# The argument of a complex number

In these notes, we examine the *argument* of a non-zero complex number $z$, sometimes called *angle* of $z$ or the *phase* of $z$. Following eq. (4.1) on p. 49 of Boas, we write:

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}\,, \tag{1}$$

where $x = \text{Re } z$ and $y = \text{Im } z$ are real numbers. The argument of $z$ is denoted by $\theta$, which is measured in radians. However, there is an ambiguity in definition of the argument. The problem is that

$$\sin(\theta + 2\pi) = \sin\theta\,, \qquad \cos(\theta + 2\pi) = \cos\theta\,,$$

since the sine and the cosine are periodic functions of $\theta$ with period $2\pi$. Thus $\theta$ is defined only up to an additive integer multiple of $2\pi$. It is common practice to establish a convention in which $\theta$ is defined to lie within an interval of length $2\pi$. The most common convention,* which we adopt in these notes, is to take $-\pi < \theta \leq \pi$. With this definition, we identify $\theta$ as the so-called *principal value* of the argument, which we denote by Arg $z$ (note the capital A). On the other hand, in many applications, it is convenient to define a multi-valued argument function,

$$\arg z \equiv \text{Arg } z + 2\pi n = \theta + 2\pi n\,, \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots\,.$$

This is a multi-valued function because for a given complex number $z$, the number arg $z$ represents an infinite number of possible values. Although Boas does not formally introduce the multi-valued argument function in Chapter 2, it will become especially useful when we study the properties of the complex logarithm and complex power functions.

## 1. Definition of the argument function

The argument of a non-zero complex number is a multi-valued function that plays a key role in understanding the properties of the complex logarithm and power functions. Any non-zero complex number $z$ can be written in polar form†

$$z = |z|e^{i\arg z}\,, \tag{2}$$

where $\arg z$ is a multi-valued function given by:

$$\arg z = \theta + 2\pi n\,, \qquad n = 0\,, \pm 1\,, \pm 2\,, \pm 3\,, \ldots\,.$$

*Another common convention adopted in some books is to take $0 \leq \theta < 2\pi$. We shall not use this convention in these notes. I leave it to you to make the appropriate modifications if you prefer the latter choice.

†Since $z = 0$ if and only if $|z| = 0$, eq. (2) remains valid despite the fact that $\arg 0$ is undefined. When studying the properties of arg $z$ and $\text{Arg}\, z$ below, we shall always assume implicitly that $z \neq 0$.

Here, $\theta \equiv \text{Arg } z$ is the so-called principal value of the argument, which by convention is taken to lie in the range $-\pi < \theta \leq \pi$. That is,

$$\arg z = \text{Arg } z + 2\pi n\,, \quad n = 0\,, \pm 1\,, \pm 2\,, \pm 3\,, \cdots\,, \quad -\pi < \text{Arg } z \leq \pi\,. \tag{3}$$

It is convenient to have an explicit formula for Arg $z$ in terms of $\arg z$. First, we introduce some notation: $[x]$ means the largest integer less than or equal to the real number $x$. That is, $[x]$ is the unique integer that satisfies the inequality

$$x - 1 < [x] \leq x\,, \qquad \text{for real } x \text{ and integer } [x]\,. \tag{4}$$

For example, $[1.5] = [1] = 1$ and $[-0.5] = -1$. With this notation, one can write Arg $z$ in terms of $\arg z$ as follows:

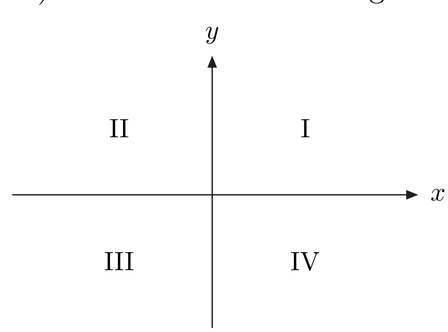
$$\text{Arg } z = \arg z + 2\pi \left[\frac{1}{2} - \frac{\arg z}{2\pi}\right]\,, \tag{5}$$

where [ ] denotes the bracket (or greatest integer) function introduced above. It is straightforward to check that Arg $z$ as defined by eq. (5) does indeed fall inside the principal interval, $-\pi < \theta \leq \pi$.

