Applications of the Wronskian to linear differential equations

1. The Wronskian

Consider a set of n continuous functions $y_i(x)$ $[i = 1, 2, 3, \ldots, n]$, each of which is differentiable at least n times. Then if there exist a set of constants λ_i that are not all zero such that

$$\lambda_i y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) = 0\,, \tag{1}$$

then we say that the set of functions $\{y_i(x)\}$ are *linearly dependent*. If the only solution to eq. (1) is $\lambda_i = 0$ for all i, then the set of functions $\{y_i(x)\}$ are *linearly independent*.

The Wronskian matrix is defined as:

$$\Phi[y_i(x)] = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}, \tag{2}$$

where

$$y_i' \equiv \frac{dy_i}{dx}\,, \quad y_i'' \equiv \frac{d^2 y_i}{dx^2}\,, \quad \cdots\,, \quad y_i^{(n-1)} \equiv \frac{d^{(n-1)} y_i}{dx^{n-1}}\,.$$

The *Wronskian* is defined to be the determinant of the Wronskian matrix,

$$W(x) \equiv \det\, \Phi[y_i(x)]\,. \tag{3}$$

In light of eq. (8.5) on p. 133 of Boas, if $\{y_i(x)\}$ is a linearly dependent set of functions then the Wronskian must vanish. However, the converse is not necessarily true, as one can find cases in which the Wronskian vanishes without the functions being linearly dependent. (For further details, see problem 3.8–16 on p. 136 of Boas.)

Nevertheless, if the $y_i(x)$ are solutions to an nth order linear differential equation for values of x that lie in some open interval (e.g., $x_0 < x < x_1$), then the converse does hold. That is, if the $y_i(x)$ are solutions of a homogeneous nth order linear differential equation and the Wronskian of the $y_i(x)$ vanishes, then $\{y_i(x)\}$ is a linearly dependent set of functions. Moreover, if the Wronskian does not vanish for some value of x, then it is does not vanish for all values of x, in which case an arbitrary linear combination of the $y_i(x)$ constitutes the most general solution to the nth order linear differential equation. A proof of this statement is given in Appendix A.

2. Applications of the Wronskian in the treatment of a second order linear differential equation

To simplify the discussion, we shall focus on the role of the Wronskian in the treatment of a second order linear differential equation,[1]

$$y'' + a(x)y' + b(x)y = 0\,. \tag{4}$$

Suppose that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of eq. (4). Then the Wronskian is non-vanishing,

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2 \neq 0\,. \tag{5}$$

Taking the derivative of the above equation,

$$\frac{dW}{dx} = \frac{d}{dx}\,(y_1 y_2' - y_1' y_2) = y_1 y_2'' - y_1'' y_2\,,$$

since the terms proportional to $y_1' y_2'$ exactly cancel. Using the fact that y_1 and y_2 are solutions to eq. (4), we have

$$y_1'' + a(x)y_1' + b(x)y_1 = 0\,, \tag{6}$$
$$y_2'' + a(x)y_2' + b(x)y_2 = 0\,. \tag{7}$$

Next, we multiply eq. (7) by y_1 and multiply eq. (6) by y_2, and subtract the resulting equations. The end result is:

$$y_1 y_2'' - y_1'' y_2 + a(x)\,[y_1 y_2' - y_1' y_2] = 0\,.$$

or equivalently,

$$\frac{dW}{dx} + a(x)W = 0\,, \tag{8}$$

This is a separable differential equation for the Wronskian W. It then follows that,

$$\frac{dW}{W} = -a(x)dx\,.$$

Integrating both sides of the above equation yields,[2]

$$\ln|W(x)| = -\int a(x)dx + \ln C\,,$$

where $\ln C$ is an integration constant. Finally, exponentiating this last result and absorbing the sign of $W(x)$ into the constant C, we end up with,

$$\boxed{W(x) = C\exp\left\{-\int a(x)dx\right\}\,.} \tag{9}$$

Eq. (9) is known as *Abel's formula*.

[1]Starting from the more general form, $c(x)y'' + a(x)y' + b(x) = 0$, one is always free to divide this equation by $c(x)$ as long as $c(x) \neq 0$ over the range of interest.

