete Winkler. Start with $a_1 = 2$. At stage n, choose an integer m from 1 to a_n uniformly at random (i.e., each number has a $1/a_n$ chance of occurring). If $m = 1$ then stop. Otherwise, set $a_{n+1} = a_n + 1$ and repeat. For example, this procedure has a $1/2$ probability of stopping after only one step, while otherwise it goes on to the second step, with $a_2 = 3$.

◆ **46.** Compute the probability that this procedure continues for at least 2 steps, at least 3 steps, and at least n steps.

◆ **47.** Compute the probability that this procedure *never* terminates.

◆ **48.** The expected (or average) number of steps that this procedure takes is

$$\sum_{n=1}^{\infty} n \cdot \Pr(\text{the procedure takes precisely } n \text{ steps}).$$

Verify that another way to write this is

$$\sum_{n=1}^{\infty} \Pr(\text{the procedure takes at least } n \text{ steps}),$$

and use this to compute the expected number of steps that this procedure takes.

Cesàro summability, named for the Italian mathematician Ernesto Cesàro (1859–1906), is a different notion of sums, given by averaging the partial sums. Define

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

We say that the series $\sum a_n$ is *Cesàro summable* to L if $\lim_{n\to\infty} \sigma_n = L$. Exercise 52 shows that if $\sum a_n$ converges to L then $\sum a_n$ is Cesàro summable to L, however, the converse does not hold. We explore Cesàro summability in Exercises 49 and 50. In both of these exercises, you should use the formulas for s_n found in the text. Exercises 52 and 53 explore the more theoretical aspects of Cesàro summability.

◆ **49.** Show that the Cesàro sum of $\sum_{n=1}^{\infty} 2^{-n}$ is 1.

◆ **50.** Show that the Cesàro sum of $\sum_{n=1}^{\infty} (-1)^{n+1}$ is $1/2$.

◆ **51.** Show that $\sum_{n=1}^{\infty} (-1)^{n+1} n$ is not Cesàro summable. (Compare this with Exercise 22 in Section 3.2.)

◆ **52.** Prove that if $\sum a_n = L$ then $\{a_n\}$ is Cesàro summable to L.

◆ **53.** Prove the result, due to Alfred Tauber (1866–1942), that if $\{a_n\}$ is a positive sequence and is Cesàro summable to L, then $\sum a_n = L$.

ANSWERS TO SELECTED EXERCISES, SECTION 2.2

1. $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} s_n = 3$
3. $2/3 + 2/4 + 2/5 + 2/6$
5. $a_n = \dfrac{3n+2}{n-4} - \dfrac{3(n-1)+2}{n-5}$ and the sum of the series is 3.
7. $a_n = (-1)^n - (-1)^{n-1}$ and the series diverges.
9. $a_n = \sin n - \sin(n-1)$ and the series diverges.
11. Diverges by the Test for Divergence
13. Diverges by the Test for Divergence
15. The terms do go to 0, so the Test for Divergence does not apply.
17. The terms do go to 0, so the Test for Divergence does not apply.
19. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)^2-(n+3)}{((n+3)+5)^3}$
21. The $n=0$ term of this series is already 0, so one answer is simply $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin n}{(n+2)^3}$. Another correct answer is $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)\sin(n-1)}{((n-1)+2)^3}$.
23. Converges at $x=-2$ (alternating harmonic series) and diverges at $x=2$ (harmonic series).
25. Diverges at $x=-2$ (harmonic series) and converges at $x=2$ (alternating harmonic series).

2.3. GEOMETRIC SERIES

One of the most important types of infinite series are geometric series. A *geometric series* is simply the sum of a geometric sequence,

$$\sum_{n=0}^{\infty} ar^n.$$

Fortunately, geometric series are also the easiest type of series to analyze. We dealt a little bit with geometric series in the last section; Example 1 showed that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

while Exercise 26 presented Archimedes' computation that

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}.$$

(Note that in this section we will sometimes begin our series at $n = 0$ and sometimes begin them at $n = 1$.)

Geometric series are some of the only series for which we can not only determine convergence and divergence easily, but also find their sums, if they converge:

Geometric Series. The geometric series

$$a + ar + ar^2 + \cdots = \sum_{n=0}^{\infty} ar^n$$

converges to

$$\frac{a}{1-r}$$

if $|r| < 1$, and diverges otherwise.

Proof. If $|r| \geqslant 1$, then the geometric series diverges by the Test for Divergence, so let us suppose that $|r| < 1$. Let s_n denote the sum of the first n terms,

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1},$$

so

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

Subtracting these two, we find that

$$\begin{array}{rcccccccccccc} s_n & = & a & + & ar & + & ar^2 & + & \cdots & + & ar^{n-1} & & \\ -rs_n & = & & - & ar & - & ar^2 & - & \cdots & - & ar^{n-1} & - & ar^n \\ \hline (1-r)s_n & = & a & & & & & & & & & - & ar^n. \end{array}$$

This allows us to solve for the partial sums s_n,

$$s_n = \frac{a - ar^n}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Now we know (see Example 5 of Section 2.1) that for $|r| < 1$, $r^n \to 0$ as $n \to \infty$, so

$$\frac{ar^n}{1-r} \to 0$$

as $n \to \infty$ as well, and thus

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a}{1-r} - \frac{ar^n}{1-r} = \frac{a}{1-r},$$

proving the result. □

An easy way to remember this theorem is

$$\text{geometric series} \sum = \frac{\text{first term}}{1 - \text{ratio between terms}}.$$

We begin with two basic examples.

Example 1. Compute $12 + 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots$.

Solution. The first term is 12 and the ratio between terms is $1/3$, so

$$12 + 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots = \frac{\text{first term}}{1 - \text{ratio between terms}} = \frac{12}{1 - \frac{1}{3}} = 18,$$

solving the problem. ●

Example 2. Compute $\displaystyle\sum_{n=6}^{\infty} (-1)^n \frac{2^{n+3}}{3^n}$.

Solution. This series is geometric with common ratio

$$r = \frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}\dfrac{2^{n+4}}{3^{n+1}}}{(-1)^n\dfrac{2^{n+3}}{3^n}} = -\frac{2}{3},$$

and so it converges because $|{-2/3}| < 1$. Its sum is

$$\sum_{n=6}^{\infty} (-1)^n \frac{2^{n+3}}{3^n} = \frac{\text{first term}}{1 - \text{ratio between terms}} = \frac{2^9/3^6}{1 + 2/3},$$

which simplifies to $512/1215$. ●

The use of the geometric series formula is of course not limited to single geometric series, as our next example demonstrates.

Example 3. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1} + 9^{n/2}}{5^n}$.

Solution. We break this series into two:

$$\sum_{n=1}^{\infty} \frac{2^{n+1} + 9^{n/2}}{5^n} = \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} + \sum_{n=1}^{\infty} \frac{9^{n/2}}{5^n}$$

(if both series converge). The first of these series has common ratio $2/5$, so it converges. To analyze the second series, note that $9^{n/2} = \left(9^{1/2}\right)^n = \sqrt{9}^n = 3^n$, so this series has common ratio $3/5$. Since both series converge, we may proceed with the addition:

$$\sum_{n=1}^{\infty} \frac{2^{n+1} + 9^{n/2}}{5^n} = \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} + \sum_{n=1}^{\infty} \frac{3^n}{5^n} = \frac{2^2/5}{1 - 2/5} + \frac{3/5}{1 - 3/5} = \frac{4}{3} + \frac{3}{2}.$$

This answer simplifies to $17/6$. ●

If a geometric series involves a variable x, then it may only converge for certain values of x. Where the series does converge, it defines a function of x, which we can compute from the summation formula. Our next example illustrates these points.

Example 4. For which values of x does the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$ converge? For those values of x, which function does this series define?

Solution. The common ratio of this series is

$$r = \frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1} x^{2n+2}}{(-1)^n x^{2n}} = -x^2.$$

Since geometric series converge if and only if $|r| < 1$, we need $|-x^2| < 1$ for this series to converge. This expression simplifies to $|x| < 1$.

Where this series does converge (i.e., for $-1 < x < 1$), its sum can be found by the geometric series formula:

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{\text{first term}}{1 - \text{ratio between terms}} = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

Although note that this only holds for $|x| < 1$. ●

The geometric series formula may also be used to convert repeating decimals into fractions, as we show next.

Example 5. Express the number $4.342342342\ldots$ as a fraction in the form p/q where p and q have no common factors.

Solution. Our first step is to express this number as the sum of a geometric series. Since the decimal seems to repeat every 3 digits, we can write this as

$$4.342342342\cdots = 4 + \frac{342}{1000} + \frac{342}{1000^2} + \cdots = 4 + \sum_{n=1}^{\infty}\frac{342}{1000^n}.$$

This series is geometric, so we can use the formula to evaluate it:

$$\sum_{n=1}^{\infty}\frac{342}{1000^n} = \frac{\text{first term}}{1 - \text{ratio between terms}} = \frac{342/1000}{1 - 1/1000} = \frac{342}{1000}\cdot\frac{1000}{999} = \frac{342}{999}.$$

Our initial answer is therefore

$$4 + \frac{342}{999} = \frac{3996+342}{999} = \frac{4338}{999}.$$

Since we were asked for a fraction with no common factors between the numerator and denominator, we now have to factor out the 9 which divides both 4338 and 999, leaving

$$4.342342342 = \frac{482}{111}.$$

To be sure that 482 and 111 have no common factors, we need to verify that $111 = 3\cdot 37$, and the prime 37 does not divide 482. ●

Our next example in some sense goes in the other direction. Here we are given a fraction and asked to use geometric series to approximate its decimal expansion.

Example 6. Use geometric series to approximate the decimal expansion of $1/48$.

Solution. First we find a number near $1/48$ with a simple decimal expansion; $1/50$ will work nicely. Now we express $1/48$ as $1/50$ times a fraction of the form $1/(1-r)$:

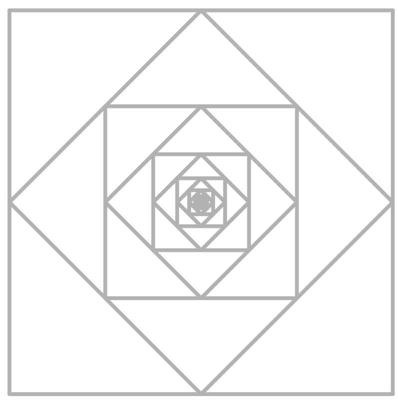
$$\frac{1}{48} = \frac{1}{50-2} = \frac{1}{50}\cdot\frac{1}{1-\frac{2}{50}}.$$

Now we can expand the fraction on the righthand side as a geometric series,

$$\frac{1}{48} = \frac{1}{50}\left(1 + \frac{2}{50} + \left(\frac{2}{50}\right)^2 + \left(\frac{2}{50}\right)^3 + \cdots\right).$$

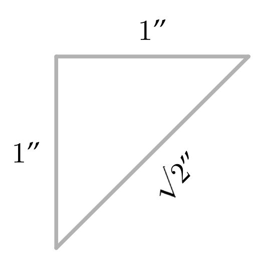
Using the first two terms of this series, we obtain the approximation $1/48 \approx 0.02(1+0.02) = 0.0204$. ●

Example 7. Suppose that you draw a 2″ by 2″ square, then you join the midpoints of its sides to draw another square, then you join the midpoints of that square's sides to draw another square, and so on, as shown below.



Would you need infinitely many pencils to continue this process forever?

Solution. If we look just at the upper left-hand corner of this figure, we see a triangle with two sides of length 1″ and a hypotenuse of length $\sqrt{2}$″:



So the sequence of side lengths of these rectangles is geometric with ratio $\sqrt{2}/2$: 2″, $\sqrt{2}$″, $1/2$″, …. Since $\sqrt{2}/2 < 1$, the sum of the side lengths of all these (infinitely many) squares therefore converges to

$$\frac{2}{1 - \frac{\sqrt{2}}{2}} = \frac{2}{\frac{2-\sqrt{2}}{2}} = 2(2+\sqrt{2}).$$

As the perimeter of a square is four times its side length, the total perimeter of this infinite construction is $8(2+\sqrt{2})''$, so we would only need finitely many pencils to draw the figure forever. ●

Our last two examples are significantly more advanced that the previous examples. On the other hand, they are also more interesting.

Example 8 (A Game of Chance). A gambler offers you a proposition. He carries a fair coin, with two different sides, heads and tails ("fair" here means that it is equally likely to land heads or tails), which he will toss. If it comes up heads, he will pay you \$1. If it comes up tails, he will toss the coin again. If, on the second toss, it comes up heads, he will pay you \$2, and if it comes up tails again he will toss it again. On the third toss, if it comes up heads, he will pay you \$3, and if it comes up tails again, he will toss it again... How much should you be willing to pay to play this game?

Solution. Would you pay the gambler \$1 to play this game? Of course. You've got to win at least a dollar. Would you pay the gamble \$2 to play this game? Here things get more complicated, since you have a 50% chance of losing money if you pay \$2 to play, but you also have a 25% chance to get your \$2 back, and a 25% chance to win money. Would you pay the gambler one million dollars to play the game? The gambler asserts that there are infinitely many positive integers greater than one million. Thus (according to the gambler, at least) you have infinitely many chances of winning more than one million dollars!

To figure out how much you should pay to play this game, we compute its average payoff (its *expectation*). In the chart below, we write H for heads and T for tails.

Outcome	Probability	Payoff	Expected Winnings
H	$1/2$	\$1	$1/2 \cdot \$1 = \$0.50$
TH	$1/4$	\$2	$1/4 \cdot \$2 = \$0.50$
TTH	$1/8$	\$3	$1/8 \cdot \$3 = \$0.375$
$TTTH$	$1/16$	\$4	$1/16 \cdot \$4 = \$0.25$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underbrace{TT\cdots T}_{n-1}H$	$1/2^n$	$\$n$	$1/2^n \cdot \$n = \$n/2^n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

There are several observations we can make about this table of payoffs. First, the sum of the probabilities of the various outcomes is

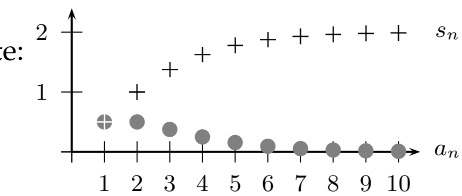
$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

which indicates that we have indeed listed all the (non-negligible) outcomes. But what if the coin *never* lands heads? The probability of this happening is $\lim_{n\to\infty} (1/2)^n = 0$, and so it is safely ignored. Therefore, we ignore this possibility.

To figure out how much you should pay to play this game, it seems reasonable to compute the sum of the expected winnings, over all possible outcomes. This sum is

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\cdots=\sum_{n=1}^{\infty}\frac{n}{2^n}.$$

We demonstrate two methods to compute this sum. The first is elementary but uses a clever trick, while the second uses calculus.



Applying the geometric series formula, we can write:

$$\begin{array}{ccccccccccc}
1 & = & 1/2 & + & 1/2^2 & + & 1/2^3 & + & 1/2^4 & + & \cdots \\
1/2 & = & & & 1/2^2 & + & 1/2^3 & + & 1/2^4 & + & \cdots \\
1/2^2 & = & & & & & 1/2^3 & + & 1/2^4 & + & \cdots \\
1/2^3 & = & & & & & & & 1/2^4 & + & \cdots \\
\vdots & = & & & & & & & & & \vdots
\end{array}$$

If we now add this array *vertically*, we obtain an equation whose left-hand side is 2 (it is the sum of a geometric series itself), and whose right-hand side, $1/2 + 2/2^2 + 3/2^3 + \cdots$, is the sum of the expected winnings, so

$$\sum_{n=1}^{\infty}\frac{n}{2^n}=2.$$

Because the expected winnings are \$2, you should be willing to pay anything less than \$2 to play the game, because in the long-run, you will make money.

Now we present a method using calculus. Since we know that

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots$$

we can differentiate this formula to get

$$\frac{1}{(1-x)^2}=0+1+2x+3x^2+4x^3+\cdots.$$

Now multiply both sides by x:

$$\frac{x}{(1-x)^2}=x+2x^2+3x^3+4x^4+\cdots.$$

Finally, if we set $x=1/2$ we obtain the sum we are looking for:

$$2=\frac{1}{2}+2\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)^3+4\left(\frac{1}{2}\right)^4+\cdots=\sum_{n=1}^{\infty}\frac{n}{2^n}.$$

One might (quite rightly) complain that we don't know that we can take the derivative of an infinite series in this way. We consider these issues in Section 3.2. ●

Example 9 (The St. Petersburg Paradox[†]). Impressed with the calculation of the expected winnings, suppose that the gambler offers you a different wager. This time, he will pay $1 for the outcome H, $2 for the outcome TH, $4 for the outcome TTH, $8 for the outcome $TTTH$, and in general, $\$2^n$ if the coin lands tails n times before landing heads. Computing the expected winnings as before, we obtain the following chart.

Outcome	Probability	Payoff	Expected Winnings
H	$1/2$	$\$1$	$\$0.50$
TH	$1/4$	$\$2$	$\$0.50$
TTH	$1/8$	$\$4$	$\$0.50$
$TTTH$	$1/16$	$\$8$	$\$0.50$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underbrace{TT\cdots T}_{n}H$	$1/2^n$	$\$2^n$	$\$0.50$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The gambler now suggests that you should be willing to pay *any* amount of money to play this game since the expected winnings, $\$0.50 + \$0.50 + \$0.50 + \cdots$, are infinite!

Solution. That is why this game is referred to as a paradox. Seemingly, there is no "fair" price to pay to play this game. Bernoulli, who introduced this paradox, attempted to resolve it by asserting that money has "declining marginal utility." This is not in dispute; imagine what your life would be like with $1 billion, and then imagine what it would be like with $2 billion. Clearly the first $1 billion makes a much bigger difference than the second $1 billion, so it has higher "utility." However, the gambler can simply adjust his payoffs, for example, he could offer you $\$2^{2^{n-1}}$ if the coin lands tails $n-1$ times before landing heads, so even taking into account the declining marginal utility of money, there is always some payoff function which results in a paradox.

A practical way out of this paradox is to note that the gambler can't keep his promises. If we assume generously that the gambler has $1 billion, then the gambler cannot pay you the full amount if the coin lands tails 30 times before landing heads ($2^{29} = 536,870,912$, but $2^{30} = 1,073,741,824$). The payouts in this case will be as above up to $n = 29$, but then beginning at $n = 30$, all you can win is $1 billion. This gives an expected winnings of

$$\sum_{n=1}^{n=29} \$0.50 + \sum_{n=30}^{\infty} \frac{\$1{,}000{,}000{,}000}{2^{n-1}} = \$14.50 + \frac{\left(\frac{\$1{,}000{,}000{,}000}{2^{30}}\right)}{1-\frac{1}{2}} \approx \$16.36.$$

[†]This problem is known as the St. Petersburg Paradox because it was introduced in a 1738 paper of Daniel Bernoulli (1700–1782) published in the *St. Petersburg Academy Proceedings*.

So by making this rather innocuous assumption, the game goes from being "priceless" to being worth the same as a new T-shirt.

Another valid point is that *you* don't have unlimited money, so there is a high probability that by repeatedly playing this game, you will go broke before you hit the rare but gigantic jackpot which makes you rich.

One amusing but nevertheless accurate way to summarize the St. Petersburg Paradox is therefore: If both you and the gambler had an infinite amount of time and money, you could earn another infinite amount of money playing this game forever. But wouldn't you have better things to do with an infinite supply of time and money? ●

EXERCISES FOR SECTION 2.3

Determine which of the series in Exercises 1–8 are geometric series. Find the sums of the geometric series.

1. $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$

2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$

4. $\sum_{n=1}^{\infty} \frac{3^n}{4^{2n-1}}$

5. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$

6. $\sum_{n=0}^{\infty} \frac{5^n}{5^{n+4}}$

7. $\sum_{n=1}^{\infty} \frac{5^n}{25^{n+4}}$

8. $\sum_{n=0}^{\infty} \frac{3^n}{n!}$

Determine which of the geometric series (or sums of geometric series) in Exercises 9–12 converge. Find the sums of the convergent series.

9. $\sum_{n=0}^{\infty} \frac{3^{2n}}{2^n}$

10. $\sum_{n=1}^{\infty} \frac{4^{n/2+1}}{3^n}$

11. $\sum_{n=1}^{\infty} \frac{2^n + 5^n}{4^{n+9}}$

12. $\sum_{n=0}^{\infty} (-1)^n \frac{3^{n+2}}{5^{n-1}}$

Find the sums of the series in Exercises 13–16.

