The complex inverse trigonometric and hyperbolic functions

In these notes, we examine the inverse trigonometric and hyperbolic functions, where the arguments of these functions can be complex numbers. These are all multivalued functions. We also carefully define the corresponding single-valued principal values of the inverse trigonometric and hyperbolic functions following the conventions employed by the computer algebra software system, Mathematica. These conventions are outlined in section 2.2.5 of Ref. 1.

The principal value of a multivalued complex function $f(z)$ of the complex variable z, which we denote by $F(z)$, is defined in such a way that it is continuous in all regions of the complex plane, except on a specific line (or lines) called branch cuts. The function $F(z)$ has a discontinuity when z crosses a branch cut. Branch cuts end at a branch point, which is unambiguous for each function $F(z)$. But the choice of branch cuts is a matter of convention. Thus, if one needs to use mathematics software to analyze the function $F(z)$, you need to know their conventions for the location of the branch cuts. The mathematical software needs to precisely define the principal value of $f(z)$ in order that it can produce a unique answer when the user types in $F(z)$ for a particular complex number z. There are often different possible candidates for $F(z)$ that differ only in the values assigned to them when z lies on the branch cut(s). These notes address these issues as they apply to the complex inverse trigonometric and hyperbolic functions.

1. The inverse trigonometric functions: arctan and arccot

We begin by examining the solution to the equation

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{1}{i}\left(\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}\right) = \frac{1}{i}\left(\frac{e^{2iw} - 1}{e^{2iw} + 1}\right) .$$

We now solve for e^{2iw},

$$iz = \frac{e^{2iw} - 1}{e^{2iw} + 1} \qquad \Longrightarrow \qquad e^{2iw} = \frac{1 + iz}{1 - iz} .$$

Taking the complex logarithm of both sides of the equation, we can solve for w,

$$w = \frac{1}{2i}\ln\left(\frac{1 + iz}{1 - iz}\right) .$$

The solution to $z = \tan w$ is $w = \arctan z$. Hence,

$$\boxed{\arctan z = \frac{1}{2i}\ln\left(\frac{1 + iz}{1 - iz}\right)} \qquad (1)$$

Since the complex logarithm is a multivalued function, it follows that the arctangent function is also a multivalued function. Using the definition of the multivalued complex logarithm,

$$\arctan z = \frac{1}{2i}\,\mathrm{Ln}\left|\frac{1+iz}{1-iz}\right| + \tfrac{1}{2}\left[\mathrm{Arg}\left(\frac{1+iz}{1-iz}\right) + 2\pi n\right], \qquad n = 0\,,\,\pm 1\,,\,\pm 2\,,\,\ldots\,, \tag{2}$$

where Arg is the principal value of the argument function.

Similarly,

$$z = \cot w = \frac{\cos w}{\sin w} = \left(\frac{i(e^{iw} + e^{-iw}}{e^{iw} - e^{-iw}}\right) = \left(\frac{i(e^{2iw} + 1}{e^{2iw} - 1}\right).$$

Again, we solve for e^{2iw},

$$-iz = \frac{e^{2iw} + 1}{e^{2iw} - 1} \qquad \Longrightarrow \qquad e^{2iw} = \frac{iz - 1}{iz + 1}.$$

Taking the complex logarithm of both sides of the equation, we conclude that

$$w = \frac{1}{2i}\ln\left(\frac{iz-1}{iz+1}\right) = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right),$$

after multiplying numerator and denominator by $-i$ to get a slightly more convenient form. The solution to $z = \cot w$ is $w = \mathrm{arccot} z$. Hence,

$$\boxed{\mathrm{arccot} z = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)} \tag{3}$$

Thus, the arccotangent function is a multivalued function,

$$\mathrm{arccot} z = \frac{1}{2i}\,\mathrm{Ln}\left|\frac{z+i}{z-i}\right| + \tfrac{1}{2}\left[\mathrm{Arg}\left(\frac{z+i}{z-i}\right) + 2\pi n\right], \qquad n = 0\,,\,\pm 1\,,\,\pm 2\,,\,\ldots\,, \tag{4}$$

Using the definitions given by eqs. (1) and (3), the following relation is easily derived:

$$\mathrm{arccot}(z) = \arctan\left(\frac{1}{z}\right). \tag{5}$$

Note that eq. (5) can be used as the *definition* of the arccotangent function. It is instructive to derive another relation between the arctangent and arccotangent functions. First, we first recall the property of the multivalued complex logarithm,

$$\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)\,, \tag{6}$$

as a set equality. It is convenient to define a new variable,

$$v = \frac{i-z}{i+z}\,, \qquad \Longrightarrow \qquad -\frac{1}{v} = \frac{z+i}{z-i}. \tag{7}$$

It follows that:

$$\arctan z + \operatorname{arccot} z = \frac{1}{2i}\left[\ln v + \ln\left(-\frac{1}{v}\right)\right] = \frac{1}{2i}\ln\left(\frac{-v}{v}\right) = \frac{1}{2i}\ln(-1)\,.$$

Since $\ln(-1) = i(\pi + 2\pi n)$ for $n = 0, \pm 1, \pm 2\ldots$, we conclude that

$$\boxed{\arctan z + \operatorname{arccot} z = \tfrac{1}{2}\pi + \pi n\,, \qquad\qquad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots} \tag{8}$$

Finally, we mention two equivalent forms for the multivalued complex arctangent and arccotangent functions. Recall that the complex logarithm satisfies

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2\,, \tag{9}$$

where this equation is to be viewed as a set equality, where each set consists of all possible results of the multivalued function. Thus, the multivalued arctangent and arccotangent functions given in eqs. (1) and (5), respectively, are equivalent to

$$\arctan z = \frac{1}{2i}\left[\ln(1 + iz) - \ln(1 - iz)\right], \tag{10}$$

$$\operatorname{arccot} z = \frac{1}{2i}\left[\ln\left(1 + \frac{i}{z}\right) - \ln\left(1 - \frac{i}{z}\right)\right], \tag{11}$$

2. The principal values Arctan and Arccot

It is convenient to define principal values of the inverse trigonometric functions, which are single-valued functions, which will necessarily exhibit a discontinuity across some appropriately chosen line in the complex plane. In Mathematica, the principal values of the complex arctangent and arccotangent functions, denoted by Arctan and Arccot respectively (with an upper case A), are defined by employing the principal values of the complex logarithms in eqs. (10) and (11),

$$\boxed{\operatorname{Arctan} z = \frac{1}{2i}\left[\operatorname{Ln}(1 + iz) - \operatorname{Ln}(1 - iz)\right], \qquad z \neq \pm i} \tag{12}$$

and

$$\boxed{\operatorname{Arccot} z = \operatorname{Arctan}\left(\frac{1}{z}\right) = \frac{1}{2i}\left[\operatorname{Ln}\left(1 + \frac{i}{z}\right) - \operatorname{Ln}\left(1 - \frac{i}{z}\right)\right], \quad z \neq \pm i\,,\, z \neq 0} \tag{13}$$

One useful feature of these definitions is that they satisfy:

$$\operatorname{Arctan}(-z) = -\operatorname{Arctan} z\,, \quad \text{for } z \neq \pm i,$$

$$\operatorname{Arccot}(-z) = -\operatorname{Arccot} z\,, \quad \text{for } z \neq \pm i \text{ and } z \neq 0. \tag{14}$$

Because the principal value of the complex logarithm Ln does not satisfy eq. (9) in all regions of the complex plane, it follows that the definitions of the complex arctangent and arccotangent functions adopted by Mathematica do not coincide with some alternative definitions employed by some of the well known mathematical reference books [for further details, see Appendix A]. Note that the points $z = \pm i$ are excluded from the above definitions, as the arctangent and arccotangent are divergent at these two points. The definition of the principal value of the arccotangent given in eq. (13) is deficient in one respect since it is not well-defined at $z = 0$. We shall address this problem shortly.

First, we shall identify the location of the discontinuity of the principal values of the complex arctangent and arccotangent functions in the complex plane. The principal value of the complex arctangent function is single-valued for all $z \neq \pm i$. These two points, called *branch points*, must be excluded as the arctangent function is singular there. Moreover, the the principal-valued logarithms, $\mathrm{Ln}\,(1 \pm iz)$ are discontinuous as z crosses the lines $1 \pm iz < 0$, respectively. We conclude that $\mathrm{Arctan}\, z$ must be discontinuous when $z = x + iy$ crosses lines on the imaginary axis such that

$$x = 0 \quad \text{and} \quad -\infty < y < -1 \quad \text{and} \quad 1 < y < \infty\,. \tag{15}$$

These two lines that lie along the imaginary axis are called the *branch cuts* of $\mathrm{Arctan}\, z$.