[2]The notation $\int f(x)dx$ refers to the indefinite integral of the function f with the integration constant omitted. The result of the integration is another function of x.

The Wronskian also appears in the following application. Suppose that one of the two solutions of eq. (4), denoted by $y_1(x)$ is known. We wish to determine a second linearly independent solution of eq. (4), which we denote by $y_2(x)$. The following equation is an algebraic identity,

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2}\,,$$

after using the definition of the Wronskian W given in eq. (5). Integrating with respect to x yields

$$\frac{y_2}{y_1} = \int \frac{W(x)\,dx}{[y_1(x)]^2}\,.$$

Hence, it follows that[3]

$$\boxed{y_2(x) = y_1(x)\int \frac{W(x)\,dx}{[y_1(x)]^2}\,.} \tag{10}$$

Note that an indefinite integral always includes an arbitrary additive constant of integration. Thus, we could have written:

$$y_2(x) = y_1(x)\left\{\int \frac{W(x)\,dx}{[y_1(x)]^2} + C'\right\}\,,$$

where C' is an arbitrary constant. Of course, since $y_1(x)$ is a solution to eq. (4), then if $y_2(x)$ is a solution, then so is $y_2(x) + C'y_1(x)$ for any number C'. Thus, we are free to choose any convenient value of C' in defining the second linearly independent solution of eq. (4). The choice of $C' = 0$ is the most common, in which case the second linearly independent solution is given by eq. (10).

Here is a simple application of eq. (10). Consider the differential equation,

$$y'' - 2ry' + r^2 y = 0\,. \tag{11}$$

The auxiliary equation has a double root given by r. This means that $y_1(x) = e^{rx}$ is one solution of eq. (11). But, what is the second linearly independent solution? To use eq. (10), we need the Wronskian, which can be obtained from Abel's formula [eq. (9)] by identifying $a(x) = -2r$. We will omit the overall factor of C since this factor simply contributes to the overall normalization of the solution that we are seeking. Hence,

$$W(x) = \exp\left\{2r\int dx\right\} = e^{2rx}\,. \tag{12}$$

Employing eq. (10), and noting that $W(x)/[y_1(x)]^2 = 1$, we end up with,

$$y_2(x) = e^{rx}\int dx = xe^{rx}\,.$$

We conclude that the most general solution to eq. (11) is given by an arbitrary linear combination of $y_1(x)$ and $y_2(x)$. That is,

$$y(x) = (A + Bx)e^{rx}\,,$$

where A and B are arbitrary constants.

[3] A second derivation of eq. (10) is given in Appendix C. This latter derivation is useful as it can be easily generalized to the case of an nth order linear differential equation.

The Wronskian also appears in the expression for the particular solution of an inhomogeneous linear differential equations. For example, consider

$$y'' + a(x)y' + b(x)y = f(x)\,, \tag{13}$$

and assume that the two linearly independent solutions to the homogeneous equation [eq. (4)], denoted by $y_1(x)$ and $y_2(x)$, are known. The most general solution to the homogeneous equation is given by

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)\,,$$

where c_1 and c_2 are arbitrary constants. Then the general solution to eq. (13) is given by

$$y(x) = y_p(x) + y_h(x)\,,$$

where $y_p(x)$, called the particular solution, is determined by the following formula,

$$\boxed{y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx\,.} \tag{14}$$

One can derive eq. (14) by employing the technique of variation of parameters.[4] Namely, one writes

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)\,, \tag{15}$$

subject to the following condition (which is chosen entirely for convenience),

$$v_1'y_1 + v_2'y_2 = 0\,. \tag{16}$$

With this choice, it follows that

$$y_p' = v_1 y_1' + v_2 y_2'\,, \tag{17}$$

$$y_p'' = v_1'y_1' + v_2'y_2' + v_1 y_1'' + v_2 y_2''\,. \tag{18}$$

Plugging eqs. (15), (17) and (18) into eq. (13), and using the fact that y_1 and y_2 satisfy the homogeneous equation [eq. (4)] one obtains,

$$v_1'y_1' + v_2'y_2' = f(x)\,. \tag{19}$$

We now have two equations, eqs. (16) and (19), which constitute two algebraic equations for v_1' and v_2', which we can write in matrix form,