13. $\sum_{n=1}^{\infty} \frac{3^n + 5^n}{7^n}$

14. $\sum_{n=1}^{\infty} e^n \pi^{-n}$

15. $\sum_{n=1}^{\infty} \frac{(9/2)^{n+2}}{3^{2n}}$

16. $\sum_{n=1}^{\infty} \frac{3^{n+2} + 4^{n/2}}{6^n}$

Determine which values of x the series in Exercises 17–20 converge for. When these series converge, they define functions of x. What are these functions?

17. $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

18. $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{3^n}$

$\boxed{\textbf{19.}}$ $\displaystyle\sum_{n=1}^{\infty} 5^{n+2}x^n$

$\boxed{\textbf{20.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^{n-1}}$

For Exercises 21–24, use geometric series to approximate these reciprocals, as in Example 6.

21. $1/99$

22. $1/102$

23. $-2/99$

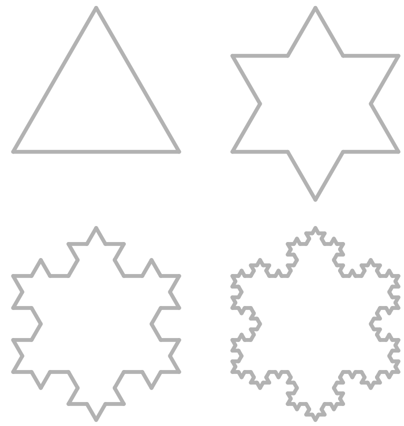
24. $1/24$

For Exercises 25 and 26, suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ is a geometric series such that the sum of the first 3 terms is 3875 and the sum of the first 6 terms is 3906.

25. What is a_6?

26. What is $\displaystyle\sum_{n=1}^{\infty} a_n$?

Exercises 27 and 28 consider the *Koch snowflake*. Introduced in 1904 by the Swedish mathematician Helge von Koch (1870–1924), the Koch snowflake is one of the earliest fractals to have been described. We start with an equilateral triangle. Then we divide each of the three sides into three equal line segments, and replace the middle portion with a smaller equilateral triangle. We then iterate this construction, dividing each of the line segments of the new figure into thirds and replacing the middle with an equilateral triangle, and then iterate this again and again, forever. The first four iterations are shown below.



◆ **27.** Write a series representing the area of the Koch snowflake, and find its value.

◆ **28.** Write a series representing the perimeter of the Koch snowflake. Does this series converge?

29. Two 50% marksmen decide to fight in a duel in which they exchange shots until one is hit. What are the odds in favor of the man who shoots first?

30. In the decimal system, some numbers have more than one expansion. Verify this by showing that $2.35999\ldots = 2.36$.

◆ **31.** The *negadecimal* number system is like the decimal system except that the base is -10. So, for example, $(12.43)_{-10} = 1(-10)^2 + 2(-10) + 4(-10)^{-1} + 3(-10)^{-2} = 97.63$ in base 10. Prove that (like the decimal system) there are non-unique expansions in the negadecimal system by showing that $(1.909090\ldots)_{-10} = (0.090909\ldots)_{-10}$.

Exercises 32 and 33 ask you to prove that there are infinitely many primes, following a proof of Euler. Exercise 34 provides the classic proof, due to Euclid.

◆ **32.** Prove, using the fact that every positive integer n has a unique prime factorization†, that

$$\sum \frac{1}{n} = \left(\sum \frac{1}{2^n}\right)\left(\sum \frac{1}{3^n}\right)\left(\sum \frac{1}{5^n}\right)\cdots$$

where the right-hand side is the product of all series of the form $\sum \frac{1}{p^n}$ for primes p.

†This refers to the fact that every positive integer n can be written as a product $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ for primes $p_1, p_2, \ldots, p_k$ and nonnegative integers $a_1, a_2, \ldots, a_k$.

◆ **33.** Use Exercise 32 to prove that there are infinitely many primes. *Hint:* the left hand-side diverges to ∞, while every sum on the right hand-side converges.

34. (Euclid's proof) Suppose to the contrary that there are only finitely many prime numbers, $\{p_1, p_2, \ldots, p_m\}$. Draw a contradiction by considering the number $n = p_1 p_2 \cdots p_m + 1$.

◆ **35.** Compute $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.
Hint: consider the derivative of $x/(1-x)^2$. For now, assume that you can differentiate this series as done in Example 8.

36. Suppose the gambler from Example 8 alters his game as follows. If the coin lands tails an even number of times before landing heads, you must pay him $\$1$. However, if the coin lands tails an odd number of times before landing heads, he pays you $\$1$. The gambler argues that there are just as many even numbers as odd numbers, so the game is fair. Would you be willing to play this game? Why or why not? (Note that 0 is an even number.)

37. Suppose the gambler alters his game once more. If the coin lands tails an even number of times before landing heads, you must pay him $\$1$. However, if the coin lands tails an odd number of times before landing heads, he pays you $\$2$. Would you be willing to play this game? Why or why not?

◆ **38.** Suppose that after a few flips, you grow suspicious of the gambler's coin because it seems to land heads more than 50% of the time (but less than 100% of the time). Design a procedure which will nevertheless produce 50-50 odds using his unfair coin. (The simplest solution to the problem is attributed to John von Neumann (1903–1957).)

◆ **39.** Consider the series $\sum 1/n$ where the sum is taken over all positive integers n which do not contain a 9 in their decimal expansion. Due to A.J. Kempner in 1914, series like this are referred to as *depleted harmonic series*. Show that this series converges. *Hint:* how many terms have denominators between 1 and 9? Between 10 and 99? More generally, between 10^{n-1} and $10^n - 1$?

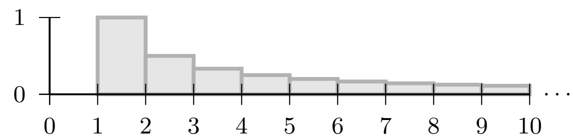
ANSWERS TO SELECTED EXERCISES, SECTION 2.3

1. Geometric series, converges to $\dfrac{2/3}{1 - 2/3} = 2$.
3. Not a geometric series.
5. Geometric series, converges to $\dfrac{1}{1 - (-1/2)} = 2/3$.
7. Geometric series, converges to $\dfrac{5/25^5}{1 - 5/25} = 4/9765625$.
9. Diverges; the ratio is $9/2 > 1$.
11. Diverges.
13. The sum is $\dfrac{3/7}{1 - 3/7} + \dfrac{5/7}{1 - 5/7} = 13/4$.
15. The sum is $\dfrac{9^2/2^3}{1 - 1/2} = 9^2/2^2$.
17. Converges for $-3 < x < 3$, to the function $\dfrac{x/3}{1 - x/3}$.
19. Converges for $-1/5 < x < 1/5$, to the function $\dfrac{5^3 x}{1 - 5x}$.

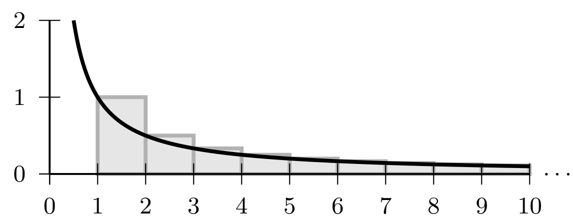
2.4. IMPROPER INTEGRALS AND THE INTEGRAL TEST

In this section we discuss a very simple, but powerful, idea: in order to prove that certain series converge or diverge, we may compare them to integrals. There are a few important caveats with this comparison, which we will make note of when we present the Integral Test formally. To motivate this test, we return to the harmonic series $\sum 1/n$. In Section 2.2, we saw Nicolas Oresme's classic proof that the harmonic series diverges. Here, we present another proof, which will generalize to handle many more series.

To begin, let us represent the series $\sum 1/n$ as the total area contained in an infinite sequence of $1 \times 1/n$ rectangles. Beginning with the first rectangle stretching from $x = 1$ to $x = 2$ and placing the rectangles next to each other, we get the following.



We now approximate the area under these rectangles. In this case, we only have to observe that the function $1/x$ lies below the tops of these rectangles for $x \geqslant 1$, as shown below.



Therefore, there is more area under the rectangles than under the function $1/x$. As we know that area under a curve is given by an integral, to find the area under $1/x$ for $x \geqslant 1$, we need to evaluate

$$\int_1^\infty \frac{1}{x}\,dx.$$

This type of an integral may be unfamiliar because it involves infinity, and for this reason, integrals of this type are called *improper integrals*†. Since we can't simply take the anti-derivative of $1/x$ and plug in ∞, we do the next best thing. We *define* the improper integral as a limit:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx.$$

†These are not the only type of improper integrals. Others involve integrating near a vertical asymptote of a function. See Exercises 36–43 for more examples of improper integrals.

Using this definition, we have another argument for why $\sum 1/n$ diverges:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &\geqslant \int_1^{\infty} \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \ln x\Big|_1^b \\ &= \lim_{b\to\infty} \ln b - \ln 1 \\ &= \infty.\end{aligned}$$

In general, if we have a function $f(x)$ defined from $x = a$ to $x = \infty$, we define

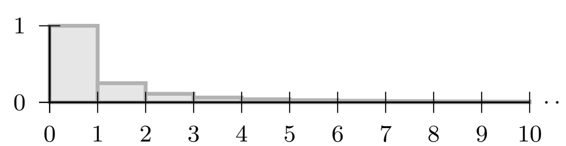
$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

and we say that this improper integral *converges* if the limit converges, and that it *diverges* if the limit diverges. The reader should note the similarity between this definition and the definition we made for the convergence of a series.

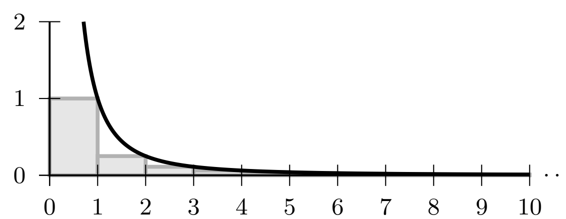
Comparing series to integrals can also be used to show that they converge, as we illustrate in the next example.

Example 1. Show that $\sum 1/n^2$ converges by comparing it to an integral.

Solution. We do roughly the same thing as we did with the harmonic series, but here, since we are to show that the series converges, we want the area in our rectangles to be *less* than the area under the curve. For this reason, we begin by placing $1 \times 1/n^2$ rectangles starting at $x = 0$:



Now we have exactly what we want, because the area under these rectangles is strictly less than the area under $1/x^2$ for $x \geqslant 0$:



This suggests that we should integrate $1/x^2$ from $x = 0$ to $x = \infty$ to get an upper bound on $\sum 1/n^2$. However, since $1/x^2$ has a vertical asymptote at $x = 0$, such an integral would be doubly improper and actually diverges, as shown by Exercises 37 and 38, so can't be used to bound $\sum 1/n^2$ from above. To get around this problem, we can simply pull off the first term of the series, and compare the rest to the integral of $1/x^2$ from $x = 1$ to $x = \infty$:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &= 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \\ &\leqslant 1 + \int_1^{\infty} \frac{1}{x^2}\,dx \\ &= 1 + \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\,dx \\ &= 1 + \lim_{b\to\infty} -\frac{1}{x}\bigg|_1^b \\ &= 2 - \lim_{b\to\infty} \frac{1}{b} \\ &= 2.\end{aligned}$$

It is tempting to conclude right now that $\sum 1/n^2$ converges, because we know that it is at most 2, but this would be reckless. Remember in Example 6 of Section 2.2 we presented a series — the series $\sum (-1)^{n+1}$ — which has bounded partial sums but does not converge.

However, in this case, $1/n^2$ is positive for all n, so the partial sums $\{s_n\}$ of $\sum 1/n^2$ are monotonically increasing. By the above argument, these partial sums are bounded: s_n lies between 0 and 2 for all n, and therefore we know by the Monotone Convergence Theorem that the sequence $\{s_n\}$ converges to a limit, and thus $\sum 1/n^2$ converges. ●

While we have shown that $\sum 1/n^2$ converges, we have not computed its value. For series that aren't geometric, such questions are generally extremely difficult, and $\sum 1/n^2$ is no exception. Finding $\sum 1/n^2$ became known as the Basel problem after it was posed by Pietro Mengoli (1626–1686) in 1644. In 1735, at the age of twenty-eight, Leohnard Euler showed that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

one of the first major results in what would be a marvelous career. We discuss his first proof (which has been incredibly influential despite the fact that it contains a flaw) in Exercises 54–58 of Section 3.3. Euler began his exploration of the Basel problem by computing the sum to 17 decimal places (which in itself was quite a feat, accomplished by viewing the series as an integral), a bit like we did in Section 2.2 to guess the sum of the alternating harmonic series. Amazingly, without any aid like the *Inverse Symbolic Calculator*, Euler recognized that this approximation looked like $\pi^2/6$! This gave Euler a significant advantage in finding the solution, since he knew what the answer should be.

Euler went on to find formulas for $\sum 1/n^p$ for all even integers p. But what about the odd integers? For a *very* long time, mathematicians could not even prove that $\sum 1/n^3$ was irrational, let alone express it in terms of well-known constants. In 1978, Roger Apéry (1916–1994) announced that he had a proof of this result. However, Apéry was not well-known and there were significant doubts that his proof could be correct. Apéry fed this suspicion by giving a strange talk announcing his proof, one of the key ingredients of which was the equation

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n!)^2}{n^3(2n)!}.$$

When asked how he derived this equation, Apéry is alleged to have replied "they grow in my garden." Nevertheless, he completed his proof, stunning the mathematical establishment. The analogous question about the irrationality of $\sum 1/n^p$ for odd integers $p \geqslant 5$ remains unsolved.

It is now time to generalize our two examples and make a test out of them. First we must decide what was special about $\sum 1/n$ and $\sum 1/n^2$ that allowed us to make the comparisons we made. In the case of $\sum 1/n$, we needed that the function $1/x$ lies below the rectangles we formed. This relies on the fact that $1/x$ is decreasing. Similarly, in the case of $\sum 1/n^2$ we needed that the function $1/x^2$ lies above the rectangles we formed. Because these rectangles were slid over by one unit, this too relies on the fact that $1/x^2$ is decreasing. We also used, in the $\sum 1/n^2$ case, the fact that $1/x^2$ is positive. Finally, we need to be able to evaluate the integrals. We could add this as a hypothesis, but in the interest of simplicity, we simply require that our functions be continuous, which guarantees that they can be integrated. Under these conditions, we have the following test.

The Integral Test. Suppose that f is a positive, decreasing, and continuous function, and that $a_n = f(n)$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

Proof. As in our two examples, we can sandwich the partial sums s_n between two improper integrals:

$$\int_1^n f(x)\,dx \leqslant s_n \leqslant a_1 + \int_1^{n-1} f(x)\,dx.$$

Now since we are proving an "if and only if" statement, we have two things to prove. First, suppose that

$$\int_1^{\infty} f(x)\,dx$$

converges. Then, by using the upper-bound, we have

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n \leqslant \lim_{n\to\infty} a_1 + \int_1^{n-1} f(x)\,dx < \infty.$$

Because $a_n = f(n)$ is positive, we know that the partial sums s_n are monotonically increasing, so since the above inequality shows that the sequence $\{s_n\}$ is bounded, the Monotone Convergence Theorem implies that $\{s_n\}$ has a limit. This proves that $\sum a_n$ converges if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

Now suppose that

$$\int_1^{\infty} f(x)\,dx$$

diverges. Using the lower-bound, we have

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n \geqslant \lim_{n\to\infty} \int_1^{n} f(x)\,dx = \infty.$$

Therefore, the sequence $\{s_n\}$ of partial sums diverges to ∞, so the series $\sum a_n$ diverges. □

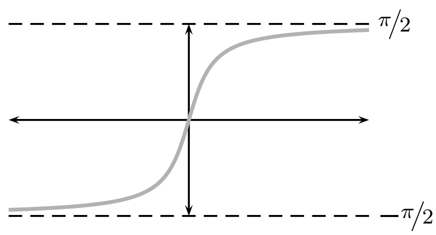
The Integral Test is a very powerful tool, but it has a serious drawback: we must be able to evaluate the improper integrals it requires. For example, how could we use the Integral Test to determine whether $\sum 4^n/n!$ converges? Nevertheless, there are numerous examples of series to which it applies.

Example 2. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge or diverge?

Solution. We first evaluate the improper integral in the Integral Test:

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x^2+1} = \lim_{b\to\infty} \arctan b - \arctan 1.$$

To evaluate this limit, it may be helpful to recall the plot of $\arctan$:



Therefore, we have

$$\lim_{b\to\infty} \arctan b - \arctan 1 = {}^{\pi}/_{2} - {}^{\pi}/_{4},$$

so the series $\sum 1/(n^2+1)$ converges by the Integral Test. ●

We began the section by considering $\sum 1/n$ and $\sum 1/n^2$. What about $\sum 1/n^p$ for other values of p? We can evaluate the integral of $1/x^p$, so the Integral Test can be used to determine which of these series converge. Because series of this form occur so often, we record this fact as its own test.

> **The p-Series Test.** The series $\sum 1/n^p$ converges if and only if $p > 1$.

Proof. When $p = 1$, we already know that the series diverges ($\sum 1/n$ is the example we began the section with). For other values of p, we simply integrate the improper integral from the Integral Test:

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \\ &= \lim_{b\to\infty} \frac{1}{-p+1}x^{-p+1}\Big|_1^b \\ &= \left(\lim_{b\to\infty}\frac{1}{-p+1}b^{-p+1}\right) - \frac{1}{-p+1}\end{aligned}$$

If $p < 1$ then the function x^{-p+1} decreases to 0 (the exponent is negative), so in this case the limit above converges to $-1/{-p+1} = 1/{1-p}$. Therefore, by the Integral Test, $\sum 1/n^p$ converges if $p < 1$. On the other hand, if $p > 1$ then the function x^{-p+1} increases without bound, so in this case the limit above diverges to ∞, and so $\sum 1/n^p$ diverges if $p > 1$. $\square$

We conclude this section with error estimates. Since improper integrals can be used to bound series, they can also be used to bound the tails of series, i.e., the error in a partial sum:

> **The Integral Test Remainder Estimates.** Suppose that f is a positive, decreasing, and continuous function, and that $a_n = f(n)$. Then the error in the nth partial sum of $\sum a_n$ is bounded by an improper integral:
> $$\left| s_n - \sum_{n=1}^{\infty} a_n \right| \leqslant \int_n^\infty f(x)\,dx.$$

The proof of the Integral Test Remainder Estimate is almost identical to the proof of the Integral Test itself, so we content ourselves with an example.

Example 3. Bound the error in using the fourth partial sum s_4 to approximate $\sum_{n=1}^{\infty} 1/n^2$.

Solution. The error in this case is the difference between s_n and the true value of the series:

$$\text{Error} = \left| s_n - \sum_{n=1}^{\infty} \frac{1}{n^2} \right|$$

By the remainder estimates, we have:

$$\begin{aligned} \text{Error} &\leqslant \int_4^{\infty} \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \int_4^b \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} -\frac{1}{x}\bigg|_4^b \\ &= \frac{1}{4}. \end{aligned}$$

This is *not* a very good bound. As we mentioned earlier, Euler approximated the value of this series to within 17 decimal places. How many terms would we need to take to get the upper bound on the error from the Integral Test Remainder Estimates under 10^{-17}? ●

EXERCISES FOR SECTION 2.4

Arrange the quantities in Exercises 1–4 in order from least to greatest.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad 1 + \displaystyle\int_1^{\infty} \frac{dx}{x^2}$ $\quad \displaystyle\int_2^{\infty} \frac{dx}{x^2}$

2. $\displaystyle\sum_{n=9}^{\infty} \frac{1}{n^2+1}$ $\quad \displaystyle\int_{10}^{\infty} \frac{dx}{x^2+1}$ $\quad \displaystyle\int_9^{\infty} \frac{dx}{x^2+1}$

3. $\displaystyle\sum_{n=1}^{10} \frac{1}{\sqrt{n^7+2}}$ $\quad \displaystyle\int_1^{11} \frac{dx}{\sqrt{x^7+2}}$ $\quad \displaystyle\int_0^{10} \frac{dx}{\sqrt{x^7+2}}$

4. $\displaystyle\sum_{n=1}^{10} -n$ $\quad \displaystyle\int_0^{10} -x\,dx$ $\quad \displaystyle\int_1^{11} -x\,dx$

For Exercises 5–20, use the Integral Test to determine if the series converge or diverge, *or* indicate why the Integral Test cannot be used.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+3}}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{4n}{(2n^2+3)^2}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{(2+5n)}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n-\sqrt{n}}{n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2-\sin n}}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2+1}$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{\sqrt{n}}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n\sqrt{n}}$

17. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

18. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

19. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n(\ln \ln n)}$

20. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n(\ln \ln n)^2}$

By the Integral Test Remainder Estimates, how many terms would you need to use to approximate the sums in Exercises 21–24 to within $1/100$?