A more useful equation for Arg $z$ can be obtained as follows. Using the polar representation of $z = x + iy$ given in eq. (1), it follows that $x = r\cos\theta$ and $y = r\sin\theta$. From these two results, one easily derives,

$$|z| = r = \sqrt{x^2 + y^2}\,, \qquad \tan\theta = \frac{y}{x} = \frac{\text{Im } z}{\text{Re } z}\,. \tag{6}$$

We identify $\theta = \text{Arg } z$ in the convention where $-\pi < \theta \leq \pi$. In light of eq. (6), it is tempting to identify Arg $z$ with $\arctan(y/x)$. However, the real function $\arctan x$ is a multi-valued function for real values of $x$. It is conventional to introduce a single-valued real arctangent function, called the principal value of the arctangent,[‡] which is denoted by Arctan $x$ and satisfies $-\frac{1}{2}\pi \leq \text{Arctan } x \leq \frac{1}{2}\pi$. Since $-\pi < \text{Arg } z \leq \pi$, it follows that Arg $z$ *cannot* be identified with $\text{Arctan}(y/x)$ in all regions of the complex plane. The correct relation between these two quantities is easily ascertained by considering the four quadrants of the complex plane separately. The quadrants of the complex plane (called regions I, II, III and IV) are illustrated in the figure below:




[‡]In defining the principal value of the arctangent, we follow the conventions of Keith B. Oldham, Jan Myland and Jerome Spanier, *An Atlas of Functions* (Springer Science, New York, 2009), Chapter 35.

Table 1: Formulae for the argument of a complex number $z = x + iy$. The range of $\operatorname{Arg} z$ is indicated for each of the four quadrants of the complex plane. For example, in quadrant I, the notation $(0, \frac{1}{2}\pi)$ means that $0 < \operatorname{Arg} z < \frac{1}{2}\pi$, etc. By convention, the principal value of the real arctangent function lies in the range $-\frac{1}{2}\pi \leq \operatorname{Arctan}(y/x) \leq \frac{1}{2}\pi$.

| Quadrant | Sign of $x$ and $y$ | range of $\operatorname{Arg} z$ | $\operatorname{Arg} z$ |
|---|---|---|---|
| I | $x > 0\,,\ y > 0$ | $(0, \frac{1}{2}\pi)$ | $\operatorname{Arctan}(y/x)$ |
| II | $x < 0\,,\ y > 0$ | $(\frac{1}{2}\pi, \pi)$ | $\pi + \operatorname{Arctan}(y/x)$ |
| III | $x < 0\,,\ y < 0$ | $(-\pi, -\frac{1}{2}\pi)$ | $-\pi + \operatorname{Arctan}(y/x)$ |
| IV | $x > 0\,,\ y < 0$ | $(-\frac{1}{2}\pi, 0)$ | $\operatorname{Arctan}(y/x)$ |

Table 2: Formulae for the argument of a complex number $z = x + iy$ when $z$ is real or pure imaginary. By convention, the principal value of the argument satisfies $-\pi < \operatorname{Arg} z \leq \pi$.

| Quadrant border | type of complex number $z$ | Conditions on $x$ and $y$ | $\operatorname{Arg} z$ |
|---|---|---|---|
| IV/I | real and positive | $x > 0\,,\ y = 0$ | $0$ |
| I/II | pure imaginary with $\operatorname{Im} z > 0$ | $x = 0\,,\ y > 0$ | $\frac{1}{2}\pi$ |
| II/III | real and negative | $x < 0\,,\ y = 0$ | $\pi$ |
| III/IV | pure imaginary with $\operatorname{Im} z < 0$ | $x = 0\,,\ y < 0$ | $-\frac{1}{2}\pi$ |
| origin | zero | $x = y = 0$ | undefined |

The principal value of the argument of $z = x + iy$ is given in Table 1, depending on in which of the four quadrants of the complex plane $z$ resides. In particular, note that $\operatorname{Arg} z = \operatorname{Arctan}(y/x)$ is valid only in quadrants I and IV. If $z$ resides in quadrant II then $y/x < 0$, in which case $-\frac{1}{2}\pi < \operatorname{Arctan}(y/x) < 0$. Thus if $z$ lies in quadrant II, then one must add $\pi$ to $\operatorname{Arctan}(y/x)$ to ensure that $\frac{1}{2}\pi < \operatorname{Arg} z < \pi$. Likewise, if $z$ resides in quadrant III then $y/x > 0$, in which case $0 < \operatorname{Arctan}(y/x) < \frac{1}{2}\pi$. Thus if $z$ lies in quadrant III, then one must subtract $\pi$ in order to ensure that $-\pi < \operatorname{Arg} z < -\frac{1}{2}\pi$.