Note that $\mathrm{Arctan}\, z$ is single-valued on the branch cut itself, since it inherits this property from the principal value of the complex logarithm. In particular, for values of $z = iy$ ($|y| > 1$) that lie on the branch cut of $\mathrm{Arctan}\, z$, eq. (12) yields,

$$\mathrm{Arctan}(iy) = \begin{cases} \frac{1}{2i}\mathrm{Ln}\left(\frac{y-1}{y+1}\right) - \frac{1}{2}\pi\,, & \text{for } -\infty < y < -1\,, \\ \frac{1}{2i}\mathrm{Ln}\left(\frac{y-1}{y+1}\right) + \frac{1}{2}\pi\,, & \text{for } \quad 1 < y < \infty\,. \end{cases} \tag{16}$$

Likewise, the principal value of the complex arccotangent function is single-valued for all complex z excluding the branch points $z \neq \pm i$. Moreover, the the principal-valued logarithms, $\mathrm{Ln}\left(1 \pm \frac{i}{z}\right)$ are discontinuous as z crosses the lines $1 \pm \frac{i}{z} < 0$, respectively. We conclude that $\mathrm{Arccot}\, z$ must be discontinuous when $z = x + iy$ crosses the branch cuts located on the imaginary axis such that

$$x = 0 \quad \text{and} \quad -1 < y < 1\,. \tag{17}$$

In particular, due to the presence of the branch cut,

$$\lim_{x\to 0^-} \mathrm{Arccot}(x + iy) \neq \lim_{x\to 0^+} \mathrm{Arccot}(x + iy)\,, \qquad \text{for } -1 < y < 1\,,$$

for real values of x, where 0^+ indicates that the limit is approached from positive real axis and 0^- indicates that the limit is approached from negative real axis. If $z \neq 0$, eq. (13) provides unique values for $\mathrm{Arccot}\, z$ for all $z \neq \pm i$ in the complex plane, including on the branch cut. Using eq. (13), one can easily show that if z is a nonzero complex number infinitesimally close to 0, then it follow that,

$$\text{Arccot}\, z \underset{z\to 0\,,\, z\neq 0}{=} \begin{cases} \tfrac{1}{2}\pi\,, & \text{for Re } z > 0\,, \\ \tfrac{1}{2}\pi\,, & \text{for Re } z = 0 \text{ and Im } z < 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for Re } z < 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for Re } z = 0 \text{ and Im } z > 0\,. \end{cases} \tag{18}$$

It is now apparent why the point $z = 0$ is problematical in eq. (13), since there is no well defined way of defining Arccot(0). Indeed, for values of $z = iy$ $(-1 < y < 1)$ that lie on the branch cut of Arccot$z$, eq. (13) yields,

$$\text{Arccot}(iy) = \begin{cases} \frac{1}{2i}\text{Ln}\left(\frac{1+y}{1-y}\right) + \frac{1}{2}\pi\,, & \text{for } \ -1 < y < 0\,, \\ \frac{1}{2i}\text{Ln}\left(\frac{1+y}{1-y}\right) - \frac{1}{2}\pi\,, & \text{for } \quad 0 < y < 1\,. \end{cases} \tag{19}$$

Mathematica supplements the definition of the principal value of the complex arccotangent given in eq. (13) by declaring that

$$\text{Arccot}(0) = \tfrac{1}{2}\pi\,. \tag{20}$$

With the definitions given in eqs. (12), (13) and (20), Arctan z and Arccot z are single-valued functions in the entire complex plane, excluding the branch points $z = \pm i$ and are continuous functions as long as the complex number z does not cross the branch cuts specified in eqs. (15) and (17), respectively.

Having defined precisely the principal values of the complex arctangent and arccotangent functions, let us check that they reduce to the conventional definitions when z is real. First consider the principal value of the real arctangent function, which satisfies

$$-\tfrac{1}{2}\pi \leq \text{Arctan}\, x \leq \tfrac{1}{2}\pi\,, \qquad \text{for } -\infty \leq x \leq \infty\,, \tag{21}$$

where x is a real variable. The definition given by eq. (12) does reduce to the conventional definition of the principal value of the real-valued arctangent function when z is real. In particular, for real values of x,

$$\text{Arctan}\, x = \frac{1}{2i}\biggl[\text{Ln}(1+ix) - \text{Ln}(1-ix)\biggr] = \tfrac{1}{2}\biggl[\text{Arg}(1+ix) - \text{Arg}(1-ix)\biggr]\,, \tag{22}$$

after noting that $\text{Ln}|1+ix| = \text{Ln}|1-ix| = \frac{1}{2}\text{Ln}(1+x^2)$. Geometrically, the quantity $\text{Arg}(1+ix) - \text{Arg}(1-ix)$ is the angle between the complex numbers $1+ix$ and $1-ix$ viewed as vectors lying in the complex plane. This angle varies between $-\pi$ and π over the range $-\infty < x < \infty$. Moreover, the values $\pm\pi$ are achieved in the limit as $x \to \pm\infty$, respectively. Hence, we conclude that the principal interval of the real-valued arctangent function is indeed given by eq. (21). For all possible values of x excluding $x = -\infty$, one can check that it is permissible to subtract the two principal-valued logarithms (or equivalently the two Arg functions) using eq. (9). In the case of $x \to -\infty$, we see that

$\text{Arg}(1+ix) - \text{Arg}(1-ix) \to -\pi$, in which case $N_- = -1$ [cf. eq. (78) below[1]] and an extra term appears when combining the two logarithms that is equal to $2\pi i N_- = -2\pi i$. The end result is,

$$\text{Arctan}(-\infty) = \frac{1}{2i}\left[\ln(-1) - 2\pi i\right] = -\tfrac{1}{2}\pi\,,$$

as required. As a final check, we can use the results of Tables 1 and 2 in the class handout, *The Argument of a Complex Number*, to conclude that $\text{Arg}(a+bi) = \text{Arctan}(b/a)$ for $a > 0$. Setting $a = 1$ and $b = x$ then yields:

$$\text{Arg}\,(1+ix) = \text{Arctan}\,x\,, \qquad\qquad \text{Arg}\,(1-ix) = \text{Arctan}(-x) = -\text{Arctan}\,x\,.$$

Subtracting these two results yields eq. (22).

In contrast to the real arctangent function, there is no generally agreed definition for the principal range of the real-valued arccotangent function. However, a growing consensus among computer scientists has led to the following choice for the principal range of the real-valued arccotangent function,

$$-\tfrac{1}{2}\pi < \text{Arccot}\,x \leq \tfrac{1}{2}\pi\,, \qquad \text{for } -\infty \leq x \leq \infty\,, \tag{23}$$

where x is a real variable. Note that the principal value of the arccotangent function does not include the endpoint $-\frac{1}{2}\pi$ [contrast this with eq. (21) for Arctan]. The reason for this behavior is that $\text{Arccot}\,x$ is *discontinuous* at $x = 0$. In particular,

$$\lim_{x\to 0^-} \text{Arccot}\,x = -\tfrac{1}{2}\pi\,, \qquad\qquad \lim_{x\to 0^+} \text{Arccot}\,x = \tfrac{1}{2}\pi\,, \tag{24}$$

as a consequence of eq. (18). In particular, eq. (23) corresponds to the convention in which $\text{Arccot}(0) = \frac{1}{2}\pi$ [cf. eq. (20)]. Thus, as x increases from negative to positive values, $\text{Arccot}\,x$ never reaches $-\frac{1}{2}\pi$ but jumps discontinuously to $\frac{1}{2}\pi$ at $x = 0$.

Finally, we examine the the analog of eq. (8) for the corresponding principal values. Employing the Mathematica definitions for the principal values of the complex arctangent and arccotangent functions, we find that

$$\text{Arctan}\,z + \text{Arccot}\,z = \begin{cases} \tfrac{1}{2}\pi\,, & \text{for Re } z > 0\,, \\ \tfrac{1}{2}\pi\,, & \text{for Re } z = 0\,, \text{ and Im } z > 1 \text{ or } -1 < \text{Im } z \leq 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for Re } z < 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for Re } z = 0\,, \text{ and Im } z < -1 \text{ or } 0 < \text{Im } z < 1\,. \end{cases} \tag{25}$$

The derivation of this result will be given in Appendix B. In Mathematica, one can confirm eq. (25) with many examples.