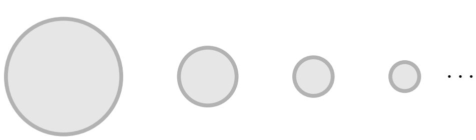
21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n\sqrt{n}}$

Exercises 25 and 26 concern an infinite sequence of circles which do not overlap and have radii 1, $1/2$, $1/3$, . . . , as shown below.



25. Is the total area inside all of the circles finite? (Note that you are not asked to find this total.)

26. Is the total circumference inside all of the circles finite? (Note that you are not asked to find this total.)

Exercises 27–32 require integration by parts. Use the Integral Test to determine if the series converge or diverge.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{7/6}}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{ne^{1/n}}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

33. Use the Integral Test to verify that the geometric series $\displaystyle\sum_{n=1}^{\infty} ar^n$ converges for $0 < r < 1$.

34. In the Integral Test, we began both the series and the integral at 1 (technically, $n = 1$ and $x = 1$, respectively). Show that this is not necessary by proving the following. Suppose that f is a positive, decreasing, and continuous function, and that $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if there is some N so that the improper integral

$$\int_N^{\infty} f(x)\, dx$$

converges.

35. Use the fact that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots \\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)\end{aligned}$$

to prove that the alternating harmonic series converges using the Integral Test.

We have focused on only one type of improper integrals, which are called improper because their domains are infinite. However, there is another type, which are called improper because their integrands have vertical asymptotes. To begin with, suppose that $f(x)$ is continuous on the interval $(a, b]$ but discontinuous at $x = a$. Then we define the integral of $f(x)$ from a to b as a limit:

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

Use this definition to evaluate the integrals in Exercises 36–39.

36. $\displaystyle\int_0^1 \sqrt{x}\,dx$

37. $\displaystyle\int_0^1 x^2\,dx$

38. $\displaystyle\int_0^\infty x^2\,dx$

39. $\displaystyle\int_0^1 x^{1.001}\,dx$

If instead $f(x)$ is continuous on the interval $[a, b)$ but discontinuous at $x = b$, then we consider the limit as the upper-bound of the integral approaches b from below:

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

Use this definition to evaluate the integrals in Exercises 40–41.

40. $\displaystyle\int_0^1 \frac{1}{1-x}\,dx$

41. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Finally, it could be the case that $f(x)$ has a vertical asymptote between the bounds a and b, say at $x = c$ for $a < c < b$. In this case, assuming that $f(x)$ is continuous on both $[a, c)$ and $(c, b]$, we break the integral in two:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

We then evaluate each of these improper integrals using the rules above. Using this definition, evaluate the integrals in Exercises 42–43.

42. $\displaystyle\int_0^2 \frac{1}{(1-x)^2}\,dx$

43. $\displaystyle\int_{-1}^1 x^{-3}\,dx$

Euler's constant γ is defined as

$$\gamma = \lim_{n\to\infty} 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n.$$

It is not clear a priori that this limit exists, and so Exercises 44 and 45 show how to prove that it does exist. Its value is approximately 0.577215, and a very readable account of research related to γ is given by Julian Havil in his book *Gamma*.

◆ **44.** Show that

$$\ln(n+1) \leqslant 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leqslant \ln n,$$

and conclude from this that the sequence $b_n = 1 + 1/2 + 1/3 + \cdots + 1/n - \ln n$ is bounded both above and below.

◆ **45.** Show that

$$\ln(n+1) - \ln n > \frac{1}{n+1},$$

and use this to conclude that the sequence $\{b_n\}$ from Exercise 44 is decreasing. This will imply that $\{b_n\}$ is a decreasing bounded sequence, so its limit, γ, exists by the Monotone Sequence Theorem.

Exercises 46 and 47 show one way to sum the alternating harmonic series, using Euler's constant γ discussed in Exercises 44 and 45.

◆ **46.** Let s_n denote the nth partial sum of the alternating harmonic series $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$. Verify that

$$\begin{aligned} s_{2n} &= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) \\ &\quad - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right). \end{aligned}$$

◆ **47.** Use Exercise 46 to show that

$$s_{2n} - \ln 2 = s_{2n} - (\ln 2n + \gamma) + (\ln n + \gamma) \to 0,$$

proving that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$.

Exercises 48–50 ask you to develop estimates for $n!$. A more precise estimate is named for James Stirling (1692–1770).

Stirling's Formula. $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\,(n/e)^n} = 1.$

◆ **48.** Using the fact that $\ln n! = \ln 2 + \ln 3 + \cdots + \ln n$, prove that

$$\ln(n-1)! < \int_1^n \ln x\,dx < \ln n!$$

Note that $\int \ln x\,dx = x\ln x - x + C$.

◆ **49.** Prove that $n! > e\,(n/e)^n$.

◆ **50.** Prove that $n! < en\,(n/e)^n$.

Exercises 51–55 consider a back-of-the-envelope calculation of the escape velocity from Earth using improper integration. These exercises are due to Professor Stephen Greenfield.

51. The continental US is about 3400 miles wide (at its widest point) and contains 4 time zones. Since there are 24 time zones in the world, show that the radius of the Earth is about 4000 miles.

52. Two masses attract each other with a force whose magnitude is proportional to the product of the masses divided by the square of the distance between them. So for a mass m, the magnitude of the force of gravity is GmM/r^2, where G is a constant, M is the mass of the Earth, and r is the distance to the center of the Earth. Since work is equal to force times distance, show that the amount of work needed to lift a mass m from the surface of the Earth to altitude R is

$$\int_{4000}^{R} \frac{GmM}{r^2}\,dr,$$

and calculate this integral. Then let $R \to \infty$ to show that $GmM/4000$ is the most work you can do to lift the mass m to anywhere in the universe (disregarding all objects besides Earth, of course).

53. Using the fact that kinetic energy is $mv^2/2$, compute how much kinetic energy we would need to supply to lift the mass m to anywhere in the universe.

54. Use the fact that acceleration due to gravity on the surface of the Earth is about 32 ft/sec^2, which is equal to $GmM/4000^2$ to solve for GM.

55. Use the answers to Exercises 53 and 54 to show that the escape velocity from the Earth is about 7 miles per second.

ANSWERS TO SELECTED EXERCISES, SECTION 2.4

1. $\displaystyle\int_2^\infty \frac{dx}{x^2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \int_1^\infty \frac{dx}{x^2}$
3. $\displaystyle\int_1^{11} \frac{dx}{\sqrt{x^7+2}} < \sum_{n=1}^{10} \frac{1}{\sqrt{n^7+2}} < \int_0^{10} \frac{dx}{\sqrt{x^7+2}}$
5. Converges by the Integral Test.
7. The Integral Test does not apply (some terms are negative).
9. Converges by the Integral Test (make a u-substitution).
11. The Integral Test does not apply (the series is not decreasing). However, this series diverges by the Test for Divergence.
13. Diverges by the Integral Test (make a u-substitution).
15. The Integral Test does not apply (the series is not decreasing). However, this series diverges by the Test for Divergence.
17. Diverges by the Integral Test (set $u = \ln x$ to evaluate the integral).
19. Diverges by the Integral Test (set $u = \ln \ln x$ to evaluate the integral).
21. 4 terms suffice.
23. e^{100} terms are enough (note that this is 2.688×10^{43})
25. The total area inside the circles is finite.

2.5. THE COMPARISON TEST

We began our systematic study of series with geometric series, proving the

- Geometric Series Test: $\sum ar^n$ converges if and only if $|r| < 1$.

Then in the last section we compared series to integrals in order to determine if they converge or diverge, and established the

- p-Series Test: $\sum 1/n^p$ converges if and only if $p > 1$.

In this section we study another type of comparison where we compare series to other series to determine convergence. The general principle is this:

- if a positive series is bigger than a positive divergent series, then it diverges, and
- if a positive series is smaller than a positive convergent series, then it converges.

For example, in the last section (Example 1) we showed that $\sum 1/n^2$ converges using the Integral Test. Then we used the Integral Test again (Example 2) to show that $\sum 1/n^2+1$ converges. But, $1/n^2+1$ is smaller than $1/n^2$ for all $n \geqslant 1$, so the convergence of $\sum 1/n^2+1$ is guaranteed by the convergence of $\sum 1/n^2$. While this approach should seem intuitively clear and simple, we caution the reader that it takes a lot of practice to become comfortable with comparisons. We state the formal test below.

The Comparison Test. Suppose that $0 \leqslant a_n \leqslant b_n$ for sufficiently large n.

- If $\sum a_n$ diverges, then $\sum b_n$ also diverges.
- If $\sum b_n$ converges, then $\sum a_n$ also converges.

Before presenting the proof of the Comparison Test, note the phrase "sufficiently large n". By "$0 \leqslant a_n \leqslant b_n$ for sufficiently large n", we mean that there is some number N such that $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$. This is just a formal way to say that we only care about tails, and should remind the reader of the Tail Observation from Section 2.2.

Proof. Suppose that for *all* n, $0 \leqslant a_n \leqslant b_n$. This seems slightly weaker than the result we have claimed, but the full result will then follow either by the Tail Observation of Section 2.2 or by an easy adaptation of this proof.

Let s_n denote the nth partial number of $\{a_n\}$ and t_n denote the nth partial sum of $\{b_n\}$, so

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n, \\ t_n &= b_1 + b_2 + \cdots + b_n. \end{aligned}$$

From our hypotheses (that $0 \leqslant a_n \leqslant b_n$ for all n), we know that $s_n \leqslant t_n$ for all n.

First suppose that $\sum a_n$ diverges. Because the terms a_n are nonnegative, the only way that $\sum a_n$ can diverge is if $s_n \to \infty$ as $n \to \infty$ (why?). Therefore the larger partial sums t_n must also tend to ∞ as $n \to \infty$, so the series $\sum b_n$ diverges as well.

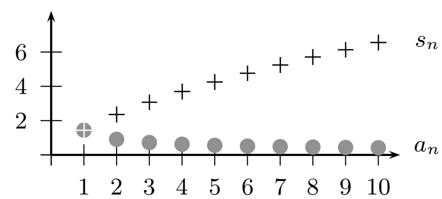
Now suppose that $\sum b_n$ converges, which implies by our definitions that $t_n \to \sum b_n$ as $n \to \infty$. The sequence $\{s_n\}$ is nonnegative and monotonically increasing because $a_n \geqslant 0$ for all n, and

$$0 \leqslant s_n \leqslant t_n \leqslant \sum b_n$$

so the sequence $\{s_n\}$ has a limit by the Monotone Convergence Theorem. This shows (again, by the definition of series summation) that the series $\sum a_n$ converges. □

The Comparison Test leaves open the question of what to compare series with. In practice, however, this choice is usually obvious, and we will almost always compare with a geometric series or a p-series. Our next four examples demonstrate the general technique.

Example 1. Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ converge or diverge?



Solution. First note that we probably shouldn't try to apply the Integral Test in this example — the function $1/\ln x$ has an antiderivative, but it has been proved that its antiderivative cannot be expressed in terms of elementary functions.

However, the Comparison Test is easy to apply in this case. Note that

$$\begin{aligned} \ln n &\leqslant n \quad \text{for } n \geqslant 2, \text{ so} \\ {}^1\!/\!_{\ln n} &\geqslant {}^1\!/\!_n \quad \text{for } n \geqslant 2. \end{aligned}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by comparison. ●

Example 2. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge or diverge?

Solution. This example can be done with the Integral Test, but it's easier to use the Comparison Test. We know that $\ln n > 1$ for $n \geqslant 3$, so

$${}^{\ln n}\!/\!_n \geqslant {}^1\!/\!_n \quad \text{for } n \geqslant 3.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ must also diverge. ●

Example 3. Does the series $\sum_{n=1}^{\infty} \left(\frac{\cos(n)}{n}\right)^2$ converge or diverge?

Solution. We can write this series as $\sum \cos^2(n)/n^2$. The numerator of this fraction, $\cos^2(n)$, is nonnegative for all n (this is important since we can't apply the Comparison Test to series with negative terms) and bounded by 1, so

$$\frac{\cos^2(n)}{n^2} \leqslant \frac{1}{n^2} \quad \text{for } n \geqslant 1.$$

Therefore since $\sum 1/n^2$ converges (it is a convergent p-series), the smaller series $\sum_{n=1}^{\infty} \left(\frac{\cos(n)}{n}\right)^2$ must converge as well. ●

Example 4. Does the series $\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$ converge or diverge?

Solution. For $n \geqslant e^2 \approx 7.39$, $\ln n \geqslant 2$, so for these values of n,

$$1/n^{\ln n} \leqslant 1/n^2.$$

Since $\sum 1/n^2$ is a convergent p-series, $\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$ converges by comparison. ●

Sometimes the inequalities we need to apply the Comparison Test seem to go the wrong way. Consider for example the series

$$\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

We would like to compare this series with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{2n},$$

but the terms in our series seem to be *smaller* than the terms of $\sum 1/2n$. Therefore we cannot naively apply the Comparison Test in this case.

Example 5. Show that the series $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges.

Solution. We have that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{2n+1} &= \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \\ &\geqslant \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \sum_{n=2}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n},\end{aligned}$$

so the series diverges by comparison to $\sum 1/n$. ●

Our next example displays a similar phenomenon. Note that $1/n^2-1 > 1/n^2$, but we are still able to compare the series.

Example 6. Show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converges.

Solution. Because $n^2 - 1 \geqslant (n-1)^2$, we have that

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{n^2-1} &= \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots \\ &\leqslant \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2},\end{aligned}$$

so the series converges by comparison to $\sum 1/n^2$. ●

Example 7. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+7}}$ diverges.

Solution. We should expect this series to diverge, because the numerator is n and the denominator behaves like n^2, but the inequality goes the wrong way. By giving up a bit in the denominator, however, we get the desired conclusion:

$$\frac{n}{\sqrt{n^4+7}} \geqslant \frac{n}{\sqrt{n^4+7n^4}} = \frac{1}{\sqrt{8}n},$$

so the series we are interested in diverges by comparison to the harmonic series. ●

In Examples 5–7, we are really *reindexing* the series. This procedure is demonstrated more formally in the example below and in Exercises 25–28. Another method for dealing with such problems, known as the Limit Comparison Test, is discussed in Exercises 42–50.

Example 8. Show that the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+3}{n^4-2}$ converges by reindexing the series with the substitution $m = n - 1$.

Solution. We want to compare this series to the series given by its leading terms, $\sum n^2/n^4$ (or some multiple of this), but the comparison seems to go the wrong way. By setting $m = n - 1$, which is equivalent to $n = m + 1$, we have

$$\sum_{n=2}^{\infty} \frac{n^2+3}{n^4-2} = \sum_{m=1}^{\infty} \frac{(m+1)^2+3}{(m+1)^4-2} = \sum_{m=1}^{\infty} \frac{m^2+2m+4}{m^4+4m^2+6m^2+4m-1}.$$

(Note here the change in the lower bound, as in the previous example.) The inequality in the numerators (we want to compare $m^2 + 2m + 4$ with m^2) still goes the wrong way, but we can take care of this by using a slightly different inequality:

$$m^2 + 2m + 4 \leqslant m^2 + 2m^2 + 4m^2 = 7m^2$$

for $m \geqslant 1$. The inequality in the denominators does go the right way:

$$m^4 + 4m^2 + 6m^2 + 4m - 1 \geqslant m^4.$$

Since we have made the numerators larger and the denominators smaller, we have made the fractions larger, and thus

$$\sum_{m=1}^{\infty} \frac{m^2+2m+4}{m^4+4m^2+6m^2+4m-1} \leqslant \sum_{m=1}^{\infty} \frac{7m^2}{m^4} = \sum_{m=1}^{\infty} \frac{7}{m^2},$$

which implies by the Comparison Test that the series in question converges, because $\sum 7/m^2 = 7 \sum 1/m^2$ is a convergent p-series. ●

Our next example doesn't require reindexing, but does require a clever bound for $\ln n$. So far we have used the facts that $\ln n \leqslant n$ for $n \geqslant 2$ (in Example 1) and $\ln n \geqslant 1$ for $n \geqslant 3$ (in Example 2). In fact, a much stronger upper bound holds. Let p be any positive real number. Then by l'Hôpital's Rule, we have

$$\lim_{x\to\infty} \frac{\ln x}{x^p} = \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0.$$

Recalling a fact from Section 2.1, this means that

$$\lim_{n\to\infty} \frac{\ln n}{n^p} = 0$$

for *every* $p > 0$. This in turn means that for every $p > 0$, $\ln n \leqslant n^p$ for sufficiently large n, a handy fact to have around for comparisons, as we demonstrate next.

Example 9. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{\sqrt{(n+3)^5}}$ converge or diverge?

Solution. As we showed above, $\ln n \leqslant n^{1/4}$ for sufficiently large n (we could give a smaller bound, but $1/4$ is good enough here) and $(n+3)^5 \geqslant n^5$, we can use the comparison

$$\frac{n \ln n}{\sqrt{(n+3)^5}} \leqslant \frac{n^{1+1/4}}{n^{5/2}} = \frac{1}{n^{5/4}}.$$

Because $\sum 1/n^{5/4}$ is a convergent p-series, $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{\sqrt{(n+3)^5}}$ converges by the Comparison Test. ●

Our last example is considerably trickier than the previous examples. The reader should pay attention to the two themes it demonstrates: first, when dealing with a variable in an exponent, it is a good idea to use e and natural log, and second, no matter how slowly a function (such as $\ln \ln n$) goes to infinity, it must eventually grow larger than 2!

Example 10. Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ converge or diverge?

Solution. As the terms have a variable in the exponent, we first manipulate the using e and $\ln$:

$$(\ln n)^{\ln n} = e^{\ln\left(\ln n^{\ln n}\right)} = e^{\ln n \ln \ln n} = n^{\ln \ln n}.$$

We now need to test $\displaystyle\sum_{n=2} \frac{1}{n^{\ln \ln n}}$ for convergence. The approach from here on is similar to Example 4: for $n \geqslant e^{e^2} \approx 1618.18$ (i.e., for large n), we have $\ln \ln n \geqslant 2$, so

$$\frac{1}{n^{\ln \ln n}} \leqslant \frac{1}{n^2},$$

and thus $\displaystyle\sum_{n=2} \frac{1}{\ln n^{\ln n}}$ converges by comparison to the convergent p-series $\sum 1/n^2$. ●

If a series converges by the Comparison Test, then we have the following remainder estimate, which we conclude the section with.

The Comparison Test Remainder Estimate. Let $\sum a_n$ and $\sum b_n$ be series with positive terms such that $a_n \leqslant b_n$ for $n \geqslant N$. Then for $n \geqslant N$, the error in the nth partial sum of $\sum a_n$, s_n, is bounded by $b_{n+1} + b_{n+2} + \cdots$:

$$\left| s_n - \sum_{n=1}^{\infty} a_n \right| \leqslant b_{n+1} + b_{n+2} + \cdots .$$

Proof. By definition,

$$\left|s_n - \sum_{n=1}^{\infty} a_n\right| = |-a_{n+1} - a_{n+2} - \cdots| .$$

Now because the terms of $\sum a_n$ are positive, this is $a_{n+1} + a_{n+2} + \cdots$, and since we have assumed that $n \geqslant N$,

$$a_{n+1} + a_{n+2} + \cdots \leqslant b_{n+1} + b_{n+2} + \cdots ,$$

proving the estimate. □

Example 11. How many terms are needed to approximate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ to within $1/10$?