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}\,.$$

Note the appearance of the Wronskian matrix above. We can solve this matrix equation using Cramer's rule. Since $W(x)$ is the determinant of the Wronskian matrix [cf. eq. (5)], it immediately follows that,

$$v_1' = -\frac{y_2(x)f(x)}{W(x)}\,, \qquad v_2' = \frac{y_1(x)f(x)}{W(x)}\,.$$

We now integrate to get v_1 and v_2 and plug back into eq. (15) to obtain eq. (14). The derivation is now complete.

[4] Boas employs an alternate method to obtain eq. (14) that makes use of Green functions discussed in Section 12 of Chapter 8 [cf. eq. (12.21) on p. 464]; this technique will not be covered in this course.

3. The particular solution of an inhomogeneous second order linear differential equation with constant coefficients

As an example, consider the inhomogeneous second order linear differential equation with constant coefficients,

$$ay'' + by' + cy = f(x)\,. \tag{20}$$

where a, b, and c are real constants ($a \neq 0$) and $f(x)$ is a given real function. In order to find the solution to eq. (20), one first obtains the solution of the corresponding homogeneous equation,

$$ay' + by + c = 0\,. \tag{21}$$

Following Section 5 of Chapter 8 on pp. 408-414 of Boas, one first finds the roots of the corresponding auxiliary equation, $ar^2 + br + c = 0$. Denoting the two roots of the auxiliary equation by r_1 and r_2, then one can immediately write down the two linearly independent solutions to eq. (21),

$$y_h(x) = \begin{cases} Ae^{r_1 x} + Be^{r_2 x}\,, & \text{for real roots, } r_1 \neq r_2, \\ (A + Bx)e^{rx}\,, & \text{for degenerate (real) roots, } r \equiv r_1 = r_2, \\ e^{\alpha x}\big[A\sin(\beta x) + B\cos(\beta x)\big]\,, & \text{for complex roots, } r_1 \equiv \alpha + i\beta \text{ and } r_2 = (r_1)^*, \end{cases} \tag{22}$$

where A and B are arbitrary constants.

In order to find the most general solution to eq. (20), one must discover a particular solution to eq. (20), denoted by $y_p(x)$. Then, the most general solution to eq. (20) is given by,

$$y(x) = y_p(x) + y_h(x)\,. \tag{23}$$

In Section 6 of Chapter 8 on pp. 417-422 of Boas, a method is provided for finding $y_p(x)$ in cases where the function $f(x)$ in eq. (20) is of the form $e^{cx}P_n(x)$, where c is some number and P_n is a polynomial of degree n. In the general case, one can employ eq. (14) to obtain a formal solution for $y_p(x)$ no matter what function $f(x)$ appears on the right hand side of eq. (20).

To employ eq. (14), we must first compute the Wronskian of $y_1(x)$ and $y_2(x)$. First, consider the case of nondegenerate real roots, where $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$. Then eq. (5) yields,

$$W(x) = (r_2 - r_1)e^{(r_1+r_2)x}\,.$$

Next we must divide eq. (20) by a in order to match the form of eq. (4), which means that $f(x)$ is replaced by $f(x)/a$. Then, eq. (14) yields,

$$y_p(x) = \frac{1}{a(r_1 - r_2)}\left\{e^{r_1 x}\int e^{-r_1 x} f(x)\,dx - e^{r_2 x}\int e^{-r_2 x} f(x)\,dx\right\}\,. \tag{24}$$

Second, in the case of degenerate roots, $y_1(x) = e^{rx}$ and $y_2(x) = xe^{rx}$. Using eq. (5), the Wronskian is given by [cf. eq. (12)],

$$W(x) = e^{2rx}\,.$$

In this case, eq. (14) yields,

$$y_p(x) = \frac{1}{a}\left\{xe^{rx}\int e^{-rx}f(x)\,dx - e^{rx}\int xe^{-rx}f(x)\,dx\right\}. \tag{25}$$