[1]See eqs. (12), (13) and (55) of the class handout entitled, *The complex logarithm, exponential and power functions.*

3. The inverse trigonometric functions: arcsin and arccos

The arcsine function is the solution to the equation:

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}\,.$$

Letting $v \equiv e^{iw}$, we solve the equation

$$v - \frac{1}{v} = 2iz\,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2izv - 1 = 0\,. \tag{26}$$

The solution to eq. (26) is:

$$v = iz + (1 - z^2)^{1/2}\,. \tag{27}$$

Since z is a complex variable, $(1-z^2)^{1/2}$ is the complex square-root function. This is a multivalued function with two possible values that differ by an overall minus sign. Hence, we do not explicitly write out the $\pm$ sign in eq. (27). To avoid ambiguity, we shall write

$$\begin{aligned} v &= iz + (1-z^2)^{1/2} = iz + e^{\frac{1}{2}\ln(1-z^2)} = iz + e^{\frac{1}{2}\left[\mathrm{Ln}|1-z^2| + i\arg(1-z^2)\right]} \\ &= iz + |1-z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\,. \end{aligned}$$

In particular, note that

$$e^{\frac{i}{2}\arg(1-z^2)} = e^{\frac{i}{2}\mathrm{Arg}(1-z^2)} e^{in\pi} = \pm e^{\frac{i}{2}\mathrm{Arg}(1-z^2)}\,, \qquad \text{for } n = 0, 1\,,$$

which exhibits the two possible sign choices.

By definition, $v \equiv e^{iw}$, from which it follows that

$$w = \frac{1}{i}\ln v = \frac{1}{i}\ln\left(iz + |1-z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\right)\,.$$

The solution to $z = \sin w$ is $w = \arcsin z$. Hence,

$$\boxed{\arcsin z = \frac{1}{i}\ln\left(iz + |1-z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\right)}$$

The arccosine function is the solution to the equation:

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2}\,.$$

Letting $v \equiv e^{iw}$, we solve the equation

$$v + \frac{1}{v} = 2z\,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2zv + 1 = 0 \,. \tag{28}$$

The solution to eq. (28) is:

$$v = z + (z^2 - 1)^{1/2} \,.$$

Following the same steps as in the analysis of arcsine, we write

$$w = \arccos z = \frac{1}{i} \ln v = \frac{1}{i} \ln \left[z + (z^2 - 1)^{1/2} \right] \,, \tag{29}$$

where $(z^2 - 1)^{1/2}$ is the multivalued square root function. More explicitly,

$$\arccos z = \frac{1}{i} \ln \left(z + |z^2 - 1|^{1/2} e^{\frac{i}{2} \arg(z^2 - 1)} \right) \,. \tag{30}$$

It is sometimes more convenient to rewrite eq. (30) in a slightly different form. Recall that

$$\arg(z_1 z_2) = \arg z + \arg z_2 \,, \tag{31}$$

as a *set equality*. We now substitute $z_1 = z$ and $z_2 = -1$ into eq. (31) and note that $\arg(-1) = \pi + 2\pi n$ (for $n = 0, \pm 1, \pm 2, \ldots$) and $\arg z = \arg z + 2\pi n$ as a set equality. It follows that $\arg(-z) = \pi + \arg z$, as a set equality. Thus,

$$e^{\frac{i}{2} \arg(z^2 - 1)} = e^{i\pi/2} \, e^{\frac{i}{2} \arg(1 - z^2)} = i e^{\frac{i}{2} \arg(1 - z^2)} \,,$$

and we can rewrite eq. (29) as follows:

$$\arccos z = \frac{1}{i} \ln \left(z + i\sqrt{1 - z^2} \right) \,, \tag{32}$$

which is equivalent to the more explicit form,

$$\boxed{\arccos z = \frac{1}{i} \ln \left(z + i|1 - z^2|^{1/2} e^{\frac{i}{2} \arg(1 - z^2)} \right)}$$

The arcsine and arccosine functions are related in a very simple way. Using eq. (27),

$$\frac{i}{v} = \frac{i}{iz + \sqrt{1 - z^2}} = \frac{i(-iz + \sqrt{1 - z^2})}{(iz + \sqrt{1 - z^2})(-iz + \sqrt{1 - z^2})} = z + i\sqrt{1 - z^2} \,,$$

which we recognize as the argument of the logarithm in the definition of the arccosine [cf. eq. (32)]. Using eq. (6), it follows that

$$\arcsin z + \arccos z = \frac{1}{i} \left[\ln v + \ln \left(\frac{i}{v} \right) \right] = \frac{1}{i} \ln \left(\frac{iv}{v} \right) = \frac{1}{i} \ln i \,.$$

Since $\ln i = i(\frac{1}{2}\pi + 2\pi n)$ for $n = 0, \pm 1, \pm 2 \ldots$, we conclude that

$$\boxed{\arcsin z + \arccos z = \tfrac{1}{2}\pi + 2\pi n \,, \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots}$$

4. The principal values Arcsin and Arccos

In Mathematica, the principal value of the arcsine function is obtained by employing the principal value of the logarithm and the principle value of the square-root function (which corresponds to employing the principal value of the argument). Thus,

$$\boxed{\operatorname{Arcsin} z = \frac{1}{i}\operatorname{Ln}\left(iz + |1-z^2|^{1/2} e^{\frac{i}{2}\operatorname{Arg}(1-z^2)}\right)} \tag{33}$$

We now examine the principal value of the arcsine for real-valued arguments. Setting $z = x$, where x is real,

$$\operatorname{Arcsin} x = \frac{1}{i}\operatorname{Ln}\left(ix + |1-x^2|^{1/2} e^{\frac{i}{2}\operatorname{Arg}(1-x^2)}\right) .$$

For $|x| < 1$, $\operatorname{Arg}(1-x^2) = 0$ and $|1-x^2| = \sqrt{1-x^2}$ defines the positive square root. Thus,

$$\begin{aligned}\operatorname{Arcsin} x &= \frac{1}{i}\operatorname{Ln}\left(ix + \sqrt{1-x^2}\right) = \frac{1}{i}\left[\operatorname{Ln}\left|ix + \sqrt{1-x^2}\right| + i\operatorname{Arg}\left(ix + \sqrt{1-x^2}\right)\right] \\ &= \operatorname{Arg}\left(ix + \sqrt{1-x^2}\right) ,\end{aligned} \tag{34}$$

since $ix + \sqrt{1-x^2}$ is a complex number with magnitude equal to 1. Moreover, $ix + \sqrt{1-x^2}$ lives either in the first or fourth quadrant of the complex plane, since $\operatorname{Re}(ix + \sqrt{1-x^2}) \geq 0$. It follows that:

$$-\frac{\pi}{2} \leq \operatorname{Arcsin} x \leq \frac{\pi}{2} , \qquad \text{for} \quad |x| \leq 1 .$$

In Mathematica, the principal value of the arccosine is defined by:

$$\operatorname{Arccos} z = \tfrac{1}{2}\pi - \operatorname{Arcsin} z . \tag{35}$$

We will demonstrate below that this definition is equivalent to choosing the principal value of the complex logarithm and the principal value of the square root in eq. (32). That is,

$$\boxed{\operatorname{Arccos} z = \frac{1}{i}\operatorname{Ln}\left(z + i|1-z^2|^{1/2} e^{\frac{i}{2}\operatorname{Arg}(1-z^2)}\right)} \tag{36}$$

It is straightforward to check that the principal values of arcsin and arccos satisfy,

$$\begin{aligned}\operatorname{Arcsin}(-z) &= -\operatorname{Arcsin} z , \\ \operatorname{Arccos}(-z) &= \pi - \operatorname{Arccos} z .\end{aligned} \tag{37}$$

We now examine the principal value of the arccosine for real-valued arguments. Setting $z = x$, where x is real,

$$\text{Arccos}\, x = \frac{1}{i}\text{Ln}\left(x + i|1-x^2|^{1/2}e^{\frac{i}{2}\text{Arg}(1-x^2)}\right) .$$

As previously noted, for $|x| < 1$ we have $\text{Arg}(1-x^2) = 0$, and $|1-x^2| = \sqrt{1-x^2}$ defines the positive square root. Thus,

$$\begin{aligned}\text{Arccos}\, x = \frac{1}{i}\text{Ln}\left(x + i\sqrt{1-x^2}\right) &= \frac{1}{i}\left[\text{Ln}\left|x + i\sqrt{1-x^2}\right| + i\text{Arg}\left(x + i\sqrt{1-x^2}\right)\right] \\ &= \text{Arg}\left(x + i\sqrt{1-x^2}\right) , \end{aligned} \tag{38}$$

since $x + i\sqrt{1-x^2}$ is a complex number with magnitude equal to 1. Moreover, the quantity $x + i\sqrt{1-x^2}$ lives either in the first or second quadrant of the complex plane, since $\text{Im}(x + i\sqrt{1-x^2}) \geq 0$. It follows that:

$$0 \leq \text{Arccos}\, x \leq \pi \,, \qquad \text{for} \quad |x| \leq 1 \,.$$

We now verify that eq. (35) is a consequence of eq. (36). Using the principal value of the square root, we define:

$$v = iz + |1-z^2|^{1/2}e^{\frac{i}{2}\text{Arg}(1-z^2)} \,, \qquad\qquad \frac{i}{v} = z + i|1-z^2|^{1/2}e^{\frac{i}{2}\text{Arg}(1-z^2)} \,.$$