Solution. We use the comparison

$$\frac{1}{2^n + n} \leqslant \left(\frac{1}{2}\right)^n$$

for all $n \geqslant 1$ to bound the error in approximating $\sum \frac{1}{2^n + n}$. The first partial sum may not be a good enough approximation:

$$\left|s_1 - \sum_{n=1}^{\infty} \frac{1}{2^n + n}\right| \leqslant \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{\left(\frac{1}{2}\right)^2}{1 - \frac{1}{2}} = \frac{1}{2}.$$

The second and third partial sums are also not guaranteed to be as close to the true sum as required:

$$\left|s_2 - \sum_{n=1}^{\infty} \frac{1}{2^n + n}\right| \leqslant \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots = \frac{\left(\frac{1}{2}\right)^3}{1 - \frac{1}{2}} = \frac{1}{4},$$

$$\left|s_3 - \sum_{n=1}^{\infty} \frac{1}{2^n + n}\right| \leqslant \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \cdots = \frac{\left(\frac{1}{2}\right)^4}{1 - \frac{1}{2}} = \frac{1}{8},$$

but the fourth partial sum is within $1/10$:

$$\left|s_4 - \sum_{n=1}^{\infty} \frac{1}{2^n + n}\right| \leqslant \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \cdots = \frac{\left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}} = \frac{1}{16}.$$

Therefore the answer is that 4 terms will certainly approximate the series within $1/10$. ●

EXERCISES FOR SECTION 2.5

In Exercises 1–4, assume that $\sum a_n$ and $\sum b_n$ are both series with positive terms.

1. If $a_n \leqslant b_n$ for sufficiently large n and $\sum b_n$ is convergent, what can you say about $\sum a_n$?

2. If $a_n \leqslant b_n$ for sufficiently large n and $\sum b_n$ is divergent, what can you say about $\sum a_n$?

3. If $a_n \geqslant b_n$ for sufficiently large n and $\sum b_n$ is convergent, what can you say about $\sum a_n$?

4. If $a_n \geqslant b_n$ for sufficiently large n and $\sum b_n$ is divergent, what can you say about $\sum a_n$?

Determine if the series in Exercises 5–18 converge or diverge.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n-3}{n^3}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+7}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+4}}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{4}{\sqrt[3]{n^5+8}}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{9}{3^n+1}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{\sqrt{n^3+1}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3-1}{n^5+1}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{2-\sin(n)}{n^{5/4}}$

15. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n+5^n}$

17. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{3n^2}\right)^2$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

Suppose that $\sum a_n$ is a convergent series with positive terms. Determine whether the series listed in Exercises 19–22 necessarily converge. If a series doesn't necessarily converge, give an example of a convergent series $\sum a_n$ with positive terms for which it diverges. It may be helpful to remember that there are only finitely many values of a_n at least 1, so these have no affect on the convergence of the series.

19. $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n} a_n$

21. $\displaystyle\sum_{n=1}^{\infty} n a_n$

22. $\displaystyle\sum_{n=1}^{\infty} a_n \sin n$

23. $\displaystyle\sum_{n=1}^{\infty} a_n^2$

24. $\displaystyle\sum_{n=1}^{\infty} \sqrt{a_n}$

In Exercises 25–28, use reindexing like we did in Examples 5–8 to determine if the given series converge or diverge.

25. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-3}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^2}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 2^n - n}$

Using the Comparison Test, determine if the series in Exercises 29–38 converge or diverge.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+\sqrt{n+1}}}$

31. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^{n^2}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{2n(\ln n)^4}{\sqrt{n^4+4}}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n(n+1)}\sqrt{\frac{\ln n}{n}}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln\ln n)^{\ln n}}$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{\ln\ln n}}$

37. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$

38. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

39. Construct an example showing that the Comparison Test need not hold if $\sum a_n$ and $\sum b_n$ are not required to have positive terms.

40. If $a_n, b_n \geqslant 0$ and $\sum a_n^2$ and $\sum b_n^2$ both converge, show that the series $\sum a_n b_n$ converges.

41. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{p-\sin n}}$$

converges for $p > 2$. What about when $p = 2$?

Another way to deal with problems like Exercises 25–28 is to apply the following test.

The Limit Comparison Test. Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite number, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

Exercise 42 leads you through the proof of the Limit Comparison Test. After that, Exercises 43–48 present applications, while Exercises 49 and 50 extend the Limit Comparison Theorem to the case where the limit is 0 or ∞.

◆ **42.** Suppose that $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is a finite number. Therefore there are positive numbers m and M with $m < c < M$ such that $m < \dfrac{a_n}{b_n} < M$ for all large n. Use this inequality and the Comparison Test to derivate the Limit Comparison Test.

43. Show that the series

$$\sum_{n=2}^{\infty} \frac{n^2+3}{n^4-2}$$

from Example 8 converges using the Limit Comparison Test.

44. Does $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ converge or diverge?

45. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-2n+1}{\sqrt[5]{n^1 1+11n}}$ converge or diverge?

46. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-2n+1}{\sqrt[5]{n^9+11n}}$ converge or diverge?

◆ **47.** Suppose that $a_n \geqslant 0$ and $a_n \to 0$. Show that $\sum \sin a_n$ converges if and only if $\sum a_n$ converges.

◆ **48.** Suppose that $0 \leqslant a_n < 1$ for all n. Prove that $\sum \arcsin a_n$ converges if and only if $\sum a_n$ converges.

◆ **49.** Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If $a_n/b_n \to 0$ and $\sum b_n$ converges, prove that $\sum a_n$ converges.

◆ **50.** Let $\sum a_n$ and $\sum b_n$ be series with positive terms. If $a_n/b_n \to \infty$ and $\sum b_n$ diverges, prove that $\sum a_n$ diverges.

ANSWERS TO SELECTED EXERCISES, SECTION 2.5

1. $\sum a_n$ converges, by the Comparison Test
3. You cannot conclude anything
5. Converges by comparison to $\sum {}^{n}/_{n^3} = \sum {}^{1}/_{n^2}$
7. Converges by comparison to $\sum {}^{1}/_{\sqrt{n^3}} = \sum {}^{1}/_{n^{3/2}}$
9. Converges by comparison to $\sum {}^{n}/_{n^{5/2}} = \sum {}^{1}/_{n^{3/2}}$
11. Converges by comparison to $\dfrac{\pi}{2} \sum \dfrac{1}{n^{3/2}}$
13. Converges by comparison to $\sum {}^{1}/_{n^2}$
15. Diverges by comparison to $\sum {}^{1}/_{\sqrt{n}}$
17. Converges by comparison to $\sum {}^{1}/_{9n^2}$
19. Converges by the Comparison Test: $a_n \leqslant \dfrac{a_n}{n}$
21. Need not converge, consider taking $a_n = {}^{1}/_{n^2}$
23. Since $\sum a_n$ converges, $a_n \leqslant 1$ for sufficiently large n. For these values of n, $a_n^2 \leqslant a_n$, so $\sum a_n^2$ converges by the Comparison Test

2.6. THE RATIO TEST

We now know how to handle series which we can integrate (the Integral Test), and series which are similar to geometric or p-series (the Comparison Test), but of course there are a great many series for which these two tests are not ideally suited, for example, the series

$$\sum_{n=1}^{\infty} \frac{4^n}{n!}.$$

Integrating the terms of this series would be difficult, especially since the first step would be to find a continuous function which agrees with $n!$ (this can be done, but the solution is not easy). We could try a comparison, but again, the solution is not particular obvious (indeed, those readers who solved Exercise 37 of the last section should feel proud). Instead, the simplest approach to such a series is the following test due to Jean le Rond d'Alembert (1717–1783).

The Ratio Test. Suppose that $\sum a_n$ is a series with positive terms and let $L = \lim_{n\to\infty} a_{n+1}/a_n$.

- If $L < 1$ then $\sum a_n$ converges.
- If $L > 1$ then $\sum a_n$ diverges.
- If $L = 1$ or the limit does not exist then the Ratio Test is inconclusive.

You shold think of the Ratio Test as a generalization of the Geometric Series Test. For example, if $\{a_n\} = \{ar^n\}$ is a geometric sequence then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$

and we know these series converge if and only if $|r| < 1$. (Note that we will only consider positive series here; we deal with mixed series in the next section.) In fact, the proof of the Ratio Test is little more than an application on the Comparison Test.

Proof. If $L > 1$ then the sequence $\{a_n\}$ is increasing (for sufficiently large n), and therefore the series diverges by the Test for Divergence.

Now suppose that $L < 1$. Choose a number r sandwiched between L and 1: $L < r < 1$. Because $a_{n+1}/a_n \to L$, there is some integer N such that

$$0 \leqslant a_{n+1}/a_n \leqslant r$$

for all $n \geqslant N$. Set $a = a_N$. Then we have

$$a_{N+1} \leqslant r a_N = ar,$$

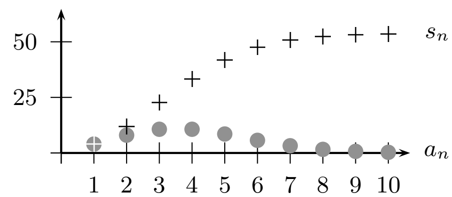
and

$$a_{N+2} \leqslant r a_{N+1} < ar^2,$$

and in general, $a_{N+k} \leqslant ar^k$. Therefore for sufficiently large n (namely, $n \geqslant N$), the terms of the series $\sum a_n$ are bounded by the terms of a convergent geometric series (since $0 < r < 1$), and so $\sum a_n$ converges by the Comparison Test. □

Since the Ratio Test involves a ratio, it is particularly effective when series contain factorials, as our first example does.

Example 1. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$ converge or diverge?



Solution. First we compute L:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{4^{n+1}}{(n+1)!}}{\frac{4^n}{n!}} = \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{4}{n+1} = 0.$$

Since $L = 0$, this series converges by the Ratio Test. ●

It is important to note that the Ratio Test is *always* inconclusive for series of the form $\sum \frac{\text{polynomial}}{\text{polynomial}}$. As an example, we consider the harmonic series and $\sum 1/n^2$.

Example 2. Show that the Ratio Test is inconclusive for $\sum 1/n$ and $\sum 1/n^2$.

Solution. For the harmonic series, we have

$$L = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1}.$$

In order to evaluate this limit, remember that we factor out the highest order term:

$$L = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{1}{1+\frac{1}{n}} = 1,$$

so the test is inconclusive.

The series $\sum 1/n^2$ fails similarly:

$$L = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2},$$

and again we factor out the highest order term, leaving

$$L = \lim_{n\to\infty} \frac{n^2}{n^2} \cdot \frac{1}{(1+\frac{1}{n})^2} = 1,$$

so neither series can be handled by the Ratio Test. ●

As Example 2 demonstrates, knowing that $a_{n+1}/a_n < 1$ is *not* enough to conclude that the sequence converges; we must know that the *limit* of this ratio is less than 1.

Example 3. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n4^{2n+1}}$ converge or diverge?

Solution. The ratio between consecutive terms is

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{10^{n+1}}{(n+1)4^{2n+3}}}{\dfrac{10^n}{n4^{2n+1}}} = \frac{10n}{4^2(n+1)} \to \frac{10}{16}$$

as $n \to \infty$. Since this limit is less than 1, we can conclude that the series converges by the Ratio Test. ●

The last example could also be handled by the Comparison Test, since

$$\frac{10^n}{n4^{2n+1}} \leqslant \frac{10^n}{4^{2n+1}} = \frac{1}{4}\left(\frac{10}{16}\right)^n,$$

so the series converges by comparison with a convergent geometric series. However, what if we moved the n from the denominator to the numerator:

$$\sum_{n=1}^{\infty} \frac{n10^n}{4^{2n+1}}?$$

Now the inequality in the comparison goes the wrong way, making the Comparison Test much harder to use. On the other hand, the limit in the Ratio Test is unchanged (you should check this for yourself). In general, it is usually a good idea to try the Ratio Test on all series with exponentials (like 10^n) or factorials.

Example 4. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{2^n n!}$ converge or diverge?

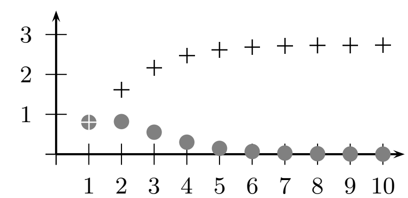
Solution. Here the ratio between consecutive terms is

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(2n+2)!}{2^{n+1}(n+1)!}}{\dfrac{(2n)!}{2^n n!}} = \frac{(2n+2)(2n+1)}{2(n+1)} = 2n+1 \to \infty$$

as $n \to \infty$. Since this limit is greater than 1 (or any other number, for that matter), the series diverges by the Ratio Test. ●

Our last example could be done using the Comparison Test (how?), but it is (probably) easier to use the Ratio Test.

Example 5. Does the series $\sum_{n=1}^{\infty} \frac{n^2+2n+1}{3^n+2}$ converge or diverge?



Solution. In this case the ratio between consecutive terms is

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)^2+2(n+1)+1}{3^{n+1}+2}}{\dfrac{n^2+2n+1}{3^n+2}} = \left(\frac{(n+1)^2+2(n+1)+1}{n^2+2n+1}\right)\left(\frac{3^n+2}{3^{n+1}+2}\right),$$

so pulling out the highest order terms, we have

$$\frac{a_{n+1}}{a_n} = \left(\frac{n^2}{n^2} \cdot \frac{\left(1+\frac{1}{n}\right)^2 + 2\left(\frac{1}{n}+\frac{1}{n^2}\right)+\frac{1}{n^2}}{1+2\frac{1}{n}+\frac{1}{n^2}}\right)\left(\frac{3^n}{3^{n+1}} \cdot \frac{1+\frac{2}{3^n}}{1+\frac{2}{3^{n+1}}}\right) \to \frac{1}{3}$$

as $n \to \infty$. Because this limit is less than 1, the series converges by the Ratio Test. ●

EXERCISES FOR SECTION 2.6

Exercises 1–4 give various values of

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

In each case, state what you conclude from the Ratio Test about the series $\sum a_n$.

1. $L = 2$

2. $L = 1$

3. $L = 1/2$

4. $L = \infty$

In Exercises 5–16, first compute

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n},$$

and then use the Ratio Test to determine if the given series converge or diverge.

5. $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

6. $\sum_{n=1}^{\infty} \frac{7^{n+2}}{2n6^n}$

7. $\sum_{n=1}^{\infty} \frac{n3^n}{n+2}$

$\boxed{8.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n3^n}{n+4^n}$

$\boxed{9.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

$\boxed{10.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n\sqrt{n}}{n!}$

$\boxed{11.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$\boxed{12.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

$\boxed{13.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{\sqrt{n!}}$

$\boxed{14.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{99^n\sqrt{n!}}$

$\boxed{15.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$\boxed{16.}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

17. Find a sequence $\{a_n\}$ of positive (in particular, nonzero) numbers such that both $\sum a_n$ and $\sum 1/a_n$ diverge.

18. Is there a sequence $\{a_n\}$ satisfying the conditions of the previous problem such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists and is not equal to 1?

A stronger test than the Ratio Test, proved by Augustin Louis Cauchy (1789–1857), is the following.

The Root Test. Suppose that $a_n \geqslant 0$ for all n and let $L = \lim_{n\to\infty} \sqrt[n]{a_n}$. The series $\sum_{n=1}^{\infty} a_n$ converges if $L < 1$ and diverges if $L > 1$. (If $L = 1$ then the Root Test is inconclusive.)

Our first task is to prove this result.

◆ **19.** Copying the beginning of the proof of the Ratio Test, give a proof of the Root Test.

Use the Root Test to determine if the series in Exercises 20–26 converge or diverge.

20. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n}{5n}\right)^{4n}$

21. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+n}\right)^{n}$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

23. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n}$

24. $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n}$

25. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

26. $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

Exercises 27 and 28 show that the Root Test is a stronger test than the Ratio Test.

◆ **27.** Show that the Root Test can handle any series that the Ratio Test can handle by proving that if $L = \lim_{n\to\infty} a_{n+1}/a_n$ exists then $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

◆ **28.** Show that there are series that the Root Test can handle but that the Ratio Test cannot handle by considering the series $\sum a_n$ where

$$a_n = \begin{cases} n/2^n & \text{if } n \text{ is odd,} \\ 1/2^n & \text{if } n \text{ is even.} \end{cases}$$

In some cases where the ratio and root tests are inconclusive, the following test due to Joseph Raabe (1801–1859) can prove useful.

Raabe's Test. Suppose that $\{a_n\}$ is a positive series. If there is some choice of $p > 1$ such that

$$\frac{a_{n+1}}{a_n} < 1 - \frac{p}{n}$$

for all large n, then $\sum_{n=1}^{\infty} a_n$ converges.

Exercises 29–31 ask you to prove Raabe's Test, while Exercises 32 and 33 consider an application of the test.

◆ **29.** Show that if $p > 1$ and $0 < x < 1$ then

$$1 - px \leqslant (1-x)^p.$$

This is called *Bernoulli's inequality*, after Johann Bernoulli (1667–1748). *Hint:* Set $f(x) = px + (1-x)^p$. Show that $f(0) = 1$ and $f'(x) \geqslant 0$ for $0 < x < 1$. Conclude from this that $f(x) \geqslant 1$ for all $0 < x < 1$.

◆ **30.** Assuming that the hypotheses of Raabe's Test hold and using Exercise 29, show that

$$\frac{a_{n+1}}{a_n} \leqslant \left(1 - \frac{1}{n}\right)^p = \frac{b_{n+1}}{b_n}$$

where $b_n = 1/(n-1)^p$.

◆ **31.** Rewrite the inequality derived in Exercise 30 as

$$\frac{a_{n+1}}{b_{n+1}} \leqslant \frac{b_{n+1}}{b_n},$$

use this to show that $a_n \leqslant M b_n$ for some positive number M and all large n, and use this to prove Raabe's Test.

◆ **32.** Show that the Ratio Test is inconclusive for the series

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}{4 \cdot 6 \cdot 8 \cdot \cdots \cdot (2k+2)}.$$

◆ **33.** Use Raabe's Test to prove that the series in Exercise 32 converges.

ANSWERS TO SELECTED EXERCISES, SECTION 2.6

1. The series diverges
3. The series converges
5. $L = 2/3$, so the series converges by the Ratio Test.
7. $L = 3$, so the series diverges by the Ratio Test.
9. $\dfrac{a_{n+1}}{a_n} = \dfrac{1}{n+1} \to 0$ as $n \to \infty$, so the series converges by the Ratio Test.
11. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)(n+1)}{(2n+2)(2n+1)} \to \dfrac{1}{4}$ as $n \to \infty$, so the series converges by the Ratio Test.
13. $\dfrac{a_{n+1}}{a_n} = \sqrt{n+1} \to \infty$ as $n \to \infty$, so the series diverges by the Ratio Test.
15. The ratio here is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} n!}{n^n (n+1)!} = \frac{(n+1)^{n+1}}{(n+1)n^n} = \frac{n^n}{(n+1)^n} = \left(1 + \frac{1}{n}\right)^n.$$

Recall from Example 11 of Section 2.1 that the limit of this ratio is $L = e$, so the series diverges by the Ratio Test because $e > 1$.

2.7. THE ALTERNATING SERIES TEST

We have focused almost exclusively on series with positive terms up to this point. In this short section we begin to delve into series with both positive and negative terms, presenting a test which works for many series whose terms alternate in sign.

The Alternating Series Test. Suppose that the sequence $\{b_n\}$ satisfies the three conditions:

- $b_n \geqslant 0$ for sufficiently large n,
- $b_{n+1} \leqslant b_n$ for sufficiently large n (i.e., $\{b_n\}$ is monotonically decreasing), and
- $b_n \to 0$ as $n \to \infty$.

Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

converges.

While we have stated the test with $(-1)^{n+1}$, it of course applies if the terms involve $(-1)^n$ instead (or $\cos n\pi$, since this is just a convoluted way to write $(-1)^n$). Also, notice that the Alternating Series Test can *not* be used to show that a series diverges (see Example 2).

Proof of the Alternating Series Test. Assume that the sequence $\{b_n\}$ is positive and decreasing for all n, and that it has limit 0. By the Tail Observation of Section 2.2, if we can prove that these series converge, the full Alternating Series Test will follow.

Let s_n denote the nth partial sum of this series. We have

$$s_{2n} = (b_1 - b_2) + (b_3 - b_4) + \cdots + (b_{2n-1} - b_{2n}).$$

Because $\{b_n\}$ is monotonically decreasing, $b_{2n-1} - b_{2n} \geqslant 0$ for all n, so this shows that s_{2n} is monotonically increasing. We can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n},$$

so since $b_{2n-2} - b_{2n-1} \geqslant 0$, $s_{2n} < b_1$. Thus the sequence $\{s_{2n}\}$ has a limit by the Monotone Convergence Theorem. Let $L = \lim_{n\to\infty} s_{2n}$. Now we consider the odd partial sums: $s_{2n+1} = s_{2n} + b_{2n+1}$, so

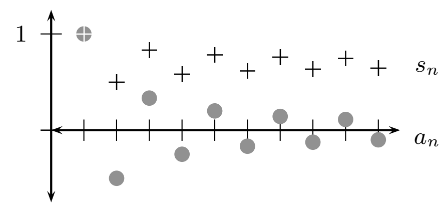
$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_n = L,$$

because $b_n \to 0$ by our hypotheses. Since both the even and odd partial sums converge to the same value, the sum of the series exists. □

Example 1 (The Alternating Harmonic Series, again). Show that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges using the Alternating Series Test.