Cases where $z$ lies on the border between two adjacent quadrants are considered separately in Table 2. To derive these results, note that for the borderline cases, the principal value of the arctangent is given by

$$\operatorname{Arctan}(y/x) = \begin{cases} 0\,, & \text{if } y = 0 \text{ and } x \neq 0\,, \\ \frac{1}{2}\pi\,, & \text{if } x = 0 \text{ and } y > 0\,, \\ -\frac{1}{2}\pi\,, & \text{if } x = 0 \text{ and } y < 0\,, \\ \text{undefined}\,, & \text{if } x = y = 0\,. \end{cases}$$

To better understand the behavior of $\operatorname{Arg} z$ in quadrants II and III, let us first consider the following example,

$$z = e^{3\pi i/4} = \cos\left(\tfrac{3}{4}\pi\right) + i\sin\left(\tfrac{3}{4}\pi\right) = \frac{-1+i}{\sqrt{2}}\,. \tag{7}$$

The complex number $z$ lies in quadrant II, since $\operatorname{Re} z < 0$ and $\operatorname{Im} z > 0$, in which case $\tfrac{1}{2}\pi < \operatorname{Arg} z < \pi$. Indeed, for $z$ defined in eq. (7), $\operatorname{Arg} z = \tfrac{3}{4}\pi$.

In contrast, by writing $z = x + iy = re^{i\theta}$, it follows that $r = 1$ and $\tan\theta = y/x = -1$. However, $\operatorname{Arctan}(-1) = -\tfrac{1}{4}\pi$, since the principal value of the arctangent lies in the range $-\tfrac{1}{2}\pi \leq \operatorname{Arctan}(y/x) \leq \tfrac{1}{2}\pi$. Thus, in this example one must use the quadrant II result given in Table 1 to obtain,

$$\operatorname{Arg} z = \pi + \operatorname{Arctan}(y/x) = \pi - \tfrac{1}{4}\pi = \tfrac{3}{4}\pi\,. \tag{8}$$

As a second example, consider

$$z = e^{-3\pi i/4} = \cos\left(\tfrac{3}{4}\pi\right) - i\sin\left(\tfrac{3}{4}\pi\right) = \frac{-1-i}{\sqrt{2}}\,. \tag{9}$$

The complex number $z$ now lies in quadrant III, since $\operatorname{Re} z < 0$ and $\operatorname{Im} z < 0$, in which case $-\pi < \operatorname{Arg} z < -\tfrac{1}{2}\pi$. Indeed, for $z$ defined in eq. (9), $\operatorname{Arg} z = -\tfrac{3}{4}\pi$. Once again, we shall write $z = x + iy = re^{i\theta}$, which yields $r = 1$ and $\tan\theta = y/x = 1$. The principal value of the arctangent in this example is $\operatorname{Arctan}(1) = \tfrac{1}{4}\pi$. Thus, we must use the quadrant III result given in Table 1 to obtain,

$$\operatorname{Arg} z = -\pi + \operatorname{Arctan}(y/x) = -\pi + \tfrac{1}{4}\pi = -\tfrac{3}{4}\pi\,. \tag{10}$$

The reader is encouraged to also consider the examples $z = e^{i\pi/4}$ (which resides in quadrant I) and $z = e^{-i\pi/4}$ (which resides in quadrant IV) and show that for both these cases, $\operatorname{Arg} z = \operatorname{Arctan}(y/x)$, as expected from the results of Table 1.

## 2. Properties of the multi-valued argument function

We can view a multi-valued function $f(z)$ evaluated at $z$ as a set of values, where each element of the set corresponds to a different choice of some integer $n$. For example, given the multi-valued function $\arg z$ whose principal value is $\operatorname{Arg} z \equiv \theta$, then $\arg z$ consists of the set of values:

$$\arg z = \{\theta\,,\ \theta + 2\pi\,,\ \theta - 2\pi\,,\ \theta + 4\pi\,,\ \theta - 4\pi\,,\ \cdots\}\,. \tag{11}$$

Consider the case of two multi-valued functions of the form, $f(z) = F(z) + 2\pi n$ and $g(z) = G(z) + 2\pi n$, where $F(z)$ and $G(z)$ are the principal values of $f(z)$ and $g(z)$ respectively. Then, $f(z) = g(z)$ if and only if for each point $z$, the corresponding set of values of $f(z)$ and $g(z)$ precisely coincide:

$$\{F(z)\,,\ F(z) + 2\pi\,,\ F(z) - 2\pi\,,\ \cdots\} = \{G(z)\,,\ G(z) + 2\pi\,,\ G(z) - 2\pi\,,\ \cdots\}\,. \tag{12}$$

Sometimes, one refers to the equation $f(z) = g(z)$ as a *set equality* since all the distinct elements of the two sets in eq. (12) must coincide. We add two additional rules to the concept of set equality. First, the ordering of terms within the set is unimportant. Second, we only care about the distinct elements of each set. That is, if our list of set elements has repeated entries, we omit all duplicate elements.