Then,

$$\begin{aligned}\text{Arcsin}\, z + \text{Arccos}\, z &= \frac{1}{i}\left[\text{Ln}|v| + \text{Ln}\left(\frac{1}{|v|}\right) + i\text{Arg}\, v + i\text{Arg}\left(\frac{i}{v}\right)\right] \\ &= \text{Arg}\ v + \text{Arg}\left(\frac{i}{v}\right) . \end{aligned} \tag{39}$$

In particular, since $\text{Re}\ (\pm iz) = \mp\text{Im}\ z$ for any complex number z,

$$\text{Re}\ v = -\text{Im}\ z + |1-z^2|^{1/2}\cos\left[\tfrac{1}{2}\text{Arg}(1-z^2)\right] , \tag{40}$$

$$\text{Re}\left(\frac{1}{v}\right) = \text{Im}\ z + |1-z^2|^{1/2}\cos\left[\tfrac{1}{2}\text{Arg}(1-z^2)\right] . \tag{41}$$

One can now prove that

$$\text{Re}\ v \geq 0 \,, \tag{42}$$

for any complex number z by considering separately the cases of $\text{Im}\ z \leq 0$ and $\text{Im}\ z > 0$. Note that $-\pi < \text{Arg}(1-z^2) \leq \pi$ implies that $\cos\left[\frac{1}{2}\text{Arg}(1-z^2)\right] \geq 0$. Thus if $\text{Im}\, z \leq 0$, then it immediately follows from eq. (40) that $\text{Re}\ v \geq 0$. Likewise, if $\text{Im}\, z > 0$, then it immediately follows from eq. (41) that Re $(1/v) > 0$. However, the sign of the real part of any complex number z is the *same* as the sign of the real part of $1/z$, since

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}\,.$$

Hence, we again conclude that Re $v > 0$, and eq. (42) is proven.

It is straightforward to check that:

$$\text{Arg } v + \text{Arg}\left(\frac{i}{v}\right) = \tfrac{1}{2}\pi\,, \qquad \text{for Re } v \geq 0\,.$$

Hence, eq. (39) yields:

$$\text{Arcsin}\, z + \text{Arccos}\, z = \tfrac{1}{2}\pi\,,$$

as claimed.

The principal value of the complex arcsine and arccosine functions are single-valued for all complex z. Moreover, due to the branch cut of the principal value square root function,[2] it follows that Arcsin z and Arccos z are both discontinuous when $z = x + iy$ crosses lines on the real axis such that

$$y = 0 \quad \text{and} \quad -\infty < x < -1 \quad \text{and} \quad 1 < x < \infty\,. \tag{43}$$

These two lines comprise the branch cuts of Arcsin z and Arccos z; each branch cut ends at a branch point located at $x = -1$ and $x = 1$, respectively (although the square root function is not divergent at these points).

5. The inverse hyperbolic functions: arctanh and arccoth

Consider the solution to the equation

$$z = \tanh w = \frac{\sinh w}{\cosh w} = \left(\frac{e^w - e^{-w}}{e^w + e^{-w}}\right) = \left(\frac{e^{2w}-1}{e^{2w}+1}\right)\,.$$

We now solve for e^{2w},

$$z = \frac{e^{2w}-1}{e^{2w}+1} \qquad \Longrightarrow \qquad e^{2w} = \frac{1+z}{1-z}\,.$$

Taking the complex logarithm of both sides of the equation, we can solve for w,

$$w = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)\,.$$

[2]One can check that the branch cut of the Ln function in eqs. (33) and (36) is never encountered for any value of z. For example, in the case of Arcsin z, the branch cut of Ln can only be reached if $iz + \sqrt{1-z^2}$ is real and negative. But this never happens since if $iz + \sqrt{1-z^2}$ is real then $z = iy$ for some real value of y, in which case $iz + \sqrt{1-z^2} = -y + \sqrt{1+y^2} > 0$.

The solution to $z = \tanh w$ is $w = \text{arctanh} z$. Hence,

$$\boxed{\text{arctanh}\, z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)} \tag{44}$$

Similarly, by considering the solution to the equation

$$z = \coth w = \frac{\cosh w}{\sinh w} = \left(\frac{e^w + e^{-w}}{e^w - e^{-w}}\right) = \left(\frac{e^{2w}+1}{e^{2w}-1}\right) .$$

we end up with:

$$\boxed{\text{arccoth} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)} \tag{45}$$

The above results then yield:

$$\text{arccoth}(z) = \text{arctanh}\left(\frac{1}{z}\right) ,$$

as a set equality.

Finally, we note the relation between the inverse trigonometric and the inverse hyperbolic functions:

$$\begin{aligned} \text{arctanh}\, z &= i \arctan(-iz) , \\ \text{arccoth}\, z &= i\, \text{arccot}(iz) . \end{aligned}$$

As in the discussion at the end of Section 1, one can rewrite eqs. (44) and (45) in an equivalent form:

$$\text{arctanh}\, z = \tfrac{1}{2}\left[\ln(1+z) - \ln(1-z)\right] , \tag{46}$$

$$\text{arccoth}\, z = \tfrac{1}{2}\left[\ln\left(1+\frac{1}{z}\right) - \ln\left(1-\frac{1}{z}\right)\right] . \tag{47}$$

6. The principal values Arctanh and Arccoth

Mathematica defines the principal values of the inverse hyperbolic tangent and inverse hyperbolic cotangent, Arctanh and Arccoth, by employing the principal value of the complex logarithms in eqs. (46) and (47). We can define the principal value of the inverse hyperbolic tangent function by employing the principal value of the logarithm,

$$\boxed{\text{Arctanh}\, z = \tfrac{1}{2}\left[\text{Ln}(1+z) - \text{Ln}(1-z)\right]} \tag{48}$$

and

$$\boxed{\text{Arccoth}\, z = \text{Arctanh}\left(\frac{1}{z}\right) = \frac{1}{2}\left[\text{Ln}\left(1+\frac{1}{z}\right) - \text{Ln}\left(1-\frac{1}{z}\right)\right]} \tag{49}$$

Note that the branch points at $z = \pm 1$ are excluded from the above definitions, as Arctanh z and Arccoth z are divergent at these two points. The definition of the principal value of the inverse hyperbolic cotangent given in eq. (49) is deficient in one respect since it is not well-defined at $z = 0$. For this special case, Mathematica defines

$$\text{Arccoth}(0) = \tfrac{1}{2} i\pi \,. \tag{50}$$

Of course, this discussion parallels that of Section 2. Moreover, alternative definitions of Arctanh z and Arccoth z analogous to those defined in Appendix A for the corresponding inverse trigonometric functions can be found in Refs. 2 and 3. In some sense there is no need to repeat all this since a comparison of eqs. (12) and (13) with eqs. (48) and (49) show that the inverse trigonometric and inverse hyperbolic tangent and cotangent functions are related by:

$$\text{Arctanh}\, z = i \text{Arctan}(-iz) \,,$$
$$\text{Arccoth}\, z = i \,\text{Arccot}(iz) \,.$$

Using these results, all other properties of the inverse hyperbolic tangent and cotangent functions can be easily derived from the properties of the corresponding arctangent and arccotangent functions.

For example the branch cuts of these functions are easily obtained from eqs. (15) and (17). Arctanh z is discontinuous when $z = x + iy$ crosses the branch cuts located on the real axis such that

$$y = 0 \quad \text{and} \quad -\infty < x < -1 \quad \text{and} \quad 1 < x < \infty \,. \tag{51}$$

Arccoth z is discontinuous when $z = x + iy$ crosses the branch cuts located on the real axis such that

$$y = 0 \quad \text{and} \quad -1 < x < 1 \,. \tag{52}$$

7. The inverse hyperbolic functions: arcsinh and arccosh

The inverse hyperbolic sine function is the solution to the equation:

$$z = \sinh w = \frac{e^w - e^{-w}}{2} \,.$$

Letting $v \equiv e^w$, we solve the equation

$$v - \frac{1}{v} = 2z \,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2zv - 1 = 0 \,. \tag{53}$$

The solution to eq. (53) is:

$$v = z + (1 + z^2)^{1/2} \,. \tag{54}$$

Since z is a complex variable, $(1 + z^2)^{1/2}$ is the complex square-root function. This is a multivalued function with two possible values that differ by an overall minus sign. Hence, we do not explicitly write out the $\pm$ sign in eq. (54). To avoid ambiguity, we shall write

$$\begin{aligned} v &= z + (1 + z^2)^{1/2} = z + e^{\frac{1}{2}\ln(1+z^2)} = z + e^{\frac{1}{2}\left[\mathrm{Ln}|1+z^2| + i\arg(1+z^2)\right]} \\ &= z + |1 + z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)} \,. \end{aligned}$$

By definition, $v \equiv e^w$, from which it follows that

$$w = \ln v = \ln\left(z + |1 + z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\right) \,.$$