Solution. The sequence $\{1/n\}$ is positive, monotonically decreasing, and has limit 0, so the alternating harmonic series converges by the Alternating Series Test. ●

Example 2. Does the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+3}{3n+4}$ converge or diverge?

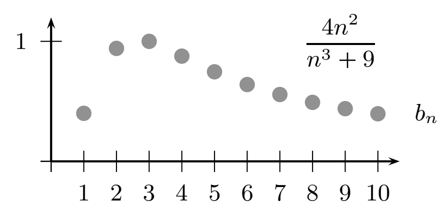
Solution. This series does alternate in sign, and $(2n+3)/(3n+4)$ is decreasing, but

$$(2n+3)/(3n+4) \to 2/3 \neq 0,$$

so the series diverges by the Test for Divergence. ●

Note that in the solution of Example 2, we did not appeal to the Alternating Series Test, but instead used the Test for Divergence. The Alternating Series Test *never* shows that series diverge.

Example 3. Show that the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{4n^2}{n^3+9}$ converges.



Solution. This series alternates in sign, and $(4n^2)/(n^3+9) \to 0$, but it is not immediately obvious that the sequence $\{(4n^2)/(n^3+9)\}$ is decreasing. Indeed, its first three terms are increasing, as indicated in the plot. Of course, we only need the sequence to be monotonically decreasing for large n. To check this condition, we take a derivative:

$$\frac{d}{dx}\frac{4x^2}{x^3+9} = \frac{(x^3+9)(8x)-(4x^2)(3x^2)}{(x^3+9)^2} = \frac{-4x^4+72x}{(x^3+9)^2}.$$

This fraction is negative for large x, so the sequence $\{(4n^2)/(n^3+9)\}$ is decreasing for large n. Therefore the series converges by the Alternating Series Test. ●

The proof of the Alternating Series Test implies the following very simple bound on remainders of these series.

The Alternating Series Remainder Estimates. Suppose that the sequence $\{b_n\}$ satisfies the three conditions of the Alternating Series Test:

- $b_n \geqslant 0$,
- $b_{n+1} \leqslant b_n$, and
- $b_n \to 0$ as $n \to \infty$

for all $n \geqslant N$. Then if $n \geqslant N$, the error in the nth partial sum of $\sum(-1)^{n+1}b_n$ is bounded by b_{n+1}:

$$\left| s_n - \sum_{n=1}^{\infty} (-1)^{n+1} b_n \right| \leqslant b_{n+1}.$$

Example 4. How many terms of the alternating series must we add to approximate the true sum with error less than $1/10000$?

Solution. Since the alternating harmonic series $\sum(-1)^{n+1}/n$ satisfies the conditions of the Alternating Series Test for all $n \geqslant 0$, the Remainder Estimates show that

$$\left| s_n - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \right| \leqslant \frac{1}{n+1}.$$

Therefore, if we want the error to be less than $1/10000$, we need

$$\text{Error} \leqslant \frac{1}{n+1} < \frac{1}{10000},$$

so we need $n > 9999$, or in other words, $n \geqslant 10000$. ●

EXERCISES FOR SECTION 2.7

In Exercises 1–12, determine if the given series converge or diverge.

$\boxed{\textbf{1.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+7}$

$\boxed{\textbf{2.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n!}$

$\boxed{\textbf{3.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$

$\boxed{\textbf{4.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n}$

$\boxed{\textbf{5.}}$ $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n-1}{n}$

$\boxed{\textbf{6.}}$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n}$

$\boxed{\textbf{7.}}$ $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$

$\boxed{\textbf{8.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-n)^n}{n^2}$

$\boxed{\textbf{9.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-n)^n}{n^{3n}}$

$\boxed{\textbf{10.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\arctan n}$

$\boxed{\textbf{11.}}$ $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2}{n^3+1}$

$\boxed{\textbf{12.}}$ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-2}{n}\right)^{3n}$

For Exercises 13–16, first determine if the given series satisfies the conditions of the Alternating Series Test. Then, if the series does satisfy the conditions, decide how many terms need to be added in order to approximate the sum to within $1/1000$.

$\boxed{\textbf{13.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^6+1}$

$\boxed{\textbf{14.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$

$\boxed{\textbf{15.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2}$

$\boxed{\textbf{16.}}$ $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n}\right)^2$

Exercises 17–19 verify that the hypotheses of the Alternating Series Test are all necessary, in the sense that if any of them is removed, then the statement becomes false.

17. Construct a sequence $\{b_n\}$ which is monotonically decreasing with limit 0 such that $\sum(-1)^{n+1}b_n$ diverges. (I.e., b_n needn't be positive.)

18. Construct a sequence $\{b_n\}$ which is positive and monotonically decreasing such that $\sum(-1)^{n+1}b_n$ diverges. (I.e., b_n needn't have limit 0.)

19. Construct a sequence $\{b_n\}$ which is positive with limit 0 such that $\sum(-1)^{n+1}b_n$ diverges. (I.e., b_n needn't be monotonically decreasing.)

Dirichlet's Test, due to Johann Peter Gustav Lejeune Dirichlet (1805–1859), is a strengthening of the Alternating Series Test (as shown in Exercise 23).

Dirichlet's Test. If $\{b_n\}$ is a positive, eventually monotonically decreasing sequence with limit 0 and the partial sums of the series $\sum a_n$ are bounded, then $\sum a_n b_n$ converges.

Exercises 20–22 ask you to develop the proof of this theorem, while Exercises 23–27 ask you to apply the test.

◆ **20.** Let $s_n = a_1 + a_2 + \cdots + a_n$. Use the fact that $s_n - s_{n-1} = a_n$ to prove

$$\sum_{n=m+1}^{\infty} a_n b_n = -s_m b_{m+1} + \sum_{n=m+1}^{\infty} s_n(b_n - b_{n+1}).$$

(This formula is often referred to as *summation by parts*.)

◆ **21.** Let $\{a_n\}$ and $\{b_n\}$ be sequences satisfying the hypotheses of Dirichlet's Test. Use Exercise 20 to show that if the partial sums of the sequence $\{a_n\}$ are at most M then

$$\left|\sum_{n=m+1}^{\infty} a_n b_n\right| \leqslant 2M\bigl|b_{m+1}\bigr|.$$

◆ **22.** Use Exercise 21 to prove Dirichlet's Test.

◆ **23.** Show that Dirichlet's Test implies the Alternating Sign Test.

24. Suppose that $\{a_n\} = \{-2, 4, 1, -3, -2, 4, 1, -3, \dots\}$ and that $b_n = {}^1\!/n$. Does $\sum a_n b_n$ diverge, converge absolutely, or converge conditionally?

◆ **25.** Use the angle addition identity

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

to derive the identity

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta)\cos(\alpha+\beta).$$

◆ **26.** Use the identity derived in Exercise 25 to show that

$$2\left(\sin {}^\pi\!/4\right)\sum_{n=1}^{m}\sin n = \sum_{n=1}^{m}\left(\cos\left(n - {}^\pi\!/4\right) - \cos\left(n + {}^\pi\!/4\right)\right),$$

then show that this is equal to $\cos {}^\pi\!/4 - \cos\left(m + {}^\pi\!/4\right)$.

◆ **27.** Use Exercise 26 to show that the partial sums of $\sum \sin n$ are bounded, and then conclude from Dirichlet's Test that the series

$$\sum_{n=1}^{\infty}\frac{\sin n}{n}$$

converges.

ANSWERS TO SELECTED EXERCISES, SECTION 2.7

1. Converges by the Alternating Series Test
3. Converges by the Alternating Series Test
5. Diverges by the Test for Divergence:
$$\frac{n-1}{n} \to 1 \neq 0$$
as $n \to \infty$.
7. Converges by the Alternating Series Test
9. Converges by the Alternating Series Test
11. Converges by the Alternating Series Test. To see that $\{b_n\}$ is decreasing for sufficiently large n, take a derivative.
13. Alternating Series Test not applicable.
15. The Alternating Series Test is applicable. Using $n = 4$ will work to approximate the sum to within $1/1000$, because
$$\frac{1}{(5!)^2} = \frac{1}{14400} < \frac{1}{1000}.$$

2.8. ABSOLUTE VS. CONDITIONAL CONVERGENCE

We are now ready to examine the strange behavior of the alternating harmonic series we first observed in Section 2.2. Remember that we showed that the alternating harmonic series converged and then we went on to bound its sum. For a lower bound, we grouped the terms in pairs, observing that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots\\ &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\cdots\\ &\geqslant 1-\frac{1}{2}\\ &= \frac{1}{2}.\end{aligned}$$

While for an upper bound, we group the terms in different pairs, showing that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots\\ &= 1-\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)-\left(\frac{1}{6}-\frac{1}{7}\right)-\cdots\\ &\leqslant 1-\frac{1}{2}+\frac{1}{3}\\ &= \frac{5}{6}=0.8333\ldots.\end{aligned}$$

(In fact, that true sum is $\ln 2 \approx 0.69315$, see Exercises 46 and 47 of Section 2.4 or Exercise 24 of Section 3.2.)

Then we showed in Example 5 of Section 2.2 that by rearranging the terms of this series, we could get it to converge to a different sum:

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right)\geqslant\left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)=\frac{389}{420}>\frac{9}{10}.$$

Our first order of business in this section is to explore this phenomenon:

When are we allowed to rearrange the terms of a series without changing the sum?

We begin by looking at series with positive terms. If $\sum a_n$ is a convergent series with positive terms, are we allowed to rearrange the terms without changing the sum? Suppose $\sum b_n$ is such a rearrangement, and consider the partial sums of each series,

$$\begin{aligned}s_n &= a_1+a_2+a_3+\cdots,\\ t_n &= b_1+b_2+b_3+\cdots.\end{aligned}$$

We would like to figure out if $\sum a_n = \sum b_n$ which, by the very definition of series summation, is equivalent to $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n$. Because $\{a_n\}$ and $\{b_n\}$ are positive sequences, the sequences of partial sums $\{s_n\}$ and $\{t_n\}$ are both increasing. Now consider any value of n. Since the sequence $\{b_n\}$ is a rearrangement of the sequence $\{a_n\}$, there must be some number N so that each of the terms

$$a_1, a_2, \ldots, a_n$$

occurs in the list

$$b_1, b_2, \ldots, b_N.$$

Since all the terms are positive, for this value of N, we have

$$s_n = a_1 + a_2 + \cdots + a_n \leqslant b_1 + b_2 + \cdots + b_N = t_N.$$

This shows that every partial sum of $\sum a_n$ is less than or equal to some partial sum of $\sum b_n$. Of course, the same argument works with the roles of a_n and b_n interchanged, so every partial sum of $\sum b_n$ is less than or equal to some partial some of $\sum a_n$. This implies that the two series converge to the same value.

So we have made some progress: convergent series with positive terms can be rearranged without affecting their sums, but rearranging the alternating harmonic series can affect its sum. What is the difference between these two examples?

Intuitively, there are two different ways for a series to converge. First, the terms could just be really small. Indeed, this is the only way that a series with positive terms can converge. But then there is a second way, illustrated by the alternating harmonic series: the terms could cancel each other out. Our next definition attempts to make precise the notion of series that converge "because their terms are really small."

Absolute Convergence. The series $\sum a_n$ is said to *converge absolutely* if $\sum |a_n|$ converges.

The first thing we should verify is that absolutely convergent series actually, well, converge. Our next theorem says even more: rearrangements don't affect the sum of an absolutely convergent series.

The Absolute Convergence Theorem. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges. Moreover, every rearrangement of $\sum a_n$ converges to the same sum.

This first part of this theorem — that absolutely convergent series converge — follows from the Comparison Test and some basic facts about series, see Exercises 25–26. The second part is more complicated, and we omit its proof.

While we have defined absolute convergence in order to investigate rearranging series, this notion is very useful on its own. Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

Even though this series is very much like $\sum 1/n^2$, it is *not* a p-series, so we can't apply the p-series Test to it. Similarly, we can't apply the Integral Test or the Comparison Test, because those tests require series to have positive terms. However, it is easy to see that this series is absolutely convergent, from which it follows that the series converges by the Absolute Convergence Theorem:

Example 1. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely.

Solution. The series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent p-series, so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely by the Absolute Convergence Theorem. ●

Our next example is another stereotypical use of the Absolute Convergence Theorem. In general when trigonometric functions appear in a series, we need to test for absolute convergence and then make a comparison.

Example 2. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n^3+1}}$ converges absolutely.

Solution. First we take the absolute values of the terms,

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{\sqrt{n^3+1}} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n^3+1}}.$$

We may use any test we like on this series (although some, like the Ratio Test in this example, might not tell us anything). Because $|\sin n| \leqslant 1$, $\sqrt{n^3+1} \geqslant \sqrt{n^3} = n^{3/2}$, and the terms of this series are positive, we can compare it:

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n^3+1}} \leqslant \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

Therefore, $\sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n^3+1}}$ is convergent by comparison to a convergent p-series, so $\sum_{n=1}^{\infty} \frac{\sin n}{\sqrt{n^3+1}}$ is absolutely convergent by the Absolute Convergence Theorem. ●

We've identified a special type of convergent series, the absolutely convergent series. But what about the others? Intuitively, these are the series which converge only because their terms happen to cancel each other out. These series are called conditionally convergent.

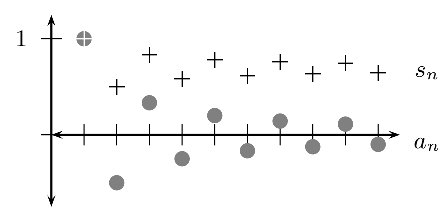
> **Conditional Convergence.** The series $\sum a_n$ is said to *converge conditionally* if $\sum a_n$ converges but $\sum |a_n|$ diverges.

If you want to show that the series $\sum a_n$ is conditionally convergent, it is important to note that this requires two steps. First you must show that $\sum a_n$ converges, and second, you must show that $\sum a_n$ is not absolutely convergent (in other words, that $\sum |a_n|$ diverges). Our first example of a conditionally convergent series should not come as a surprise.

Example 3 (The Alternating Harmonic Series, last time). Show that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is conditionally convergent.



Solution. The alternating harmonic series converges by the Alternating Series Test because the sequence $\{1/n\}$ is monotonically decreasing, positive, and has limit 0.

The alternating harmonic series does not converge absolutely because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

(the harmonic series) diverges. Therefore the alternating harmonic series is conditionally convergent. ●

Example 4. Show that the series $\sum_{n=1}^{\infty} (-1)^n \frac{4n^2}{n^3+9}$ converges conditionally.

Solution. We saw in Example 3 of the previous section that this series converges, so we only need to show that it does not converge absolutely. To test for absolute convergence, we take the absolute value:

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{4n^2}{n^3+9} \right| = \sum_{n=1}^{\infty} \frac{4n^2}{n^3+9}.$$

There are at least two different ways to show that this series diverges.

With the Integral Test: We must evaluate the integral

$$\int_1^{\infty} \frac{4x^2}{x^3+9}\,dx = \lim_{b\to\infty} \int_1^b \frac{4x^2}{x^3+9}\,dx.$$

Setting $u = x^3 + 9$ gives $du = 3x^2\,dx$, so $dx = du/3x^2$. Making these substitutions leaves us with

$$\lim_{b\to\infty} \int_{x=1}^{x=b} \frac{4}{3u}\,du = \lim_{b\to\infty} \frac{4}{3} \ln u \Big|_{x=1}^{x=b} = \lim_{b\to\infty} \frac{4}{3} \ln\left(b^3+9\right) - \frac{4}{3}\ln(10) = \infty,$$

so the series diverges by the Integral Test.

With the Comparison Test: Here we can use the bound

$$\frac{4x^2}{x^3+9} \geqslant \frac{4x^2}{x^3+9x^3} = \frac{4}{10x}$$

to see that the series diverges by comparison to $\frac{4}{10}\sum \frac{1}{n}$. ●

We've seen one example of how by rearranging the terms of the alternating harmonic series we can change its sum. What if we wanted to rearrange the series to make it sum to a specific number? Would that be possible? Yes! We begin with a specific example, and then discuss how to generalize this example.

Example 5. Rearrange the terms of the alternating harmonic series to get a series which converges to 1.

Solution. The positive terms of this series are

$$1 + 1/3 + 1/5 + 1/7 + 1/9 + \cdots$$

while the negative terms are

$$-1/2 - 1/4 - 1/6 - 1/8 - 1/10 - \cdots.$$

Note that both of these series diverge. By our Tail Observation of Section 2.2, this means that all tails of these series diverge as well.

Now, how are we going to rearrange the series to make it sum to 1? First, we make the series sum to *more* than 1:

$$1 + 1/3 \quad = \quad 1.3333\ldots \quad > \quad 1.$$

Next we use negative terms to make the series sum to *less* than 1:

$$1 + 1/3 - 1/2 \quad = \quad 0.8333\ldots \quad < \quad 1.$$

Then we use as many of the positive terms that we haven't used yet to make the series sum to more than 1 again:

$$1 + 1/3 - 1/2 + 1/5 \quad = \quad 1.0333\ldots \quad > \quad 1,$$

and then use negative terms to make it sum to less than 1:

$$1 + 1/3 - 1/2 + 1/5 - 1/4 \quad = \quad 0.7833\ldots \quad < \quad 1.$$

In doing so we obtain with the rearrangement

$$\mathrlap{\overbrace{}^{>1}}\mathrlap{\overbrace{}^{>1}}\mathrlap{\underbrace{}_{<1}}\mathrlap{\underbrace{}_{<1}}\mathrlap{\underbrace{}_{<1}}\overbrace{1 + 1/3}^{>1} - 1/2 + 1/5 - 1/4 + 1/7 + 1/9 - 1/6 + \cdots,$$

but does this rearrangement really sum to 1? Mightn't we get stuck at some point and not be able to continue the construction?

We certainly won't get stuck under 1. No matter how many of the positive terms we have used up to that point, the positive terms that we have remaining will sum to ∞ (they are a tail of the divergent series $1 + 1/3 + 1/5 + \cdots$). Similarly, we can't get stuck over 1. Therefore, we will be able to create partial sums which are alternatively greater than 1 and less than 1, but will they converge to 1? This follows because the terms we are using are getting smaller. If we add the term $1/93$ to get a partial sum over 1, that means that our previous partial sum was *under* 1, which means that the new partial sum is within $1/93$ of 1. As we use up the larger terms of the series, we will have no choice but to get closer and closer to 1. Therefore this construction (if we carried it out forever) would indeed yield a sum of 1. ●

Now we know we can rearrange the alternating harmonic series to sum to 1, but what was so special about 1? Absolutely nothing, in fact. If you replace the number 1 in the previous argument with any other number S, everything works just fine. Now, what was so special about the alternating harmonic series? First, we needed that the positive terms formed a divergent series and that the negative terms formed a divergent series (so that our partial sums wouldn't get stuck under or above 1). This fact is actually true for all conditionally convergent series though (why?). Then we needed that the terms get increasingly small, to prove that the limit of the partial sums was really 1. But if the terms didn't get close to 0, then the series would diverge by the Test for Divergence, so this is true for all conditionally convergent series as well.

We have just sketched the proof of a famous theorem of Bernhard Riemann (1826–1866).

> **Reimann's Rearrangement Theorem.** If $\sum a_n$ is a conditionally convergent series and S is any real number, then there is a rearrangment of $\sum s_n$ which converges to S.

We conclude with a more formal proof.

Proof of Reimann's Rearrangement Theorem We begin by dividing the terms of the sequence $\{a_n\}$ into two groups. Let $\{b_n\}$ denote the sequence which contains the positive terms of $\{a_n\}$ and $\{c_n\}$ denote the sequence which contains the negative terms of $\{a_n\}$.