Finally, in the case of complex roots, $y_1(x) = e^{\alpha x}\sin(\beta x)$ and $f_2(x) = e^{\alpha x}\cos(\beta x)$. Then, eq. (5) yields,

$$W(x) = -\beta e^{2\alpha x}\,.$$

In this case, eq. (14) yields,

$$y_p(x) = \frac{\beta e^{\alpha x}}{a}\left\{\sin(\beta x)\int e^{-\alpha x}\cos(\beta x)f(x) - \cos(\beta x)\int e^{-\alpha x}\sin(\beta x)f(x)\right\}. \tag{26}$$

In summary, the solution to eq. (20) is given by eq. (23), where $y_h(x)$ is given by eq. (22) in the three cases of real nondegenerate, real degenerate and complex roots of the auxiliary equation, and $y_p(x)$ is given in the three corresponding cases by eqs. (24), (25) and (26), respectively.

4. Explicit examples: finding the particular solution of an inhomogeneous second order linear differential equation

As an example, consider the following equation considered by Boas in Example 7 in Section 6 of Chapter 8 on p. 422 of Boas,

$$y'' + y - 2y = 18xe^x\,. \tag{27}$$

The auxiliary equation is $z^2 + z - 2 = (z+2)(z-1) = 0$, which has two real nondegenerate roots, $r_1 = 1$ and $r_2 = -2$. Hence, eq. (24) yields,

$$y_p(x) = 6\left\{e^x\int x\,dx - e^{-2x}\int xe^{3x}\,dx\right\} = 3x^2e^x - 2xe^x - \tfrac{2}{3}e^x\,.$$

Note that Ce^x is a solution to the homogeneous equation, $y'' + y - 2y = 0$ for any constant C [cf eq. (22)]. Hence, we can delete the term $\frac{2}{3}e^x$ from the above result. That is,

$$y_p(x) = (3x^2 - 2c)e^x\,,$$

is a particular solution to eq. (27), in agreement with eq. (6.26) on p. 422 of Boas.

Indeed, it is not difficult to verify the general result of the method of undetermined coefficients quoted in eq. (6.24) on p. 421 of Boas based on our results given in eqs. (24), (25) and (26). I leave it to the reader to determine which method is more efficient. Of course, one advantage of eqs. (24), (25) and (26) is that these results can be applied to any function $f(x)$, and not only to those functions of the special form specified in eq. (6.24) on p. 421 of Boas.

Finally, we provide one last simple example that encapsulates the main results of Section 2. Consider the differential equation,

$$y'' - \frac{y'}{x} + \frac{y}{x^2} = x\,. \tag{28}$$

We first examine the corresponding homogeneous equation,

$$y'' - \frac{y'}{x} + \frac{y}{x^2} = 0\,. \tag{29}$$

By inspection, $y_1(x) = x$ is one possible solution. Using Abel's formula [cf. eq. (9)], the Wronskian is given by

$$W(x) = \exp\left\{\int \frac{dx}{x}\right\} = x\,,$$

where we have omitted the overall factor of C in eq. (9) since this factor simply contributes to the overall normalization of the solution that we are seeking. Then, eq. (10) yields a second solution to eq. (29),

$$y_2(x) = x\int\frac{dx}{x} = x\ln|x|\,. \tag{30}$$

Finally, eq. (14) provides a particular solution to eq. (28),

$$\begin{aligned} y_p(x) &= -x\int x\ln|x|\,dx \;+\; x\ln|x|\int x\,dx \\ &= -x\left(\tfrac{1}{2}x^2\ln|x| - \tfrac{1}{4}x^2\right) + \tfrac{1}{2}x^3\ln|x| = \tfrac{1}{4}x^3\,. \end{aligned}$$

Therefore, the most general solution to eq. (28) is,

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) = \tfrac{1}{4}x^3 + \big(c_1 + c_2\ln|x|\big)x\,,$$

where c_1 and c_2 are arbitrary constants.

APPENDIX A: General formula for the Wronskian

To demonstrate that the Wronskian either vanishes for all values of x or it is never equal to zero, if the $y_i(x)$ are solutions to an nth order linear differential equation, we shall derive a general formula for the Wronskian.