The solution to $z = \sinh w$ is $w = \mathrm{arcsinh} z$. Hence,

$$\boxed{\mathrm{arcsinh} z = \ln\left(z + |1 + z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\right)} \tag{55}$$

The inverse hyperbolic cosine function is the solution to the equation:

$$z = \cos w = \frac{e^w + e^{-w}}{2} \,.$$

Letting $v \equiv e^w$, we solve the equation

$$v + \frac{1}{v} = 2z \,.$$

Multiplying by v, one obtains a quadratic equation for v,

$$v^2 - 2zv + 1 = 0 \,. \tag{56}$$

The solution to eq. (56) is:

$$v = z + (z^2 - 1)^{1/2} \,.$$

Following the same steps as in the analysis of inverse hyperbolic sine function, we write

$$w = \mathrm{arccosh}\, z = \ln v = \ln\left[z + (z^2 - 1)^{1/2}\right] \,, \tag{57}$$

where $(z^2 - 1)^{1/2}$ is the multivalued square root function. More explicitly,

$$\boxed{\mathrm{arccosh}\, z = \ln\left(z + |z^2 - 1|^{1/2} e^{\frac{i}{2}\arg(z^2-1)}\right)}$$

The multivalued square root function satisfies:

$$(z^2 - 1)^{1/2} = (z + 1)^{1/2}(z - 1)^{1/2} \,.$$

Hence, an equivalent form for the multivalued inverse hyperbolic cosine function is:

$$\operatorname{arccosh} z = \ln\left[z + (z+1)^{1/2}(z-1)^{1/2}\right] ,$$

where we again remind the reader that the multivalued square-root functions are employed above. More precisely,

$$\operatorname{arccosh} z = \ln\left(z + |z^2-1|^{1/2} e^{\frac{i}{2}\arg(z+1)} e^{\frac{i}{2}\arg(z-1)}\right) . \tag{58}$$

Finally, we note the relations between the inverse trigonometric and the inverse hyperbolic functions:

$$\operatorname{arcsinh} z = i \arcsin(-iz) , \tag{59}$$

$$\operatorname{arccosh} z = i \arccos z , \tag{60}$$

where the above equalities are interpreted as set inequalities for the multivalued functions. In deriving the second relation above, we have employed eqs. (29) and (57).

8. The principal values Arcsinh and Arccosh

The principal value of the inverse hyperbolic sine function, Arcsinh z, defined by Mathematica is obtained from eq. (55) by replacing the complex logarithm and argument functions by their principal value. This is equivalent to choosing the principal value of the square-root function in eq. (54). That is,

$$\boxed{\operatorname{Arcsinh} z = \operatorname{Ln}\left(z + |1+z^2|^{1/2} e^{\frac{i}{2}\operatorname{Arg}(1+z^2)}\right)} \tag{61}$$

For the principal value of the inverse hyperbolic cosine function Arccosh z, Mathematica chooses eq. (58) with the complex logarithm and argument functions replaced by their principal values. That is,

$$\boxed{\operatorname{Arccosh} z = \operatorname{Ln}\left(z + |z^2-1|^{1/2} e^{\frac{i}{2}\operatorname{Arg}(z+1)} e^{\frac{i}{2}\operatorname{Arg}(z-1)}\right)} \tag{62}$$

The relation between the principal values of the inverse trigonometric and the inverse hyperbolic sine functions is given by

$$\operatorname{Arcsinh} z = i\operatorname{Arcsin}(-iz) ,$$

as one might expect in light of eq. (59). Unfortunately, a comparison of eqs. (36) and (62) reveals that in contrast to the simple behavior of eq. (60),

$$\operatorname{Arccosh} z = \begin{cases} i\operatorname{Arccos} z , & \text{for either } \operatorname{Im} z > 0 \text{ or for } \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \leq 1 , \\ -i\operatorname{Arccos} z , & \text{for either } \operatorname{Im} z < 0 \text{ or for } \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \geq 1 . \end{cases} \tag{63}$$

Note that both formulae above are true for the case of Im $z = 0$ and Re $z = 1$, since for this special point,

$$\text{Arccosh}(1) = \text{Arccos}(1) = 0\,. \tag{64}$$

For a derivation of eq. (63), see Appendix C.

The principal value of the inverse hyperbolic sine and cosine functions are single-valued for all complex z. Moreover, due to the branch cut of the principal value square root function,[3] it follows that Arcsinh z is discontinuous when $z = x + iy$ crosses lines on the imaginary axis such that

$$x = 0 \quad \text{and} \quad -\infty < y < -1 \quad \text{and} \quad 1 < y < \infty\,. \tag{65}$$

These two lines comprise the branch cuts of Arcsinh z, and each branch cut ends at a branch point located at $z = -i$ and $z = i$, respectively (although the square root function is not divergent at these points). The branch cut for Arccosh z derives from the branch cuts of the square root function and the branch cut of the complex logarithm. In particular, for real z satisfying $|z| < 1$, we have a branch cut due to $(z+1)^{1/2}(z-1)^{1/2}$, whereas for real z satisfying $-\infty < z \leq -1$, the branch cut of the complex logarithm takes over. Hence, it follows that Arccosh z is discontinuous when $z = x + iy$ crosses lines on the real axis such that

$$y = 0 \quad \text{and} \quad -\infty < x < 1\,. \tag{66}$$

This branch cut ends at a branch point located at $z = 1$.

APPENDIX A: Alternative definitions for Arctan and Arccot

The well-known reference book for mathematical functions by Abramowitz and Stegun (see Ref. 2) and the more recent NIST Handbook of Mathematical Functions (see Ref. 3) define the principal values of the complex arctangent and arccotangent functions as follows,

$$\text{Arctan}\, z = \tfrac{1}{2} i \,\text{Ln}\left(\frac{1 - iz}{1 + iz}\right)\,, \tag{67}$$

$$\text{Arccot}\, z = \text{Arctan}\left(\frac{1}{z}\right) = \tfrac{1}{2} i \,\text{Ln}\left(\frac{z - i}{z + i}\right)\,. \tag{68}$$

With these definitions, the branch cuts are still given by eqs. (15) and (17), respectively. Comparing the above definitions with those of eqs. (12) and (13), one can check that the two definitions differ only on the branch cuts. One can use eqs. (67) and (68) to

[3] One can check that the branch cut of the Ln function in eq. (61) is never encountered for any value of z. In particular, the branch cut of Ln can only be reached if $z + \sqrt{1 + z^2}$ is real and negative. But this never happens since if $z + \sqrt{1 + z^2}$ is real then z is also real. But for any real value of z, we have $z + \sqrt{1 + z^2} > 0$.

define the single-valued functions by employing the standard conventions for evaluating the complex logarithm on its branch cut [namely, by defining $\text{Arg}(-x) = \pi$ for any real positive number x].[4] For example, for values of $z = iy$ ($|y| > 1$) that lie on the branch cut of $\text{Arctan}\, z$, eq. (67) yields,[5]

$$\text{Arctan}(iy) = \frac{i}{2}\text{Ln}\left(\frac{y+1}{y-1}\right) - \tfrac{1}{2}\pi\,, \quad \text{for } |y| > 1\,. \tag{69}$$

This result differs from eq. (16) when $1 < y < \infty$.

It is convenient to define a new variable,

$$v = \frac{1-iz}{1+iz} = \frac{i+z}{i-z}\,, \qquad \Longrightarrow \qquad -\frac{1}{v} = \frac{z-i}{z+i}\,. \tag{70}$$

Then, we can write:

$$\begin{aligned} \text{Arctan}\, z + \text{Arccot}\, z &= \frac{i}{2}\left[\text{Ln}\, v + \text{Ln}\left(-\frac{1}{v}\right)\right] \\ &= \frac{i}{2}\left[\text{Ln}|v| + \text{Ln}\left(\frac{1}{|v|}\right) + i\text{Arg}\, v + i\text{Arg}\left(-\frac{1}{v}\right)\right] \\ &= -\frac{1}{2}\left[\text{Arg}\, v + \text{Arg}\left(-\frac{1}{v}\right)\right]\,. \end{aligned} \tag{71}$$

It is straightforward to check that for any nonzero complex number v,

$$\text{Arg}\, v + \text{Arg}\left(-\frac{1}{v}\right) = \begin{cases} \pi\,, & \text{for Im } v \geq 0\,, \\ -\pi\,, & \text{for Im } v < 0\,. \end{cases} \tag{72}$$

Using eq. (70), we can evaluate Im v by computing

$$\frac{i+z}{i-z} = \frac{(i+z)(-i-z^*)}{(i-z)(-i-z^*)} = \frac{1-|z|^2-2i\,\text{Re } z}{|z|^2+1-2\,\text{Im } z}\,.$$

Writing $|z|^2 = (\text{Re } z)^2 + (\text{Im } z)^2$ in the denominator,

$$\frac{i+z}{i-z} = \frac{1-|z|^2-2i\,\text{Re } z}{(\text{Re } z)^2 + (1-\text{Im } z)^2}\,.$$

Hence,

$$\text{Im } v \equiv \text{Im}\left(\frac{i+z}{i-z}\right) = \frac{-2\,\text{Re } z}{(\text{Re } z)^2 + (1-\text{Im } z)^2}\,.$$

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[4]Ref. 3 does not assign a unique value to Arctan or Arccot for values of z that lie on the branch cut. However, computer programs such as Mathematica do not have this luxury since it must return a unique value for the corresponding functions evaluated at any complex number z.