Clearly $\sum |a_n| = \sum b_n - \sum c_n$, so since $\sum a_n$ is not absolutely convergent, at least one of $\sum b_n$ or $\sum c_n$ must diverge. But $\sum a_n$ is conditionally convergent, so if $\sum b_n$ diverges (to ∞), $\sum c_n$ must also diverge (to $-\infty$), and vice versa. Therefore both $\sum b_n$ and $\sum c_n$ diverge, to ∞ and $-\infty$, respectively.

Suppose that a target sum S is given. Choose N_1 to be the minimal integer such that

$$b_1 + \cdots + b_{N_1} > S$$

(note that if S is negative, then N_1 will be 0). We can be certain that N_1 exists because $\sum b_n$ diverges to ∞. Note that, because $b_1 + \cdots + b_{N_1-1} < S$, $b_1 + \cdots + b_{N_1}$ is within b_{N_1} of S. Next choose M_1 minimal so that

$$(b_1 + \cdots + b_{N_1}) + (c_1 + \cdots + c_{M_1}) < S.$$

Again, M_1 must exist because $\sum c_n$ diverges to $-\infty$. Note that any partial sum of the form $(b_1 + \cdots + b_{N_1}) + (c_1 + \cdots + c_n)$ where $n \leqslant M_1$ must be within b_{N_1} of S. Next choose N_2 so that

$$(b_1 + \cdots + b_{N_1}) + (c_1 + \cdots + c_{M_1}) + (b_{N_1+1} + \cdots + b_{N_2}) > S.$$

Next we choose M_2 so that

$$(b_1 + \cdots + b_{N_1}) + (c_1 + \cdots + c_{M_1}) + (b_{N_1+1} + \cdots + b_{N_2}) - (c_{M_1+1} + \cdots + c_{M_2}) < S.$$

Continuing in this manner, define $N_3, M_3, \ldots$. At each stage, our partial sums will be within b_{N_i} or c_{M_i} of S for some i, and so since $b_n \to \infty$ and $c_n \to \infty$ (why?) we obtain a rearrangement that sums to S, as desired. $\square$

EXERCISES FOR SECTION 2.8

For Exercises 1–12, determine if the given series converge absolutely, converge conditionally, or diverge. Note that these exercises may require the use of all the tests we have learned thus far.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n+1)!}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln n}$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n)(\ln n)(\ln \ln n)}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos(1/n)$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n^4 + \sin n^5}{n^2}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{e^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln \ln \ln \ln \ln n}$

Determine if the series in Exercises 13–16 converge at $x = -1$ and at $x = 5$.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n \sqrt{n}}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n n^2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{3^n \sqrt{n}}$

Determine whether the series in Exercises 17–20 converge at $x = -2$ and at $x = 4$.

17. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{\sqrt{n^9+5}} \left(\frac{x-1}{3}\right)^n$

18. $\displaystyle\sum_{n=0}^{\infty} (x-1)^2 \frac{n^3}{3^n n^7 + 2n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{2n+3^n} (x-1)^n$

20. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin n}{n^3} \left(\frac{x-1}{3}\right)^n$

Determine whether the series in Exercises 21–24 converge at $x = 0$ and at $x = 4$.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{2^n n^{3/2}}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln n} \left(\frac{x-2}{2}\right)^n$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(2-x)^n}{n 2^n + 2^n}$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n \sin n}{2^n}$

Exercises 25 and 26 prove the first part of the Absolute Convergence Theorem: absolutely convergent series converge.

25. Verify the inequality

$$0 \leqslant a_n + |a_n| \leqslant 2|a_n|$$

and use this to prove that if $\sum a_n$ is absolutely convergent, then the series $\sum (a_n + |a_n|)$ converges.

26. Use the conclusion of Exercise 25 and Exercise 27 from Section 2.2 to prove that all absolutely convergent series converge.

ANSWERS TO SELECTED EXERCISES, SECTION 2.8

1. Absolutely convergent (use a comparison on the absolute values)
3. Absolutely convergent (use the Ratio Test on the absolute values)
5. Absolutely convergent (use the Integral Test)
7. Conditionally convergent (use the Alternating Series Test, and then use a comparison on the absolute values)
9. Conditionally convergent (use the Alternating Series Test, and then use a comparison on the absolute values)
11. Absolutely convergent (use the Ratio Test on the absolute values)
13. Diverges at both $x = -1$ and $x = 5$
15. Converges at $x = -1$, diverges at $x = 5$

3. POWER SERIES

3.1. SERIES AS FUNCTIONS

At the end of Chapter 1 we saw that Taylor polynomials of "infinite degree" might be valuable for approximating functions. First we needed to understand what it meant to add infinitely many numbers together, a notion that we formalized and studied in Chapter 2. With these prerequisites covered, we return to the analysis of functions, beginning by studying *power series*, which are series involving powers of x, such as

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots {}^{\dagger}.$$

This power series is *centered* at $x = 0$. Our definition below is slightly more general.

> **Power Series.** A *power series centered at $x = a$* is a series of the form
>
> $$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots .$$

The first question we should ask is:

Given a power series, for what values of x does it converge?

As the next three examples show, the techniques we have developed to analyze series are capable of answering this question as well.

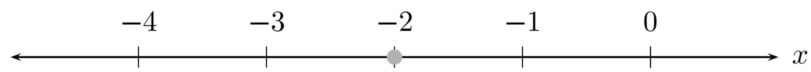
Example 1. Find the values of x for which the power series $\displaystyle\sum_{n=0}^{\infty} n!(x+2)^n$ converges and plot them on a number line.

[†] The reader may have noticed that we have switched (mostly) from sums starting at $n = 1$ to sums starting at $n = 0$. This is because when dealing with regular series, it is natural to index the first term as a_1, while with power series it is more convenient to index the terms based on the power of x.

Solution. We test the series for absolute convergence using the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!(x+2)^{n+1}}{n!(x+2)^n}\right| = |(n+1)(x+2)| \to \infty \text{ unless } x = -2.$$

Therefore the series converges only when $x = -2$, so our plot is a single point,



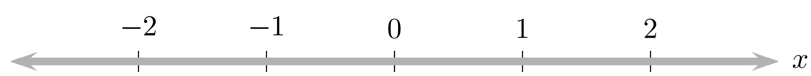
showing that the power series converges only at $x = -2$. ●

Example 2. For what values of x does the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$ converge?

Solution. We again test the series for absolute convergence using the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-1)^{n+1}}{(n+1)!}}{\frac{(x-1)^n}{n!}}\right| = \left|\frac{x-1}{n+1}\right| \to 0 \text{ for all } x.$$

Therefore this series converges (absolutely) for every x, so our number line contains all real numbers,



We can also write this as the interval $(-\infty, \infty)$, or we may simply express it as the set of all real numbers, $\mathbb{R}$. ●

Our third and final example is a bit more interesting.

Example 3. For what values of x does the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{(n+1)4^n}$ converge?

Solution. Again we begin by testing the series for absolute convergence with the Ratio Test, although in this case we will need to work more afterward:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x+3)^{n+1}}{(n+2)4^{n+1}}}{\frac{(x+3)^n}{(n+1)4^n}}\right| = \left|\frac{x+3}{4}\cdot\frac{n+2}{n+1}\right| \to \left|\frac{x+3}{4}\right|.$$

For what values of x is $\left|\frac{x+3}{4}\right| < 1$? This inequality can be rewritten as

$$-1 < \frac{x+3}{4} < 1,$$

or, simplifying,

$$-4 < x + 3 < 4,$$

so the given power series converges by the Ratio Test if $-7 < x < 1$, or in other words, if x lies in the interval $(-7, -1)$. The power series diverges if $x < -7$ or $x > 1$. But when $x = -7$ or $x = 1$, the Ratio Test is inconclusive, so we have to test these *endpoints* individually. This is typical for power series, not specific to this example.

Plugging in $x = -7$, our series simplifies to

$$\sum_{n=0}^{\infty} \frac{(-4)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

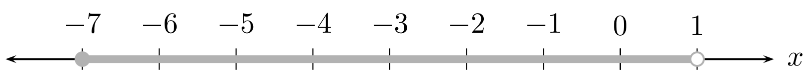
Since this is the alternating harmonic series, we know that it converges (conditionally). So, our power series converges at $x = -7$.

Plugging in $x = 1$, our series simplifies to

$$\sum_{n=0}^{\infty} \frac{4^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}.$$

This is the harmonic series which we know diverges, so our power series diverges at $x = 1$.

Putting this all together, the given power series converges if and only if $-7 \leqslant x < 1$, which we can also write as the interval $[-7, 1)$. The number line plot of this interval is:



(Here the closed circle means that $x = -7$ is included, while the open circle means that $x = -1$ is excluded.) ●

These examples have demonstrated three different types of convergence. As our next theorem shows, *every* power series exhibits one of these three behaviors.

Radius Theorem. Every power series $\sum c_n(x-a)^n$ satisfies one of the following:

(1) The series converges only when $x = a$, and this convergence is absolute.

(2) The series converges for all x, and this convergence is absolute.

(3) There is a number $R > 0$ such that the series converges absolutely when $|x - a| < R$ and diverges when $|x - a| > R$. Note that the series may converge absolutely, converge conditionally, or diverge when $|x - a| = R$.

The proof of the Radius Theorem is outlined in Exercises 40–45.

Case (1) of this theorem holds when the coefficients c_n are "large", while case (2) holds when these coefficients are "small". Case (3) holds for coefficients which lie somewhere in between these two extremes. Note that for series which satisfy case (3), the interval of convergence is *centered at* $x = a$. We call the number R in this case the *radius of convergence*. (In case (1) we might say that the radius of convergence is 0, while in case (2) we might say that it is ∞.)

When case (3) holds, there are four possibilities for the interval of convergence:

$$(a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R), \quad [a-R, a+R].$$

In order to decide which of these is the interval of convergence, we must test the endpoints one-by-one, as we did in Example 3. Therefore the general procedure for determining the interval of convergence of a given power series is:

1. Identity the center of the power series.

2. Use the Ratio Test to determine the radius of convergence. (There are rare instances in which the Ratio Test is not sufficient, in which case the Root Test should be used instead, see Exercises 20–26 of Section 2.6.)

3. If the series has a positive, finite radius of convergence (case (3)), then we need to test the endpoints $a - R$ and $a + R$. These two series may be tested with any method from the last chapter.

When a power series converges, it defines a function of x, so our next question is:

What can we say about functions defined as power series?

The short answer is that *inside its radius of convergence*, a power series can be treated like a long polynomial. In particular, we can differentiate power series like polynomials:

Term-by-Term Differentiation. Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

converges for all x in the interval $(a - R, a + R)$. Then f is differentiable for all values of x in the interval $(a - R, a + R)$, and

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots .$$

In particular, the radius of convergence of $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ is at least R.

This theorem says quite a lot about the behavior of a power series inside its radius of convergence. Not only can we differentiate such a power series, but the derivative has *at least as large a radius of convergence*. Well then, there's nothing stopping us from taking the derivative of this derivative, and so on. Therefore, inside its radius of convergence, a power series defines an *infinitely differentiable*, or *smooth*, function of x. Such functions are extremely well-behaved. For one, remember that in order to be differentiable, a function must first be continuous, so inside its radius of convergence, a power series defines a continuous function.

Integration can be handled the same way, by treating a power series as a long polynomial inside its radius of convergence.

Term-by-Term Integration. Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

converges for all x in the interval $(a-R, a+R)$. Then

$$\begin{aligned} \int f(x)\,dx &= \sum_{n=1}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C \\ &= c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots + C, \end{aligned}$$

and this series converges for all x in the interval $(a-R, a+R)$.

Term-by-term differentiation and integration should *not* seem obvious, and their justification takes quite a bit of work, even in more advanced courses. For now, we take them for granted.

EXERCISES FOR SECTION 3.1

Find the intervals of convergence of the series in Exercises 1–10.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[7]{n}}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+2)^3}$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n (x+2)^n}{(n+2)^3}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(4x)^n}{n^4}$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(-4x)^n}{n^4}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{(-4x+2)^n}{n^4}$

7. $\displaystyle\sum_{n=0}^{\infty} x^n n!$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n n!}{2^n}$

9. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x+2}{3n}\right)^n$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n+1)}{n!} x^n$

In Exercises 11–20, construct a power series with the given intervals of converge, or explain why one does not exist.

11. $(-2,2)$

12. $(-4,0)$

13. $[0,2]$

14. $(-\infty,\infty)$

15. $[0,\infty)$

16. $(1,\infty)$

17. $[3,7)$

18. $(3,7]$

19. $(-\infty,2]$

20. $\{7\}$

Rewrite the expressions in Exercises 21–24 as series in which the generic term involves x^n.

21. $\displaystyle\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$

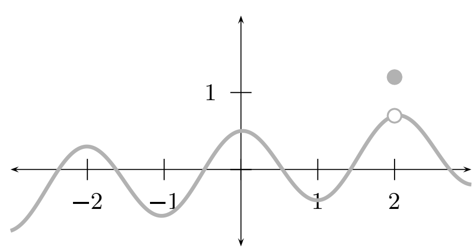
22. $\displaystyle\sum_{n=0}^{\infty} c_n x^{n+3}$

23. $\displaystyle\sum_{n=1}^{\infty} nc_n x^{n-1} + 2x\sum_{n=0}^{\infty} a_n x^n$

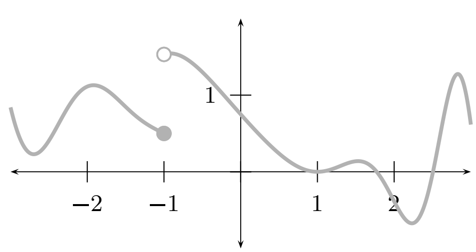
24. $\displaystyle x\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$

Explain why none of the functions plotted in Exercises 25–28 are equal to power series on the interval $(-3,3)$.

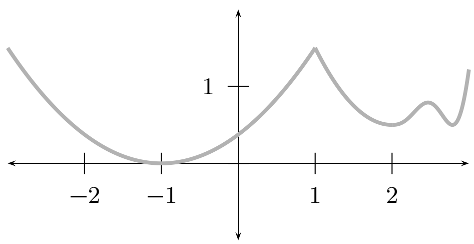
25.



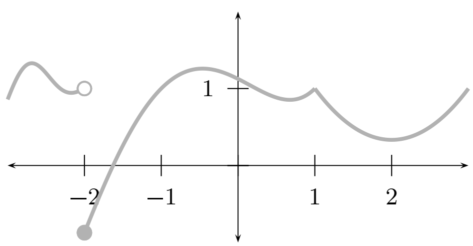
26.



27.



28.



29. Explain why the radius of convergence of the power series for $f(x) = \tan x$ centered at $a = 0$ is at most $\pi/2$.

30. Explain why a power series can converge conditionally for at most two points.

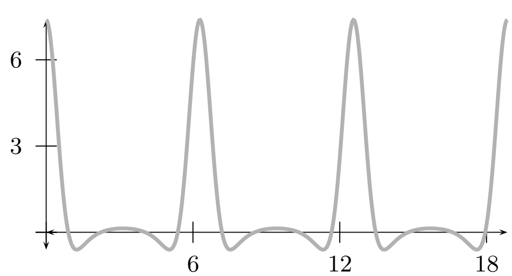
Exercises 31–33 concern the series

$$\sum_{n=0}^{\infty} \frac{2^n \cos nx}{n!}.$$

Note that this is *not* a power series. Below, the 100th partial sum,

$$\sum_{n=0}^{100} \frac{2^n \cos nx}{n!}$$

of this series is plotted.



31. Show that this series converges for all x.

32. Does this series define a periodic function of x, as the plot above seems to demonstrate?

33. Verify that the actual series is within $1/100$ of the partial sum plotted above for all values of x.

34. Suppose you know, from using the Ratio Test, that the radius of convergence of $\sum c_n x^n$ is $R = 6$. What is the radius of convergence of $\sum c_n n^3 x^n$?

35. Suppose you know, from using the Ratio Test, that the radius of convergence of $\sum c_n x^n$ is $R = 6$. What is the radius of convergence of $\sum c_n x^n / 3^n$?

36. Suppose that the radius of convergence of $\sum c_n x^n$ is $R \geqslant 1$. Then what is the radius of convergence of $\sum s_n x^n$ where $s_n = c_0 + c_1 + \cdots + c_n$?

37. Suppose that the radius of convergence of $\sum c_n x^n$ is $R < 1$. Then what is the radius of convergence of $\sum s_n x^n$ where $s_n = c_0 + c_1 + \cdots + c_n$?

Use the bounds given by Exercises 50 and 50 of Section 2.4 to find the radii of convergence of the series in Exercises 38 and 39.

38. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n} x^n$

39. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n!)}{\sqrt{3n^2 + 2n + 1}} x^n$

Exercises 40–45 detail the proof of the Radius Theorem. For simplicity, we assume that the series is centered at 0, that is, that $a = 0$, but the proof easily extends to other centers by making a change of variables, setting $y = x - a$.

40. Suppose that the power series $\sum c_n x^n$ converges at $x = s$. Prove that there is an integer N so that $|c_n| < 1/|s|^n$ for all $n \geqslant N$.

41. With N as in the previous exercise, prove that if $|x| < |s|$ and $n \geqslant N$, $|c_n x^n| < |x/s|^n$.

42. Using the previous two exercises and the Comparison Test, prove that if the power series $\sum c_n x^n$ converges at $x = s$ then it converges absolutely whenever $|x| < |s|$.

43. Prove that if the power series $\sum c_n x^n$ diverges at $x = t$ then it diverges whenever $|x| > |t|$.

44. Define C to be the set of values of x for which $\sum c_n x^n$ converges. Prove that either C contains all real numbers or C is bounded.

45. Use the fact that every bounded set of real numbers has a least upper bound (this is called the Completeness Property) to prove the Radius Theorem. (The least upper bound b of the set C is the least number such that $c \leqslant b$ for all $c \in C$.)

ANSWERS TO SELECTED EXERCISES, SECTION 3.1

1. $[-1, 1)$
3. $[-7/3, -5/3]$
5. $[-1/4, 1/4]$
7. Converges only for $x = 0$
9. Converges for all real numbers
11. One example is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$
13. One example is $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n^2}$
15. No such power series exists, by the Radius Theorem
17. One example is $\displaystyle\sum_{n=0}^{\infty} \frac{(x-5)^n}{\sqrt{n}2^n}$
19. No such power series exists, by the Radius Theorem
21. $\displaystyle\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n$
23. $\displaystyle\sum_{n=0}^{\infty} (n+1)c_{n+1}x^n + \sum_{n=1}^{\infty} 2a_{n-1}x^n$
25. Because the function has a discontinuity at $x = 2$
27. Because the function has a sharp corner at $x = 1$ (so its first derivative is not defined there)
29. Because $\tan x$ has a vertical asymptote at $x = \pi/2$

3.2. MANIPULATION AND DERIVATION

We concluded the previous section by noting that power series can be differentiated and integrated term-by-term (within their radii of convergence). This is quite a strong property of power series (which does not hold for series in general). We begin this section by showing that term-by-term differentiation and integration can be used to *find* power series.

Our starting point will always (in this section, at least) be a geometric series of some kind, the simplest example of such being

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x| < 1,$$

which we know from our study of geometric series in Section 2.3. By differentiating (term-by-term) both sides of this equation, we obtain our first new power series:

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2} \text{ for } |x| < 1.$$

This is one of the three basic forms of power series we derive in this section. The other two are given in Examples 1 and 2.

Example 1. Find the power series centered at $x = 0$ for $\ln(1+x)$ and its radius of convergence.

Solution. Recall that $\ln(1+x)$ is the antiderivative of $1/(1+x)$:

$$\int \frac{1}{1+x}\,dx = \ln(1+x).$$

Furthermore, we can express $1/(1+x)$ as a geometric power series (for $|x| < 1$):

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n.$$

Therefore, all we have to do to get the power series for $\ln(1+x)$ is integrate this series term-by-term,

$$\begin{aligned}
\ln(1+x) &= \int \frac{1}{1+x}\,dx \\
&= \int \sum_{n=0}^{\infty}(-1)^n x^n \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C.
\end{aligned}$$

But what is C, the constant of integration? To find C we substitute $x = 0$ into both sides. We know that $\ln(1+0) = \ln 1 = 0$, so $C = 0$. This gives

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

The geometric series we integrated had radius of convergence $R = 1$, so the radius of convergence of this series for $\ln(1+x)$ is also $R = 1$. ●

The power series for $\ln(1+x)$ that we found in Example 1 is known as the *Mercator series*, after Nicholas Mercator (1620–1687). Note that by substituting $x = 1$ into this series, we obtain

$$\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \ ?$$

This seems to indicate that the sum of the alternating harmonic series is $\ln 2$. However, there is a problem with this line of reasoning: term-by-term integration is only guaranteed to work *inside* the interval of convergence, and $x = 1$ is an endpoint of the interval of convergence for the Mercator series. Nevertheless, this computation can be made rigorous, as shown by Abel, see Exercise 33. (Another proof of this result, using Euler's constant γ is given in Exercises 46 and 47 of Section 2.4.)