Consider the differential equation,

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0\,. \tag{31}$$

We are interested in solving this differential equation for values of x that lie in an open interval of the real axis, $x_0 < x < x_1$, in which $a_0(x) \neq 0$. We can rewrite eq. (31) as a first order matrix differential equation. Defining the vector

$$\vec{\boldsymbol{Y}} = \begin{pmatrix} y \\ y' \\ y'' \\ \ldots \\ y^{(n-1)} \end{pmatrix},$$

It is straightforward to verify that eq. (31) is equivalent to

$$\frac{d\vec{\boldsymbol{Y}}}{dx} = A(x)\vec{\boldsymbol{Y}}\,,$$

where the matrix $A(x)$ is given by,

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{a_n(x)}{a_0(x)} & -\dfrac{a_{n-1}(x)}{a_0(x)} & -\dfrac{a_{n-2}(x)}{a_0(x)} & -\dfrac{a_{n-3}(x)}{a_0(x)} & \cdots & -\dfrac{a_1(x)}{a_0(x)} \end{pmatrix} . \tag{32}$$

It immediately follows that if the $y_i(x)$ are linearly independent solutions to eq. (31), then the Wronskian matrix Φ, defined in eq. (2), satisfies the first order matrix differential equation,

$$\frac{d\Phi}{dx} = A(x)\Phi \,. \tag{33}$$

Using eq. (39) of Appendix B, it follows that

$$\frac{d}{dx}\det\Phi = \det\Phi\,\mathrm{Tr}\left(\Phi^{-1}\frac{d\Phi}{dx}\right) = \det\Phi\,\mathrm{Tr}\big(\Phi^{-1}A(x)\Phi\big) = \det\Phi\,\mathrm{Tr}\;A(x)\,,$$

after employing eq. (33) and the cyclicity property of the trace (i.e. the trace is unchanged by cyclically permuting the matrices inside the trace). Hence, in terms of the Wronskian, $W \equiv \det\Phi$, defined in eq. (3),

$$\frac{dW}{dx} = W\,\mathrm{Tr}\;A(x)\,. \tag{34}$$

This is a separable first order differential equation for W that is easily integrated,

$$W(x) = W(x_0)\exp\left\{\int_{x_0}^{x}\mathrm{Tr}\,A(t)dt\right\} .$$

Using eq. (32), it follows that $\mathrm{Tr}\,A(t) = -a_1(t)/a_0(t)$. Hence, we arrive at *Abel's formula*,

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x}\frac{a_1(t)}{a_0(t)}dt\right\} . \tag{35}$$

Note that if $W(x_0) \neq 0$, then the result for $W(x)$ is strictly positive or strictly negative depending on the sign of $W(x_0)$. This confirms our assertion that the Wronskian either vanishes for all values of x or it is never equal to zero.

Of course, eq. (35) is equivalent to the version of Abel's formula obtained in eq. (9) in the case of the second order linear differential equation given by eq. (4).

Reference:

Daniel Zwillinger, *Handbook of Differential Equations*, 3rd Edition (Academic Press, San Diego, CA, 1998).

APPENDIX B: Derivative of the determinant of a matrix

Recall that for any matrix A, the determinant can be computed by the cofactor expansion. The adjugate of A, denoted by adj A is equal to the transpose of the matrix of cofactors. In particular,

$$\det A = \sum_j a_{ij} (\text{adj } A)_{ji}\,, \qquad \text{for any fixed } i\,, \tag{36}$$

where the a_{ij} are elements of the matrix A and $(\text{adj } A)_{ji} = (-1)^{i+j} M_{ij}$ where the minor M_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.