[5]In light of footnote 4, the result obtained in eq. (4.23.27) of Ref. 3 for $\text{Arctan}(iy)$ is not single-valued, in contrast to eq. (69).

We conclude that

$$\text{Im } v \geq 0 \quad \Longrightarrow \quad \text{Re } z \leq 0\,, \qquad\qquad \text{Im } v < 0 \quad \Longrightarrow \quad \text{Re } z > 0\,.$$

Therefore, eqs. (71) and (72) yield:

$$\text{Arctan}\, z + \text{Arccot}\, z = \begin{cases} -\tfrac{1}{2}\pi\,, & \text{for Re } z \leq 0 \text{ and } z \neq \pm i\,, \\ \tfrac{1}{2}\pi\,, & \text{for Re } z > 0 \text{ and } z \neq \pm i\,. \end{cases} \tag{73}$$

which differs from eq. (25) when z lives on one of the branch cuts, for Re $z = 0$ and $z \neq \pm i$. Moreover, there is no longer any ambiguity in how to define Arccot(0). Indeed, for values of $z = iy$ ($-1 < y < 1$) that lie on the branch cut of Arccot z, eq. (68) yields,

$$\text{Arccot}(iy) = \frac{i}{2}\text{Ln}\left(\frac{1-y}{1+y}\right) - \tfrac{1}{2}\pi\,, \quad \text{for } |y| < 1, \tag{74}$$

which differs from the result of eq. (19) when $-1 < y < 0$. That is, by employing the definition of the principal value of the arccotangent function given by eq. (68), Arccot(iy) is a continuous function of y on the branch cut. In particular, plugging $z = 0$ into eq. (68) yields,

$$\text{Arccot}(0) = \tfrac{1}{2} i\, \text{Ln}(-1) = -\tfrac{1}{2}\pi\,. \tag{75}$$

Unfortunately, this result is the negative of the convention proposed in eq. (20).

One disadvantage of the definition of the principal value of the arctangent given by eq. (67) concerns the value of Arctan($-\infty$). In particular, if $z = x$ is real,

$$\left|\frac{1-ix}{1+ix}\right| = 1\,. \tag{76}$$

Since Ln 1 = 0, it would follow from eq. (67) that for all real x,

$$\text{Arctan}\, x = -\tfrac{1}{2}\text{Arg}\left(\frac{1-ix}{1+ix}\right)\,. \tag{77}$$

Indeed, eq. (77) is correct for all finite real values of x. It also correctly implies that Arctan $(-\infty) = -\frac{1}{2}\text{Arg}(-1) = -\frac{1}{2}\pi$, as expected. However, if we take $x \to \infty$ in eq. (77), we would also get Arctan $(\infty) = -\frac{1}{2}\text{Arg}(-1) = -\frac{1}{2}\pi$, in contradiction with the conventional definition of the principal value of the real-valued arctangent function, where Arctan $(\infty) = \frac{1}{2}\pi$. This slight inconsistency is not surprising, since the principal value of the argument of any complex number z must lie in the range $-\pi < \text{Arg}\, z \leq \pi$. Consequently, eq. (77) implies that $-\frac{1}{2}\pi \leq \text{Arctan}\, x < \frac{1}{2}\pi$, which is not quite consistent with eq. (21) as the endpoint at $\frac{1}{2}\pi$ is missing.

Some authors finesse this defect by defining the value of Arctan (∞) as the limit of Arctan (x) as $x \to \infty$. Note that

$$\lim_{x\to\infty} \text{Arg}\left(\frac{1-ix}{1+ix}\right) = -\pi\,,$$

since for any finite real value of $x > 1$, the complex number $(1 - ix)/(1 + ix)$ lies in Quadrant III[6] and approaches the negative real axis as $x \to \infty$. Hence, eq. (77) yields

$$\lim_{x\to\infty} \text{Arctan}\,(x) = \tfrac{1}{2}\pi\,.$$

With this interpretation, eq. (67) is consistent with the definition for the principal value of the real arctangent function.[7]

It is instructive to consider the difference of the two definitions of Arctan z given by eqs. (12) and (67). Using eqs. (13) and (55) of the class handout entitled, *The complex logarithm, exponential and power functions*, it follows that

$$\text{Ln}\left(\frac{1-iz}{1+iz}\right) - [\text{Ln}(1-iz) - \text{Ln}(1+iz)] = 2\pi i N_-\,,$$

where

$$N_- = \begin{cases} -1\,, & \text{if} \quad \text{Arg}(1-iz) - \text{Arg}(1+iz) > \pi\,, \\ \;\;0\,, & \text{if} \quad -\pi < \text{Arg}(1-iz) - \text{Arg}(1+iz) \leq \pi\,, \\ \;\;1\,, & \text{if} \quad \text{Arg}(1-iz) - \text{Arg}(1+iz) \leq -\pi\,. \end{cases} \tag{78}$$

To evaluate N_- explicitly, we must examine the quantity $\text{Arg}(1 - iz) - \text{Arg}(1 + iz)$ as a function of the complex number $z = x + iy$. Hence, we shall focus on the quantity $\text{Arg}(1 + y - ix) - \text{Arg}(1 - y + ix)$ as a function of x and y. If we plot the numbers $1 + y - ix$ and $1 - y + ix$ in the complex plane, it is evident that for finite values of x and y and $x \neq 0$ then

$$-\pi < \text{Arg}(1+y-ix) - \text{Arg}(1-y+ix) < \pi\,.$$

The case of $x = 0$ is easily treated separately, and we find that

$$\text{Arg}(1+y) - \text{Arg}(1-y) = \begin{cases} -\pi\,, & \text{if } y > 1\,, \\ \;\;0\,, & \text{if } -1 < y < 1\,, \\ \;\;\pi\,, & \text{if } y < -1\,. \end{cases}$$

Note that we have excluded the points $x = 0$, $y = \pm 1$, which correspond to the branch points where the arctangent function diverges. Hence, it follows that in the finite complex

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[6]This is easily verified. We write:

$$z \equiv \frac{1-ix}{1+ix} = \frac{1-ix}{1+ix}\cdot\frac{1-ix}{1-ix} = \frac{1-x^2-2ix}{1+x^2}\,.$$

Thus, for real values of $x > 1$, it follows that $\text{Re}\, z < 0$ and $\text{Im}\, z < 0$, i.e. the complex number z lies in Quadrant III. Moreover, as $x \to \infty$, we see that $\text{Re}\, z \to -1$ and $\text{Im}\, z \to 0^-$, where 0^- indicates that one is approaching 0 from the negative side. Some authors write $\lim_{x\to\infty}(1 - ix)/(1 + ix) = -1 - i0$ to indicate this behavior, and then define $\text{Arg}(-1 - i0) = -\pi$.

[7]This is strategy adopted in Ref. 3 since this reference does not assign a unique value to Arctan z and Arccot z on their respective branch cuts.

plane excluding the branch points at $z = \pm i$,

$$N_- = \begin{cases} 1\,, & \text{if } \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z > 1\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

This means that in the finite complex plane, the two possible definitions for the principal value of the arctangent function given by eqs. (12) and (67) differ only on the branch cut along the positive imaginary axis above $z = i$. That is, for finite $z \neq \pm i$,

$$\tfrac{1}{2}i \operatorname{Ln}\left(\frac{1-iz}{1+iz}\right) = \begin{cases} -\pi + \tfrac{1}{2}i\left[\operatorname{Ln}(1-iz) - \operatorname{Ln}(1+iz)\right]\,, & \text{if } \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z > 1\,, \\ \\ \tfrac{1}{2}i\left[\operatorname{Ln}(1-iz) - \operatorname{Ln}(1+iz)\right]\,, & \text{otherwise}\,. \end{cases} \tag{79}$$

Additional discrepancies between the two definitions can arise when x and/or y become infinite. For example, since $\operatorname{Arg}(a + i\infty) = \frac{1}{2}\pi$ and $\operatorname{Arg}(a - i\infty) = -\frac{1}{2}\pi$ for any real number a, it follows that $N_- = 1$ for $x = \infty$.