We move on to another example of using integration to derive a power series.

Example 2. Find the power series centered at $x = 0$ for $\arctan x$ and its radius of convergence.

Solution. For this we need to recall that

$$\arctan x = \int \frac{1}{1+x^2}\,dx.$$

Again, we can write $1/(1+x^2)$ as a geometric power series (for $|x| < 1$):

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

Now we integrate this series term-by-term:

$$\begin{aligned}
\arctan x &= \int \frac{1}{1+x^2}\,dx \\
&= \int \sum_{n=0}^{\infty}(-1)^n x^{2n} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + C.
\end{aligned}$$

Finally, we substitute $x = 0$ into both sides of this equation to see that $C = 0$, giving

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ for } |x| < 1.$$

The geometric series we integrated had radius of convergence $R = 1$, so the radius of convergence of this series for $\arctan x$ is also $R = 1$. ●

As in Example 1, this series suggests an intriguing equality:

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \; ?$$

But once again, $x = 1$ lies on the endpoint of the interval of convergence for this series, so this equation does not necessarily follow from what we have done. Nevertheless, as with the previous example, this can be made precise using Abel's Theorem, see Exercise 34. (This is called the GregoryLeibniz formula for π, after Gottfried Leibniz (1646–1716) and James Gregory (1638–1675)).

But what if we wanted power series for $\ln(1 + 3x^2)$ or $\arctan 2x^3$? We could write them as integrals and then integrate some form of geometric power series as in Examples 1 and 2, but this is tedious and error-prone. More worryingly, what about more complicated functions like $\ln(1 + x)/(1 + 2x)$?

Just as with differentiation and integration, it turns out that *within their radii of convergence* we may treat power series just like polynomials when

- substituting,
- multiplying, and
- dividing.

Here even stating the theorems is technical; we instead illustrate the point with examples.

Example 3. Find the power series centered at $x = 0$ for $\dfrac{1}{(1-8x^3)^2}$.

Solution. We know from the beginning of the section that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} \text{ for } |x| < 1,$$

so to get the power series for $1/(1 - 8x^3)^2$, we simply replace x with $8x^3$:

$$\frac{1}{(1-8x^3)^2} = \sum_{n=0}^{\infty} n \left(8x^3\right)^{n-1} = \sum_{n=0}^{\infty} n8^{n-1}x^{3n-3}.$$

As the power series for $1/(1-x)^2$ held when $|x| < 1$, this power series holds when $\left|8x^3\right| < 1$, which simplifies to $|x| < 1/2$. ●

Example 4. Find the power series centered at $x = 0$ for $\ln(4 + 3x^2)$.

Solution. First we need to get the function in the form $\ln(1 + \text{something})$:

$$\ln(4 + 3x^2) = \ln\left(4 \cdot \left(1 + \frac{3x^2}{4}\right)\right) = \ln 4 + \ln\left(1 + \frac{3x^2}{4}\right).$$

Now we substitute $3x^2/4$ into the power series we found for $\ln(1 + x)$ in Example 1:

$$\ln 4 + \ln\left(1 + \frac{3x^2}{4}\right) = \ln 4 + \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{3x^2}{4}\right)^{n+1}}{n+1} = \ln 4 + \sum_{n=0}^{\infty}(-1)^n \frac{3^{n+1}x^{2n+2}}{4^{n+1}(n+1)}.$$

Note that our series for $\ln(1 + x)$ was valid for $|x| < 1$, so this new series is valid for $\left|3x^2/4\right| < 1$, or $|x| < \sqrt{4/3}$. ●

Example 5. Find the power series centered at $x = 0$ for $\displaystyle\int \frac{\arctan 2x^3}{x^3}\, dx$.

Solution. First we substitute $2x^3$ into our power series for $\arctan x$ from Example 2 to find a power series for $\arctan 2x^3$:

$$\arctan 2x^3 = \sum_{n=0}^{\infty}(-1)^n \frac{\left(2x^3\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}x^{6n+3}}{2n+1}.$$

Now we divide each of the terms of this series by x^3 and integrate term-by-term:

$$\begin{aligned}
\int \frac{\arctan 2x^3}{x^3}\, dx &= \int \frac{\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}x^{6n+3}}{2n+1}}{x^3}\, dx \\
&= \int \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}x^{6n}}{2n+1}\, dx \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}x^{6n+1}}{(2n+1)(6n+1)} + C.
\end{aligned}$$

Since our series for $\arctan x$ was valid for $|x| < 1$, this series is valid for $\left|2x^3\right| < 1$, or $|x| < \sqrt[3]{1/2}$. ●

In practice, it can be quite tedious to find many coefficients by multiplication. However, the first few coefficients are the most important.

Example 6. Compute the first four nonzero terms of the power series for $\dfrac{\ln(1+x)}{1+2x}$.

Solution. We simply need to multiply the geometric power series for $1/(1+2x)$ with the power series for $\ln(1+x)$ that we found in Example 1:

$$\begin{aligned} \frac{\ln(1+x)}{1-2x} &= \left(1 - 2x + 4x^2 - 8x^3 + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) \\ &= x - \frac{5x^2}{2} + \frac{16x^3}{3} - \frac{131x^4}{12} + \cdots \end{aligned}$$

Since the series for $1/(1+2x)$ has radius of convergence $1/2$ and the series for $\ln(1+x)$ has radius of convergence 1, the radius of convergence of their product is the minimum of these two values, $1/2$. ●

In the next section we use division to derive the power series for $\tan x$. In order to show how division works in this section, we repeat the previous example, dividing instead of multiplying.

Example 7. Use division to compute the first four nonzero terms of the power series for the function $f(x) = \dfrac{\ln(1+x)}{1+2x}$.

Solution. We use long division to divide the power series for $\ln(1+x)$ by $1+2x$:

$$\begin{array}{r|llllll}
 & x - & 5x^2/2 & + 16x^3/3 - & 131x^4/12 & + \cdots \\ \hline
1+2x & x - & x^2/2 & + \ x^3/3 \ - & x^4/4 & + \cdots \\
 & x + & 2x^2 & & & \\ \hline
 & & -5x^2/2 + & x^3/3 \ - & x^4/4 & + \cdots \\
 & & -5x^2/2 - & 5x^3 & & \\ \hline
 & & & 16x^3/3 - & x^4/4 & + \cdots \\
 & & & 16x^3/3 + & 32x^4/3 & + \cdots \\ \hline
 & & & & -131x^4/12 & + \cdots
\end{array}$$

Note that this agrees with our computation in the previous example. ●

EXERCISES FOR SECTION 3.2

Find power series centered at $x = 0$ for the functions in Exercises 1–16 and give their radii of convergence.

$\boxed{1.}$ $f(x) = \dfrac{1}{1+x}$

$\boxed{2.}$ $f(x) = \dfrac{2}{3+x}$

$\boxed{3.}$ $f(x) = \dfrac{3}{1-x^3}$

$\boxed{4.}$ $f(x) = \dfrac{4}{2x^3+3}$

$\boxed{5.}$ $f(x) = \dfrac{2}{(1+x)^2}$

$\boxed{6.}$ $f(x) = \dfrac{x}{(1+x)^2}$

$\boxed{7.}$ $f(x) = \dfrac{x^2}{(1+x^5)^2}$

$\boxed{8.}$ $f(x) = \dfrac{2}{(4-2x^2)^2}$

$\boxed{9.}$ $f(x) = \ln(1-x)$

$\boxed{10.}$ $f(x) = \ln(1-2x^3)$

$\boxed{11.}$ $f(x) = \ln(e - e^2x^2)$

$\boxed{12.}$ $f(x) = x^2 \arctan(3x^3)$

$\boxed{13.}$ $f(x) = \arctan(x/2)$

$\boxed{14.}$ $f(x) = \dfrac{\arctan(x)}{1+x}$

15. $f(x) = (1+x)\ln(1+x)$

16. $f(x) = \ln(1+x)^{1+x} + \ln(1-x)^{1-x}$

17. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln 2$. *Hint:* try substituting an appropriate value of x into the series for $\ln(1+x)$.

18. Use division of power series to give another proof that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$.

Exercises 19 and 20 explore the Fibonacci numbers.

19. Define

$$f(x) = \sum_{n=0}^{\infty} f_n x^n,$$

where f_n denotes the nth Fibonacci number. Use the recurrence relation and initial conditions for $\{f_n\}$ to show that

$$f(x) = 1 + xf(x) + x^2 f(x),$$

and derive from this that

$$f(x) = \frac{1}{1-x-x^2}.$$

20. Use Exercise 19, partial fractions, and geometric series to derive Binet's formula.

Use partial fractions and geometric series to find formulas for the coefficients of the functions in Exercises 21–24.

21. $f(x) = \dfrac{x}{1-5x+6x^2}$

22. $f(x) = \dfrac{2-5x}{1-5x+6x^2}$

23. $f(x) = \dfrac{x}{1-3x-2x^2}$

24. $f(x) = \dfrac{244-246x}{1-4x+3x^2}$

25. Using the fact that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ for } |x| < 1,$$

show that

$$\lim_{x\to 1^-} \sum_{n=1}^{\infty} \frac{x^n}{n} = \infty,$$

thereby proving (again) that the harmonic series diverges.

◆ **26.** Let $\sum a_n$ be a series with partial sums $\{s_n\}$ and suppose that the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges for $|x| < 1$. Prove that

$$\sum_{n=0}^{\infty} s_n x^n = \frac{f(x)}{1-x}.$$

27. The *harmonic numbers* $\{H_n\}$ are defined by $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$. Show that

$$\sum_{n=1}^{\infty} H_n x^n = \frac{1}{1-x} \ln \frac{1}{1-x}.$$

The sequence $\{a_n\}$ is said to be *Abel summable* to L if $\sum a_n x^n$ converges on (at least) the interval $[0,1)$ and

$$\lim_{x \to 1^-} a_n x^n = L.$$

(Note that the first term of these sequences is a_0.) In Exercises 28–32 consider Abel summability. This concept is due to Niels Henrik Abel (1802–1829).

28. Show that the sequence $\{2^{-n}\}$ is Abel summable to 2.

29. Show that $\{(-1)^n\}$ is Abel summable to $1/2$. (C.f. Exercise 50 from Section 2.2.)

30. Show that $\left\{\dfrac{(-1)^n}{n+1}\right\}$ is Abel summable to $\ln 2$.

31. Show that $\{(-1)^{n+1} n\}$ is Abel summable to $1/4$. (C.f. Exercise 51 in Section 2.2.)

32. Show that $\{(-1)^n n(n-1)\}$ is Abel summable to $1/4$.

Abel's Theorem guarantees that convergent series are Abel summable to their true values:

Abel's Theorem. If the series $\sum a_n$ converges to a finite value L, then $\{a_n\}$ is Abel summable to L.

Assume the truth of Abel's Theorem in Exercises 33 and 34.

33. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$

34. Show that $4 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \pi.$

35. Exercise 33 shows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$$

(this is also proved in Exercises 46 and 47 of Section 2.4), while Exercise 17 gives a different series that converges to $\ln 2$. Which of these two series converges "faster"?

In 1995, after a several month long search by computer, Bailey, Borwein, and Plouffe discovered a remarkable formula for π which allows one to compute *any* binary or hexadecimal digit of π without computing any of the digits that come before it. For details about how the formula was discovered, we refer the reader to the book *Mathematics by Experiment* by Borwein and Bailey. Exercises 36–38 establish the formula.

36. Show that

$$\int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8}\, dx = \pi.$$

Hint: Once we substitute $u = \sqrt{2}x$, the integral becomes

$$\int_0^1 \frac{16u - 16}{u^4 - 2u^3 + 4u - 4}\, du,$$

which, using partial fractions, is equal to

$$\int_0^1 \frac{4u}{u^2-2}\, du - \int_0^1 \frac{4u-8}{u^2-2u+2}\, du.$$

The first integral requires another substitution, while the second must be split into two integrals. One can be evaluated by substitution, the other can be done by writing $u^2 - 2u + 2$ as $(u-1)^2 + 1$.

37. Let k be a fixed integer less than 8. Show that

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\, dx = \frac{1}{2^{k/2}} \sum_{n=0}^{\infty} \frac{1}{16^n(8n+k)}.$$

38. Use Exercises 36 and 37 to show that π is equal to

$$\sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right).$$

This is the Bailey-Borwein-Plouffe formula for π.

ANSWERS TO SELECTED EXERCISES, SECTION 3.2

1. $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n$

3. $\displaystyle\sum_{n=0}^{\infty} 3x^{3n}$

5. $\displaystyle\sum_{n=0}^{\infty} 2n(-1)^n x^{n-1}$

7. $\displaystyle\sum_{n=0}^{\infty}(-1)^n n x^{5n-3}$

9. $\displaystyle\sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}$

11. $\displaystyle 1-\sum_{n=0}^{\infty}\frac{e^{n+1}x^{2n+2}}{n+1}$

13. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2^{2n+1}(2n+1)}$

3.3. TAYLOR SERIES

Having explored sequences, series, and power series, we are now ready to return to our original motivation: Taylor polynomials. Recall that the Taylor polynomial of degree n for $f(x)$ centered at $x = a$, $T_n(x)$, is the unique polynomial of degree n which matches $f(x)$ and its first n derivatives at $x = a$.

Back in Section 1.1, we observed that higher degree Taylor polynomials tended to give better approximations to $f(x)$. Then in Section 1.2 we made this observation precise with the Remainder Theorem.

The Remainder Theorem. Suppose that f is $n+1$ times differentiable and let R_n denote the difference between $f(x)$ and the Taylor polynomial of degree n for $f(x)$ centered at a. Then

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x.

Now that we have established the notion of series, we are ready to consider Taylor polynomials of "infinite degree", which we will call *Taylor series*†. These are a special type of power series.

Taylor Series. Suppose that the function $f(x)$ is infinitely differentiable (smooth) at $x = a$. The Taylor series for $f(x)$ centered at $x = a$ is then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Assuming that we can compute all the derivatives of $f(x)$ at $x = a$, this definition is very concrete, but it leaves some unanswered questions:

Where does the Taylor series for $f(x)$ converge?
When the Taylor series for $f(x)$ does converge, does it converge to $f(x)$?

In general, these two questions must be answered on a function-by-function basis, and such answers revolve around the Remainder Theorem.

†Taylor series centered at $x = 0$ are sometimes referred to as *Maclaurin series* after the Scottish mathematician Colin Maclaurin (1698–1746).

For our first case study, we consider the function $f(x) = \sin x$, centered at $x = 0$. Since the derivatives of $\sin x$ repeat in the pattern $\cos x$, $-\sin x$, $-\cos x$, $\sin x$, the sequence of derivatives evaluated at $x = 0$ is $\{1, 0, -1, 0, \dots\}$. Therefore the Taylor series for $\sin x$ centered at $x = 0$ is:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

We could apply the Ratio Test to this series (as in Section 3.1) to show that it converges for all x, but there is a more important question: Does this series converge to $\sin x$ for all x?

By the Remainder Theorem, the difference between $\sin x$ and its Taylor polynomial of degree n centered at $x = 0$, $T_n(x)$, is

$$\sin x - T_n(x) = R_n(x) = \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some c between 0 and x. Since the derivatives of $\sin x$ only take on values between -1 and 1, we have that

$$|R_n(x)| \leqslant \frac{|x|^{n+1}}{(n+1)!}.$$

We would like to show that the Taylor series for $\sin x$ converges to $\sin x$ for all x, which is equivalent to showing that the limit of the remainders $R_n(x)$ is 0 (as $n \to \infty$). This is really just an exercise from Section 2.1, although we have delayed it now.

Fact 1. For all values of x, $\displaystyle\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$.

Proof. First let us consider a special case, where $|x| = 4$. In this case, we are interested in the sequence $\{4^n/n!\}$. For the first 4 terms, this sequence is actually increasing:

n	$4^n/n!$
0	1
1	4
2	$4^2/2! = 8$
3	$4^3/3! = 10.6666\dots$
4	$4^4/4! = 10.6666\dots$

But then the sequence starts decreasing. To get the $n = 5$ term, we multiply the $n = 4$ term by $4/5$; to get the $n = 6$ term we multiply this by $4/6$; to get the $n = 7$ term we multiply this by $4/7$, and so on. For $n \geqslant 5$, we therefore have the bound

$$\frac{4^n}{n!} \leqslant \left(\frac{4}{5}\right)^{n-4} \frac{4^4}{4!} \to 0 \text{ as } n \to \infty.$$

Our proof for generic x follows the same general approach. Choose an integer $m > |x|$. We have, for $n \geqslant m$,

$$\frac{|x|^n}{n!} = \frac{|x|^n}{\underbrace{n \cdot (n-1) \cdots (m+1) \cdot m}_{n-m \text{ terms, all } \geqslant m > |x|} \cdot (m-1)!} \leqslant \frac{|x|^n}{m^{n-m}(m-1)!}.$$

Ideally we would like to have m^n in the denominator, so we multiply top and bottom by m^m:

$$\frac{|x|^n}{n!} \leqslant \frac{|x|^n m^m}{m^n(m-1)!} = \frac{|x|^n}{m^n} \cdot \frac{m^m}{(m-1)!}.$$

Our goal is to prove that the sequence $\{|x|^n/n!\}$ converges to 0 by sandwiching it between the constant sequence $\{0\}$ and the sequence above.

Remember that m is not changing, so $m^m/(m-1)!$ is just a constant. Since $m > |x| \geqslant 0$, $|x|/m$ is between 0 and 1, so Example 5 of Section 2.1 shows that the geometric sequence $(|x|/m)^n \to 0$. Therefore, by the Sandwich Theorem, $|x|^n/n! \to 0$ as $n \to \infty$ for all values of x. □

Our next example collects these observations to show that the Taylor series for $\sin x$ centered at $x = 0$ converges[†] to $\sin x$.

Example 2. Show that the Taylor series centered at $x = 0$ for $\sin x$ converges to $\sin x$ for all values of x.

Solution. Proving this statement is equivalent to showing that for any fixed value of x, the remainder term $R_n(x)$ tends to 0 as $n \to \infty$. Fix a value of x. By the Remainder Theorem, our previous bounds, and Fact 1,

$$|R_n(x)| = |\sin x - T_n(x)| \leqslant \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty,$$

which is precisely what we wanted to show. ●

Example 2 is in some sense completely bizarre. Remember that we defined Taylor polynomials (and therefore series as well) by mathcing derivatives at $x = a$ (the center). Therefore, this example shows that if a power series matches all of the derivatives of $\sin x$ at $x = 0$, then the series is *equal* to $\sin x$ for *all* values of x. Why should $\sin x$ be completely determined by its derivatives at 0? With our limited tools, the best answer we will be able to provide here is that $\sin x$ is a very well-behaved function; the technical term for such functions is *analytic*.

The same approach would work for the function $f(x) = \cos x$, but there is an easier way. Now that we know that the Taylor series for $\sin x$ centered at $x = 0$ converges to $\sin x$

[†]This type of convergence is known as *pointwise convergence*, because the Taylor series converges to $\sin x$ at each individual point. There is a stronger, more global type of convergence known as *uniform convergence*.

for all values of x, we may differentiate it term-by-term to obtain:

$$\begin{aligned}
\cos x &= \frac{d}{dx}\sin x \\
&= \frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}(-1)^n(2n+1)\frac{x^{2n}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \\
&= 1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots.
\end{aligned}$$

For our next case study, we consider the function $f(x) = e^x$, again centered at $x = 0$. Since every derivative of e^x is e^x itself, $f^{(n)}(0) = 1$ for every n, and thus the Taylor series centered at $x = 0$ for e^x is

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots.$$

Example 3. Show that the Taylor series centered at $x = 0$ for e^x converges to e^x for all values of x.