Suppose that the elements a_{ij} depend on a variable x. Then, by the chain rule,

$$\frac{d}{dx} \det A = \sum_{i,j} \frac{\partial \det A}{\partial a_{ij}}\, \frac{da_{ij}}{dx}\,. \tag{37}$$

Using eq. (36), and noting that $(\text{adj } A)_{ji}$ does not depend on a_{ij} (since the ith row and jth column are removed before computing the minor determinant),

$$\frac{\partial \det A}{\partial a_{ij}} = (\text{adj } A)_{ji}\,.$$

Hence, eq. (37) yields *Jacobi's formula:*[5]

$$\frac{d}{dx} \det A = \sum_{i,j} (\text{adj } A)_{ji} \frac{da_{ij}}{dx} = \text{Tr}\left[(\text{adj } A)\frac{dA}{dx}\right]\,. \tag{38}$$

If A is invertible, then we can use the formula

$$A^{-1} \det A = \text{Adj } A\,,$$

to rewrite eq. (38) as[6]

$$\frac{d}{dx} \det A = \det A \,\text{Tr}\left(A^{-1}\frac{dA}{dx}\right)\,, \tag{39}$$

which is the desired result.

Reference:

M.A. Goldberg, *The derivative of a determinant*, The American Mathematical Monthly, Vol. 79, No. 10 (Dec. 1972) pp. 1124–1126.

[5]Recall that if $A = [a_{ij}]$ and $B = [b_{ij}]$, then the ij matrix element of AB are given by $\sum_k a_{ik} b_{kj}$. The trace of AB is equal to the sum of its diagonal elements, or equivalently

$$\text{Tr}(AB) = \sum_{jk} a_{jk} b_{kj}\,.$$

[6]Note that $\text{Tr}\,(cB) = c\,\text{Tr}\,B$ for any number c and matrix B. In deriving eq. (39), $c = \det A$.

APPENDIX C: Another derivation of eq. (10)

Given a second order linear differential equation

$$y'' + a(x)y' + b(x)y = 0\,, \tag{40}$$

with a known solution $y_1(x)$, then one can derive a second linearly independent solution $y_2(x)$ by the method of variations of parameters.[7] Indeed, this is the method employed in Section 7 of Chapter 8 on p. 434 of Boas [corresponding to Case (e), which Boas calls *reduction of order*].

In this context, the idea of this method is to define a new variable v,

$$y_2(x) = y_1(x)v(x) = y_1(x)\int w(x)dx\,, \tag{41}$$

where $w \equiv v'$. It follows that,

$$y_2' = y_1'v + y_1 w\,, \qquad\qquad y_2'' = y_1''v + y_1 w' + 2y_1'w\,.$$

Since y_2 is a solution to eq. (40), it follows that

$$y_1 w' + [2y_1' + a(x)y_1]w + [y_1'' + a(x)y_1' + b(x)y_1]v = 0\,,$$

Using the fact that y_1 is a solution to eq. (40), the coefficient of v vanishes and we are left with a first order differential equation for w

$$y_1 w' + [2y_1' + a(x)y_1]w = 0\,.$$

This is a separable first order differential equation. Thus, we can write

$$\frac{w'}{w} = -\left[\frac{2y_1'}{y_1} + a(x)\right]dx\,. \tag{42}$$

Integrating eq. (42) and then exponentiating the resulting equation then yields,

$$w(x) = C\,\exp\left\{-\int\left(\frac{2y_1'(x)}{y_1(x)} + a(x)\right)dt\right\} = C\,e^{-2\ln|y_1(x)|}\exp\left\{-\int a(x)\,dx\right\}\,,$$

where C is an arbitrary constant. Using Abel's formula for the Wronskian given in eq. (9), it follows that

$$w(x) = \frac{C}{[y_1(x)]^2}\exp\left\{-\int xa(x)dx\right\} = \frac{W(x)}{[y_1(x)]^2}\,. \tag{43}$$

Thus, the second solution to eq. (40) defined by eq. (41) is given by

$$y_2(x) = y_1(x)\int\frac{W(x)}{[y_1(x)]^2}\,dt\,,$$

after employing eq. (43).

[7] This method is easily extended to the case of an nth order linear differential equation. In particular, if a non-trivial solution to eq. (31) is known, then this solution can be employed to reduce the order of the differential equation by one. This procedure is called reduction of order. For further details, see pp. 352–354 of Daniel Zwillinger, *Handbook of Differential Equations*, 3rd Edition (Academic Press, San Diego, CA, 1998).