Likewise one can determine the difference of the two definitions of $\operatorname{Arccot} z$ given by eqs. (13) and (68). Using the relation $\operatorname{Arccot} z = \operatorname{Arctan}(1/z)$ [which holds for both sets of definitions], eq. (79) immediately yields:

$$\tfrac{1}{2}i \operatorname{Ln}\left(\frac{z-i}{z+i}\right) = \begin{cases} -\pi + \dfrac{i}{2}\left[\operatorname{Ln}\left(1-\dfrac{i}{z}\right) - \operatorname{Ln}\left(1+\dfrac{i}{z}\right)\right]\,, & \text{if } \operatorname{Re} z = 0 \text{ and } -1 < \operatorname{Im} z < 0\,, \\ \dfrac{i}{2}\left[\operatorname{Ln}\left(1-\dfrac{i}{z}\right) - \operatorname{Ln}\left(1+\dfrac{i}{z}\right)\right]\,, & \text{otherwise}\,. \end{cases} \tag{80}$$

It follows that the two possible definitions for the principal value of the arccotangent function given by eqs. (13) and (68) differ only on the branch cut along the negative imaginary axis above $z = -i$.

So which set of conventions is best? Of course, there is no one right or wrong answer to this question. As a practical matter, I always employ the Mathematica definitions, as this is a program that I often use in my research. In contrast, the authors of Refs. 4–6 argue for choosing eq. (12) to define the principal value of the arctangent but use a slight variant of eq. (68) to define the principal value of the arccotangent function,[8]

$$\operatorname{Arccot} z = \frac{1}{2i} \operatorname{Ln}\left(\frac{z+i}{z-i}\right)\,. \tag{81}$$

This new definition has the benefit of ensuring that $\operatorname{Arccot}(0) = \frac{1}{2}\pi$ [in contrast to eq. (75)]. But, adopting eq. (81) will lead to modifications of $\operatorname{Arccot} z$ (compared to alternative definitions previously considered) when evaluated on the branch cut, $\operatorname{Re} z = 0$ and $|\operatorname{Im} z| < 1$. For example, with the definitions of $\operatorname{Arctan} z$ and $\operatorname{Arccot} z$ given by eqs. (12) and (81), respectively, it is straightforward to show that a number of relations, such as $\operatorname{Arccot} z = \operatorname{Arctan}(1/z)$, are modified.

[8] The right hand side of eq. (81) can be identified with $-\operatorname{Arccot}(-z)$ in the convention where $\operatorname{Arccot} z$ is defined by eq. (68).

For example, one can easily derive,

$$\operatorname{Arccot} z = \begin{cases} \pi + \operatorname{Arctan}\left(\frac{1}{z}\right) , & \text{if } \operatorname{Re} z = 0 \text{ and } 0 < \operatorname{Im} z < 1 , \\ \operatorname{Arctan}\left(\frac{1}{z}\right) , & \text{otherwise} , \end{cases}$$

excluding the branch points $z = \pm i$ where $\operatorname{Arctan} z$ and $\operatorname{Arccot} z$ both diverge. Likewise, the expression for $\operatorname{Arctan} z + \operatorname{Arccot} z$ previously obtained will also be modified,

$$\operatorname{Arctan} z + \operatorname{Arccot} z = \begin{cases} \frac{1}{2}\pi , & \text{for } \operatorname{Re} z > 0 , \\ \frac{1}{2}\pi , & \text{for } \operatorname{Re} z = 0 , \text{ and } \operatorname{Im} z > -1 , \\ -\frac{1}{2}\pi , & \text{for } \operatorname{Re} z < 0 , \\ -\frac{1}{2}\pi , & \text{for } \operatorname{Re} z = 0 , \text{ and } \operatorname{Im} z < -1 . \end{cases} \tag{82}$$

Other modifications of the results of this Appendix in the case where eq. (81) is adopted as the definition of the principal value of the arccotangent function are left as an exercise for the reader.

CAUTION!!

The principal value of the arccotangent is given in terms the principal value of the arctangent,

$$\operatorname{Arccot} z = \operatorname{Arctan}\left(\frac{1}{z}\right) , \tag{83}$$

for both the Mathematica definition [eq. (13)] or the alternative definition presented in eq. (68). However, many books define the principal value of the arccotangent differently via the relation,

$$\operatorname{Arccot} z = \tfrac{1}{2}\pi - \operatorname{Arctan} z . \tag{84}$$

This relation should be compared with the corresponding relations, eqs. (25), (73) and (82), that are satisfied with the definitions of the principal value of the arccotangent introduced in eqs. (13), (68) and (81), respectively. Eq. (84) has been adopted by the Maple computer algebra system (see Ref. 7), which is one of the main competitors of Mathematica.

The main motivation for eq. (84) is that the principal value of the real cotangent function satisfies $0 \leq \operatorname{Arccot} x \leq \pi$, instead of the interval quoted in eq. (23). One advantage of this latter definition is that for real values of x, $\operatorname{Arccot} x$ is continuous at $x = 0$, in contrast to eq. (83) which exhibits a discontinuity at $x = 0$. Note that if one adopts eq. (84) as the the definition of the principal value of the arccotangent, then the branch cuts of $\operatorname{Arccot} z$ are the same as those of $\operatorname{Arctan} z$ [cf. eq. (15)]. The disadvantages of the definition given in eq. (84) are discussed in detail in Refs. 4 and 5.

Which convention does your calculator and/or your favorite mathematics software use? Try evaluating $\operatorname{Arccot}(-1)$. In the convention of eq. (13) or eq. (68), we have $\operatorname{Arccot}(-1) = -\frac{1}{4}\pi$, whereas in the convention of eq. (84), we have $\operatorname{Arccot}(-1) = \frac{3}{4}\pi$.

APPENDIX B: Derivation of eq. (25)

To derive eq. (25), we will make use of the computations provided in Appendix A. Start from eq. (73), which is based on the definitions of the principal values of the arctangent and arccotangent given in eqs. (67) and (68), respectively. We then use eqs. (79) and (80) which allow us to translate between the definitions of eqs. (67) and (68) and the Mathematica definitions of the principal values of the arctangent and arccotangent given in eqs. (12) and (13), respectively. Eqs. (79) and (80) imply that the result for $\text{Arctan}\, z + \text{Arccot}\, z$ does not change if $\text{Re}\, z \neq 0$. For the case of $\text{Re}\, z = 0$, $\text{Arctan}\, z + \text{Arccot}\, z$ changes from $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$ if $0 < \text{Im}\, z < 1$ or $\text{Im}\, z < -1$. This is precisely what is exhibited in eq. (25).

APPENDIX C: Derivation of eq. (63)

Consider the multivalued square root function, denoted by $z^{1/2}$. Let us introduce the principal value of the square foot function, which we shall denote by the symbol $\sqrt{z}$. Then,

$$\sqrt{z} = \sqrt{|z|}e^{\frac{1}{2}\text{Arg}\, z}\,, \tag{85}$$

where $\sqrt{|z|}$ denotes the unique positive squared root of the real number $|z|$. In this notation,

$$i\text{Arccos}\, z = \text{Ln}\left(z + i\sqrt{1-z^2}\right)\,, \tag{86}$$

$$\text{Arccosh}\, z = \text{Ln}\left(z + \sqrt{z+1}\sqrt{z-1}\right)\,. \tag{87}$$

Our first task is to relate $\sqrt{z+1}\sqrt{z-1}$ to $\sqrt{z^2-1}$. Of course, these two quantities are equal for all real numbers $z \geq 1$. But, as these quantities are principal values of complex numbers, one must be more careful in the general case. We shall make use of eqs. (13) and (77) of the class handout entitled, *The complex logarithm, exponential and power functions*, in which the following formula is obtained:

$$\sqrt{z_1 z_2} = e^{\frac{1}{2}\text{Ln}(z_1 z_2)} = e^{\frac{1}{2}(\text{Ln}\, z_1 + \text{Ln}\, z_2 + 2\pi i N_+)} = \sqrt{z_1}\sqrt{z_2}\, e^{\pi i N_+}\,,$$

where

$$N_+ = \begin{cases} -1\,, & \text{if} \quad \text{Arg}\, z_1 + \text{Arg}\, z_2 > \pi\,, \\ 0\,, & \text{if} \quad -\pi < \text{Arg}\, z_1 + \text{Arg}\, z_2 \leq \pi\,, \\ 1\,, & \text{if} \quad \text{Arg}\, z_1 + \text{Arg}\, z_2 \leq -\pi\,. \end{cases}$$

That is,

$$\sqrt{z_1 z_2} = \varepsilon\sqrt{z_1}\sqrt{z_2}\,, \qquad \varepsilon = \pm 1\,, \tag{88}$$

where the choice of sign is determined by:

$$\varepsilon = \begin{cases} +1\,, & \text{if } -\pi < \text{Arg}\, z_1 + \text{Arg}\, z_2 \leq \pi\,, \\ -1\,, & \text{otherwise.} \end{cases}$$