Solution. We want to show that for fixed x, $|R_n(x)| \to 0$ as $n \to \infty$. The Remainder Theorem shows that

$$|R_n(x)| = |e^x - T_n(x)| = \left|\frac{e^c}{(n+1)!}x^{n+1}\right|$$

for some c between 0 and x (here we used the fact that the $n + 1$st derivative of e^x is e^x itself). We have to be a little careful here about giving a bound for e^c, since the bound changes depending on whether x is positive or not:

$$e^c \leqslant \begin{cases} e^x & \text{if } x \geqslant 0, \\ 1 & \text{if } x < 0. \end{cases}$$

Still, since we are considering a *fixed* value of x, our bound on e^c is just a constant, and Fact 1 shows that

$$\frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty.$$

Therefore $|R_n(x)| \to 0$ as $n \to \infty$, proving that the Taylor series for e^x centered at $x = 0$ converges to e^x for all values of x. ●

Substituting $x = 1$ into the Taylor series from Example 3 gives an appealing identity:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

Because $n!$ grows so quickly, this series converges to e exceptionally fast. For example, using just the first 10 terms, we get e correct to six decimal places:

$$\begin{aligned} e &= 2.7182818\ldots \\ \sum_{n=0}^{9} \frac{1}{n!} &= 2.7182815\ldots \end{aligned}$$

The speed of this convergence (and it really is fast — at $n = 59$, we are adding the reciprocal of the estimated atoms in the universe, about 10^{-80}) allows us to prove that e is irrational in Exercises 50–52.

As with general power series, we are allowed to substitute into Taylor series and to multiply and divide them. This can be quite useful for expressing integrals as power series (our next example), evaluating limits (Example 5), computing derivatives (Example 6), and finding Taylor series for quotients (Example 7).

Example 4. Express $\displaystyle\int \sin x^2 \, dx$ as a power series.

Solution. We know from Example 2 that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

so

$$\sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

All that remains is to integrate this series term-by-term:

$$\int \sin x^2 = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + C.$$

It has been proved that the antiderivative of $\sin x^2 \, dx$ is not an "elementary function", so this one of the nicest possible ways to express this integral. ●

Example 5. Evaluate $\displaystyle\lim_{x \to 0} \frac{\cos x - 1 + x^2/2}{x^4}$.

Solution. While this example could be solved with l'Hôpital's Rule, it is easier to use power series. We know that

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ,$$

so

$$\begin{aligned} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} &= \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - 1 + \frac{x^2}{2}}{x^4} \\ &= \frac{\frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}{x^4} \\ &= \frac{1}{4!} - \frac{x^2}{6!} + \cdots . \end{aligned}$$

As $x \to 0$, this quantity approaches $1/4! = 1/24$, so this is the limit. ●

Example 6. Compute the 102nd derivative of $f(x) = e^{2x^3}$ at $x = 0$.

Solution. Example 3 shows that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

so by substitution, we have that

$$e^{2x^3} = \sum_{n=0}^{\infty} \frac{\left(2x^3\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{3n}}{n!}.$$

Since this is the Taylor series for $f(x) = e^{2x^3}$ centered at $x = 0$, by the definition of Taylor series, we know that it is equal to

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Therefore if we are interested in $f^{(102)}(0)$, we need only look at the coefficient of x^{102}. We get the coefficient of x^{102} in

$$\sum_{n=0}^{\infty} \frac{2^n x^{3n}}{n!}$$

by substituting $n = 102/3 = 34$, so

$$\frac{2^{34}}{34!} = \frac{f^{102}(0)}{102!}.$$

Solving for $f^{(102)}(0)$ shows that

$$f^{(102)}(0) = \frac{2^{34}102!}{34!}.$$

We could have solved this problem simply by taking 102 derivatives, but finding the Taylor series is *much* easier. ●

Example 7. Find the first four nonzero terms of the Taylor series centered at $x = 0$ for the function $f(x) = \tan x$.

Solution. It would be possible to solve this problem by taking derivatives of $\tan x$, but this approach gets quite messy. An easier method is to use the series we have for $\sin x$ and $\cos x$ together with the Division Theorem from Section 3.2. Since

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - x^3/3! + x^5/5! - x^7/7! + \cdots}{1 - x^2/2! + x^4/4! - x^6/6! + \cdots},$$

we can use long division

$$\begin{array}{r|l}
 & x + x^3/3 + 2x^5/15 + 17x^7/315 + \cdots \\ \hline
1 - x^2/2 + x^4/24 - x^6/720 + \cdots & x - x^3/6 + x^5/120 - x^7/5040 + \cdots \\
 & x - x^3/2 + x^5/24 - x^7/720 + \cdots \\ \cline{2-2}
 & \quad x^3/3 - x^5/30 + x^7/840 - \cdots \\
 & \quad x^3/3 - x^5/6 + x^7/72 - \cdots \\ \cline{2-2}
 & \qquad 2x^5/15 - 4x^7/315 + \cdots \\
 & \qquad 2x^5/15 - x^7/15 + \cdots \\ \cline{2-2}
 & \qquad\qquad 17x^7/315 + \cdots
\end{array}$$

to find that the first four nonzero terms of the Taylor series centered at $x = 0$ for $\tan x$ are

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315}.$$

This is also therefore the Taylor polynomial of $\tan x$ of degree 7, centered at $x = 0$. ●

We can also use the Taylor series for e^x to *define* the function e^x for more quantities than just real numbers. For example, we can define e^x for *complex numbers* x. Recall that a complex number is a number of the form $a + bi$ where a and b are real numbers, and i is the (imaginary) square-root of -1. This leads to the following formula, which the Nobel Prize winning physicist Richard Feynman (1918–1988) referred to as "one of the most remarkable, almost astounding, formulas in all of mathematics":

Example 8 (Euler's Formula). Prove Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$.

Solution. We begin by substituting $i\theta$ into the Taylor series for e^x:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n}\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i^{2n+1}\theta^{2n+1}}{(2n+1)!}.$$

Now we need to compute the powers of i:

$$\begin{aligned} i^2 &= -1, \\ i^3 &= i \cdot i^2 = -i, \\ i^4 &= i^2 \cdot i^2 = (-1)\cdot(-1) = 1, \\ i^5 &= i \cdot i^4 = i. \end{aligned}$$

This shows that the powers of i form a periodic sequence $\{i, -1, -i, 1, \dots\}$, which allows us to simplify our series for $e^{i\theta}$ above:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} = \cos\theta + i\sin\theta.$$

An alternative proof, using derivatives, is outlined in Exercise 45. ●

A particularly wondrous special case of Euler's Formula, known as Euler's Identity, is,

$$e^{\pi i} + 1 = 0,$$

an identity relating five of the most important numbers in all of mathematics. In Exercises 46–49 we use a similar approach to define e^M for a matrix M.

Euler's formula is also a very convenient way to prove various trigonometric identities. For example, the angle addition formulas, which show how to evaluate $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ in terms of $\sin\alpha$, $\sin\beta$, $\cos\alpha$, and $\cos\beta$, have a particularly straightforward derivation using Euler's formula. As the French mathematician Jacques Hadamard (1865–1963) wrote, "the shortest route between two truths in the real domain sometimes passes through the complex domain."

Example 9. Derive the angle addition formulas.

Solution. We begin with the trigonometric functions we are interested in, convert to exponentials, simply the expression, and then convert back to trigonometric functions:

$$\begin{aligned} \cos(\alpha+\beta) + i\sin(\alpha+\beta) &= e^{(\alpha+\beta)i} \\ &= e^{\alpha i}\cdot e^{\beta i} \\ &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta). \end{aligned}$$

Now equate real and imaginary parts of both sides to see that

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta,\\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta,\end{aligned}$$

as desired. ●

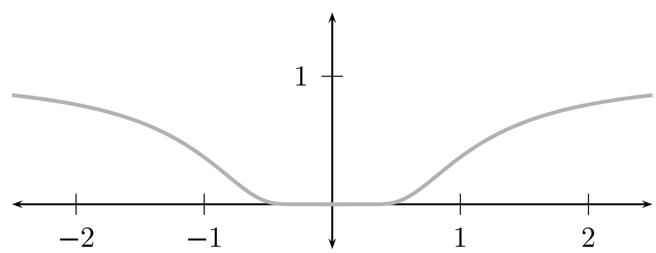
We know from Section 3.1 that power series define infinitely differentiable (smooth) functions when they converge, so not every function is equal to its Taylor series (in fact, a function much be infinitely differentiable for us to even define its Taylor series). Our success in this section and the last with finding Taylor series for infinitely differentiable functions suggests a final question:

Is every infinitely differentiable function equal to its Taylor seres?

The answer is **no**. A counterexample is provided by the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0,\\ 0 & \text{if } x = 0,\end{cases}$$

whose plot is shown below.



It is possible (using l'Hôpital's Rule) to establish that every derivative of this function at $x = 0$ is 0, so therefore its Taylor series is simply 0, which does not converge to this function. Therefore, infinite differentiability is a necessary condition for a function to equal its Taylor series, but it is not a sufficient condition.

EXERCISES FOR SECTION 3.3

Derive Taylor series for the functions in Exercises 1–8 at the specified centers.

1. $f(x) = \sin 2x$ centered at $x = 0$

2. $f(x) = \cos x$ centered at $x = \pi/2$

3. $f(x) = x^2 \cos(\sqrt{2}x^3)$ centered at $x = 0$

4. $f(x) = (e^x)^2$ centered at $x = 0$

5. $f(x) = 2xe^x$ centered at $x = 0$

6. $f(x) = 2xe^x$ centered at $x = 1$

7. $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$ centered at $x = 0$

8. $f(x) = \cosh x = \dfrac{e^x + e^{-x}}{2}$ centered at $x = 0$

Compute the limits in Exercises 9–16 using Taylor series.

9. $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{x^3}$

10. $\displaystyle\lim_{x\to 0} \frac{x - \arctan x}{x^3}$

11. $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x}}{x}$

12. $\displaystyle\lim_{x\to 0} \frac{\sin x - \arctan x}{x^3}$

13. $\displaystyle\lim_{x\to 0} \frac{(2 - 2\cos x)^3}{x^6}$

14. $\displaystyle\lim_{x\to 0} \frac{\ln(1 + x^3)}{x^3}$

15. $\displaystyle\lim_{x\to 0} \frac{(\sin x - x)^2}{(\cos 5x - 1)^3}$

16. $\displaystyle\lim_{x\to 0} \frac{(\sin 3x - 3x)^2}{(e^{2x} - 1 - 2x)^3}$

Use Taylor series of known functions to evaluate the sums in Exercises 17–20.

17. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

18. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{2^n(2n+1)!}$

19. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^n}{n!}$

20. $\displaystyle\sum_{n=0}^{\infty} (-7)^n \frac{\pi^{2n}}{(2n)!}$

Derive Taylor series for the integrals in Exercises 21–24 (centered at $x = 0$). Note that none of these integrals has an answer in terms of elementary functions.

21. $\displaystyle\int \frac{\cos(x^3) - 1}{x^2}\, dx$

22. $\displaystyle\int \frac{\sin x}{x}\, dx$

23. $\displaystyle\int e^{-x^2/2}\, dx$

24. $\displaystyle\int \frac{\ln(x + 1)}{x}\, dx$

Use the Remainder Theorem in Exercises 25–30.

25. Give an upper bound on the error when using $T_3(x) = 1 + x + x^2/2 + x^3/6$ to approximate $f(x) = e^x$ for $0 \leqslant x \leqslant 1/2$.

26. Give an upper bound on the error when using $T_1(x) = x$ to approximate $f(x) = \tan x$ for $-\pi/4 \leqslant x \leqslant \pi/4$.

27. Give an upper bound on the error when using $T_3(x) = 1 - 2(x - 1) + 3(x - 1)^2 - 4(x - 1)^3$ to approximate $f(x) = 1/x^2$ for $3/4 \leqslant x \leqslant 5/4$.

28. Give an upper bound on the error when using $T_2(x) = 2 + (x - 4)/4 - (x - 4)^2/64$ to approximate $f(x) = \sqrt{x}$ for $4 \leqslant x \leqslant 4.1$.

29. Give an upper bound on the error when using $T_3(x) = (x - 1) + (x - 1)^2/2 - (x - 1)^3/6$ to approximate $f(x) = x \ln x$ for $1/2 \leqslant x \leqslant 3/2$.

30. Give an upper bound on the error when using $T_4(x) = x - x^2/2 + x^3/3 - x^4/4$ to approximate $f(x) = \ln 1 + x$ for $0 \leqslant x \leqslant 1$.

31. Prove that $x^n/n! \to 0$ as $n \to \infty$ for every fixed value of x. (Note that this is a bit stronger than Fact 1.)

For Exercises 32–35, give the Taylor polynomials of degree 8 (centered at $x = 0$) for the specified functions.

32. $f(x) = e^{\cos x^4}$

33. $f(x) = (1 + x + x^2) \sin x^3$

34. $f(x) = \dfrac{\sin x}{1 - 2x}$

35. $f(x) = \ln(\cos x)$

Exercises 36–39 investigate another way to bound the error in using Taylor polynomials to approximate functions. Consider first the example of using $T_3(x) = 1 + x + x^2/2 + x^3/6$ to approximate $e^{1/2}$. We could of course use the Remainder Theorem for this problem (as is requested in Exercise 25), but since we know that

$$e^{1/2} = 1 + 1/2 + \frac{(1/2)^2}{2} + \frac{(1/2)^3}{6} + \sum_{n=4}^{\infty} \frac{(1/2)^n}{n!},$$

another way to estimate this error is simply to bound the tail by comparing it to a geometric series:

$$\begin{aligned}
\sum_{n=4}^{\infty} \frac{(1/2)^n}{n!} &= \frac{(1/2)^4}{4!} + \frac{(1/2)^5}{5!} + \frac{(1/2)^6}{6!} + \cdots \\
&< \frac{(1/2)^4}{4!} + \frac{(1/2)^5}{4 \cdot 4!} + \frac{(1/2)^6}{4^2 \cdot 4!} + \cdots \\
&= \frac{(1/2)^4}{4!} \left(1 + \frac{1/2}{4} + \frac{1/2}{4^2} + \cdots\right) \\
&= \frac{(1/2)^4}{4!} \left(\frac{1}{1 - \frac{1/2}{4}}\right).
\end{aligned}$$

(This gives an error estimate of about 0.003, whereas the Remainder Theorem shows that the error is at most 0.004.)

36. Bound the error involved in using the Taylor polynomial of degree 5 to approximate $\sin x$ near $x = 1/2$ without using the Remainder Theorem.

37. Bound the error involved in using the Taylor polynomial of degree 5 to approximate

$$\int_0^1 e^{-x^2/2}\, dx$$

without using the Remainder Theorem.

38. Show that the Remainder Theorem cannot give any practical bounds on the error involved in approximating $f(x) = -\ln(1-x)$ at $x = 1/2$ by its Taylor polynomial of degree 3, $T_4(x) = x + x^2/2 + x^3/3$.

39. Give a bound on the error involved in the estimate in Exercise 38 by observing that

$$\begin{aligned}
R_3(1/2) &= \frac{(1/2)^4}{4} + \frac{(1/2)^5}{5} + \frac{(1/2)^6}{6} + \cdots \\
&< \frac{(1/2)^4}{4} \left(1 + (1/2) + (1/2)^2 + \cdots\right).
\end{aligned}$$

Exercises 40 and 41 concern the series

$$\sum_{n=2}^{\infty} \frac{\pi^n}{n!}.$$

40. Compute the 10th partial sum of this series.

41. Prove that the sum of this series is *not* 19.

42. Use Euler's formula to prove the two identities

$$\begin{aligned}
\cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \\
\sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}.
\end{aligned}$$

43. Use Euler's formula to prove De Moivre's Formula,

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta.$$

for all integers n.

44. Give an example showing that De Moivre's Formula does not necessarily hold when n is not an integer.

45. Define

$$f(\theta) = (\cos\theta + i\sin\theta)\, e^{-i\theta}.$$

Compute the derivative of $f(\theta)$ and use this to give another proof of Euler's Formula.

As with complex numbers, we also use the Taylor series for e^x to define e^M for matrices square M as

$$I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots,$$

where I denotes the identity matrix (which has 1s along the diagonal and 0s everywhere else). Use this to compute e^M for the matrices in Exercises 46–49.

46. $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

47. $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

48. $M = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}$

49. $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Suppose that e were a rational number, so $e = a/b$ for positive integers a and b. In Exercises 50–52 we will draw a contradiction, thereby proving that e is irrational. This proof is often attributed to Charles Hermite (1822–1901).

50. Define

$$a_n = n!\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right),$$

and prove that a_n is an integer for all $n \geqslant b$.

51. Show that $a_n = \dfrac{1}{n+1} + \dfrac{1}{(n+1)(n+2)} + \cdots$.

52. Show that $0 < a_n < 1$, and therefore a_n can *not* be an integer, contradicting our assumption that e is rational.

Is there an irrational number which raised to an irrational power is rational? Exercise 54 proves that there is, although it doesn't tell us precisely what it is. I thank Professor Steven Landsburg for making me aware of these problems.

53. Verify that $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = 2$. (One method is to take the logarithm of both sides.)

54. Using the result of the previous exercise and the fact that $\sqrt{2}$ is irrational, prove that there is an irrational number which raised to an irrational number is rational.

Exercises 55–59 present Euler's original computation of $\sum 1/n^2$, from 1735. While this proof was considered a breakthrough at the time, Euler mistakenly assumed that he could apply a fact about polynomials to power series (see Exercise 58). The fact he assumed, however, does not hold for arbitrary series, as Exercise 59 shows. It was not until about 150 years later that Karl Weierstrass (1815–1897) "corrected" this proof by proving the Weierstrass Factorization (or, Product) Theorem. It should also be noted that Euler gave two other correct proofs for this calculation in the same paper, and a fourth proof in 1741, but his "incorrect" proof is the one that is best remembered. As the American author Henry Mencken (1880–1956) wrote, "For every problem, there is one solution which is simple, neat, and wrong."

55. Find a series centered at 0 which is equal to

$$f(x) = \frac{\sin\sqrt{x}}{\sqrt{x}}$$

for $x > 0$.

56. Find all the roots of the power series from Exercise 55. This should involve two steps; first show that this series has no negative roots, and then find all the positive roots, which are (by the previous problem) also roots of

$$f(x) = \frac{\sin\sqrt{x}}{\sqrt{x}}.$$

57. If a polynomial of degree 3 has roots r_1, r_2, and r_3, then it is given by $p(x) = c(x-r_1)(x-r_2)(x-r_3)$ for some constant c. By expanding this product, verify that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = -\frac{\text{coefficient of } x}{\text{constant term}}.$$

(This generalizes to polynomials of any degree.)

58. Assume, as Euler did, that Exercise 57 holds for series to show that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \cdots = \frac{1}{6}$$

where $r_1, r_2, r_3, \ldots$ are the roots from Exercise 56. Conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

59. The function $f(x) = 2 - 1/(1-x)$ has a single root, $x = 1/2$. Derive its power series (centered at $x = 0$) and conclude that, contrary to Euler's assumption, Exercise 57 cannot be applied to arbitrary series.

ANSWERS TO SELECTED EXERCISES, SECTION 3.3

1. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}x^{2n+1}}{(2n+1)!}$

3. $\displaystyle x^2\sum_{n=0}^{\infty}(-1)^n\frac{\left(\sqrt{2}x^3\right)^{2n}}{(2n)!}=x^2\sum_{n=0}^{\infty}(-1)^n\frac{2^nx^{6n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{2^nx^{6n+2}}{(2n)!}$

5. $\displaystyle 2x\sum_{n=0}^{\infty}\frac{x^n}{n!}=\sum_{n=0}^{\infty}2\frac{x^{n+1}}{n!}$

7. $\displaystyle \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$

9. The series is $\dfrac{-1}{3!}+\dfrac{x^2}{5!}-\dfrac{x^4}{7!}+\cdots$, so the limit is $-1/3!=-1/6$.

11. The series is $2+\dfrac{2x^2}{3!}+\dfrac{2x^4}{5!}+\cdots$, so the limit is 2.

13. The series is $1-\dfrac{x^2}{4}+\cdots$, so the limit is 1.

15. The series is $\dfrac{-2}{140625}-\dfrac{41x^2}{468750}-\cdots$, so the limit is $-2/140625$.

17. $\cos\pi=0$

19. $e^{-\pi}$

21. $\displaystyle\int\sum_{n=1}^{\infty}(-1)^n\frac{x^{6n-2}}{(2n)!}\,dx=\sum_{n=1}^{\infty}(-1)^n\frac{x^{6n-1}}{(6n-1)(2n)!}+C$

23. $\displaystyle\int\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^nn!}\,dx=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2^n(2n+1)n!}+C$