Thus, we must determine in which interval the quantity $\text{Arg}(z+1)+\text{Arg}(z-1)$ lies as a function of z. The special cases of $z=\pm 1$ must be treated separately, since $\text{Arg}\,0$ is not defined. By plotting the complex points $z+1$ and $z-1$ in the complex plane, one can easily show that for $z\neq\pm 1$,

$$-\pi < \text{Arg}\,(z+1)+\text{Arg}\,(z-1)\leq\pi\,,\qquad \text{if}\ \begin{cases}\text{Im}\ z>0\ \text{ and }\ \text{Re}\ z\geq 0\,,\\ \qquad\qquad \text{or}\\ \text{Im}\ z=0\ \text{ and }\ \text{Re}\ z>-1\,,\\ \qquad\qquad \text{or}\\ \text{Im}\ z<0\ \text{ and }\ \text{Re}\ z>0\,.\end{cases}$$

If the above conditions do not hold, then $\text{Arg}(z+1)+\text{Arg}(z-1)$ lies outside the range of the principal value of the argument function. Hence, we conclude that if $z_1=z+1$ and $z_2=z-1$ then if $\text{Im}\ z\neq 0$ then ε in eq. (88) is given by:

$$\varepsilon=\begin{cases}+1\,, & \text{if Im}\ z>0\,,\ \text{Re}\ z\geq 0\ \text{ or }\ \text{Im}\ z<0, \text{Re}\ z>0\,,\\ -1\,, & \text{if Im}\ z>0\,,\ \text{Re}\ z<0\ \text{ or }\ \text{Im}\ z<0\,,\ \text{Re}\ z\leq 0\,.\end{cases}$$

In the case of $\text{Im}\ z=0$, we must exclude the points $z=\pm 1$, in which case we also have

$$\varepsilon=\begin{cases}+1\,, & \text{if Im}\ z=0\ \text{ and }\ \text{Re}\ z>-1\ \text{ with }\ \text{Re}\ z\neq 1\,,\\ -1\,, & \text{if Im}\ z=0\ \text{ and }\ \text{Re}\ z<-1\,.\end{cases}$$

It follow that $\text{Arccosh}\, z=\text{Ln}(z\pm\sqrt{z^2-1})$, where the sign is identified with ε above. Noting the identity:

$$z-\sqrt{z^2-1}=\frac{1}{z+\sqrt{z^2-1}}\,,$$

we can relate the two logarithms by recalling that [cf. eq.(57) from the class handout on the complex logarithm]

$$\text{Ln}(1/z)=\begin{cases}-\text{Ln}(z)+2\pi i\,, & \text{if } z \text{ is real and negative}\,,\\ -\text{Ln}(z)\,, & \text{otherwise}\,.\end{cases}$$

Since $z+\sqrt{z^2-1}$ is real and negative if and only if $\text{Im}\ z=0$ and $\text{Re}\ z\leq -1$,[9] one finds:

$$\text{Ln}(z-\sqrt{z^2-1})=\begin{cases}2\pi i-\text{Ln}(z+\sqrt{z^2-1})\,, & \text{for Im}\ z=0 \text{ and Re}\ z\leq -1\,,\\ -\text{Ln}(z+\sqrt{z^2-1})\,, & \text{otherwise}\,.\end{cases}$$

[9]Let $w=z+\sqrt{z^2-1}$, and assume that $\text{Im}\ w=0$ and $\text{Re}\ w\neq 0$. That is, w is real and nonzero, in which case $\text{Im}\ w^2=0$. But

$$0=\text{Im}\ w^2=\text{Im}\ \left[2z^2-1+2z\sqrt{z^2-1}\right]=\text{Im}\ (2zw-1)=2w\text{Im}\ z\,,$$

which confirms that $\text{Im}\ z=0$, i.e. z must be real. If we require in addition that that $\text{Re}\ w<0$, then we also must have $\text{Re}\ z\leq -1$.

To complete this part of the analysis, we must consider separately the points $z = \pm 1$. At these two points, eq. (87) yields $\text{Arccosh}(1) = 0$ and $\text{Arccosh}(-1) = \text{Ln}(-1) = \pi i$. Collecting all of the above results then yields:

$$
\text{Arccosh}\, z = \begin{cases} \text{Ln}(z + \sqrt{z^2-1})\,, & \text{if Im } z > 0\,,\ \text{Re } z \geq 0 \ \text{ or } \ \text{Im } z = 0\,,\ \text{Re } z \geq -1 \\ & \qquad \text{or } \ \text{Im } z < 0, \text{Re } z > 0\,, \\ -\text{Ln}(z + \sqrt{z^2-1})\,, & \text{if Im } z > 0\,,\ \text{Re } z < 0 \ \text{ or } \ \text{Im } z < 0\,,\ \text{Re } z \leq 0\,, \\ 2\pi i - \text{Ln}(z + \sqrt{z^2-1})\,, & \text{if Im } z = 0, \text{Re } z \leq -1\,. \end{cases} \tag{89}
$$

Note that the cases of $z = \pm 1$ are each covered twice in eq. (89) but in both respective cases the two results are consistent.

Our second task is to relate $i\sqrt{1-z^2}$ to $\sqrt{z^2-1}$. To accomplish this, we first note that for any nonzero complex number z, the principal value of the argument of $-z$ is given by:

$$
\text{Arg}(-z) = \begin{cases} \text{Arg}\, z - \pi\,, & \text{if Arg}\, z > 0\,, \\ \text{Arg}\, z + \pi\,, & \text{if Arg}\, z \leq 0\,. \end{cases} \tag{90}
$$

This result is easily checked by considering the location of the complex numbers z and $-z$ in the complex plane. Hence, by making use of eqs. (85) and (90) along with $i = e^{i\pi/2}$, it follows that:

$$
i\sqrt{1-z^2} = \sqrt{|z^2-1|}\, e^{\frac{1}{2}\left[\pi + \text{Arg}(1-z^2)\right]} = \eta\sqrt{z^2-1}\,, \qquad \eta = \pm 1\,,
$$

where the sign η is determined by:

$$
\eta = \begin{cases} +1\,, & \text{if Arg}(1-z^2) \leq 0\,, \\ -1\,, & \text{if Arg}(1-z^2) > 0\,, \end{cases}
$$

assuming that $z \neq \pm 1$. If we put $z = x + iy$, then $1 - z^2 = 1 - x^2 + y^2 - 2ixy$, and we deduce that

$$
\text{Arg}(1-z^2) \text{ is } \begin{cases} \text{positive}\,, & \text{either if } xy < 0 \quad \text{or} \quad \text{if } y = 0 \text{ and } |x| > 1\,, \\ \text{zero}\,, & \text{either if } x = 0 \quad \text{or} \quad \text{if } y = 0 \text{ and } |x| < 1\,, \\ \text{negative}\,, & \text{if } xy > 0\,. \end{cases}
$$

We exclude the points $z = \pm 1$ (corresponding to $y = 0$ and $x = \pm 1$) where $\text{Arg}(1 - z^2)$ is undefined. Treating these two points separately, eq. (86) yields $\text{Arccos}(1) = 0$ and $i\text{Arccos}(-1) = \text{Ln}(-1) = \pi i$. Collecting all of the above results then yields:

$$
i\text{Arccos}\, z = \begin{cases} \text{Ln}(z + \sqrt{z^2-1})\,, & \text{if Im } z > 0\,,\ \text{Re } z \geq 0 \ \text{ or } \ \text{Im } z < 0\,,\ \text{Re } z \leq 0 \\ & \qquad \text{or } \ \text{Im } z = 0\,,\ |\text{Re } z| \leq 1\,, \\ -\text{Ln}(z + \sqrt{z^2-1})\,, & \text{if Im } z > 0\,,\ \text{Re } z < 0 \ \text{ or } \ \text{Im } z < 0\,,\ \text{Re } z > 0\,, \\ & \qquad \text{or } \ \text{Im } z = 0\,,\ \text{Re } z \geq 1\,, \\ 2\pi i - \text{Ln}(z + \sqrt{z^2-1})\,, & \text{if Im } z = 0\,,\ \text{Re } z \leq -1\,. \end{cases} \tag{91}
$$

Note that the cases of $z = \pm 1$ are each covered twice in eq. (91) but in both respective cases the two results are consistent.

Comparing eqs. (89) and (91), we conclude that:

$$\text{Arccosh}\, z = \begin{cases} i\text{Arccos}\, z\,, & \text{for either Im } z > 0 \quad \text{or} \quad \text{for Im } z = 0 \text{ and Re } z \leq 1\,, \\ -i\text{Arccos}\, z\,, & \text{for either Im } z < 0 \quad \text{or} \quad \text{for Im } z = 0 \text{ and Re } z \geq 1\,. \end{cases}$$

which is identical to eq. (63).

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