To see how the set equality of two multi-valued functions works, let us consider the multi-valued function $\arg z$. One can prove that:

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \,, \qquad \text{for } z_1,\, z_2 \neq 0, \tag{13}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \,, \qquad \text{for } z_1,\, z_2 \neq 0, \tag{14}$$

$$\arg\left(\frac{1}{z}\right) = \arg z^* = -\arg z \,, \qquad \text{for } z \neq 0. \tag{15}$$

To prove eq. (13), consider $z_1 = |z_1| e^{i\arg z_1}$ and $z_2 = |z_2| e^{i\arg z_2}$. The arguments of these two complex numbers are: $\arg z_1 = \text{Arg } z_1 + 2\pi n_1$ and $\arg z_2 = \text{Arg } z_2 + 2\pi n_2$, where $n_1$ and $n_2$ are arbitrary integers. [One can also write $\arg z_1$ and $\arg z_2$ in set notation as in eq. (11).] Thus, one can also write $z_1 = |z_1| e^{i\text{Arg } z_1}$ and $z_2 = |z_2| e^{i\text{Arg } z_2}$, since $e^{2\pi i n} = 1$ for any integer $n$. It then follows that

$$z_1 z_2 = |z_1 z_2| e^{i(\text{Arg } z_1 + \text{Arg } z_2)} \,,$$

where we have used $|z_1||z_2| = |z_1 z_2|$. Thus, $\arg(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2\pi n_{12}$, where $n_{12}$ is also an arbitrary integer. Therefore, we have established that:

$$\arg z_1 + \arg z_2 = \text{Arg } z_1 + \text{Arg } z_2 + 2\pi(n_1 + n_2) \,,$$

$$\arg(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2\pi n_{12} \,,$$

where $n_1$, $n_2$ and $n_{12}$ are arbitrary integers. Thus, $\arg z_1 + \arg z_2$ and $\arg(z_1 z_2)$ coincide as sets, and so eq. (13) is confirmed. One can easily prove eqs. (14) and (15) by a similar method. In particular, if one writes $z = |z| e^{i\arg z}$ and employs the definition of the complex conjugate (which yields $z^* = |z| e^{-i\arg z}$ and $|z^*| = |z|$), then it follows that $\arg(1/z) = \arg z^* = -\arg z$. As an instructive example, consider the last relation in the case of $z = -1$. It then follows that

$$\arg(-1) = -\arg(-1) \,,$$

as a set equality. This is not paradoxical, since the sets,

$$\arg(-1) = \{\pm\pi \,,\, \pm 3\pi \,,\, \pm 5\pi \,,\, \ldots\} \qquad \text{and} \qquad -\arg(-1) = \{\mp\pi \,,\, \mp 3\pi \,,\, \mp 5\pi \,,\, \ldots\} \,,$$

coincide, as they possess precisely the same list of elements.

Now, for a little surprise:

$$\arg z^2 \neq 2 \arg z \,. \tag{16}$$

To see why this statement is surprising, consider the following false proof. Use eq. (13) with $z_1 = z_2 = z$ to derive:

$$\arg z^2 = \arg z + \arg z \stackrel{?}{=} 2\arg z\,, \qquad \text{[FALSE!!]}. \tag{17}$$

The false step is the one indicated by the symbol $\stackrel{?}{=}$ above. Given $z = |z|e^{i\arg z}$, one finds that $z^2 = |z|^2 e^{2i(\text{Arg } z + 2\pi n)} = |z|^2 e^{2i\text{Arg } z}$, and so the possible values of $\arg(z^2)$ are:

$$\arg(z^2) = \{2\text{Arg } z\,,\ 2\text{Arg } z + 2\pi\,,\ 2\text{Arg } z - 2\pi\,,\ 2\text{Arg } z + 4\pi\,,\ 2\text{Arg } z - 4\pi\,,\ \cdots\}\,,$$

whereas the possible values of $2\arg z$ are:

$$\begin{aligned} 2\arg z &= \{2\text{Arg } z\,,\ 2(\text{Arg } z + 2\pi)\,,\ 2(\text{Arg } z - 2\pi)\,,\ 2(\text{Arg } z + 4\pi)\,,\ \cdots\} \\ &= \{2\text{Arg } z\,,\ 2\text{Arg } z + 4\pi\,,\ 2\text{Arg } z - 4\pi\,,\ 2\text{Arg } z + 8\pi\,,\ 2\text{Arg } z - 8\pi\,,\ \cdots\}\,. \end{aligned}$$

Thus, $2\arg z$ is a *subset* of $\arg(z^2)$, but half the elements of $\arg(z^2)$ are missing from $2\arg z$. These are therefore unequal sets, as indicated by eq. (16). Now, you should be able to see what is wrong with the statement:

$$\arg z + \arg z \stackrel{?}{=} 2\arg z\,. \tag{18}$$

When you add $\arg z$ as a set to itself, the element you choose from the first $\arg z$ need not be the same as the element you choose from the second $\arg z$. In contrast, $2\arg z$ means take the set $\arg z$ and multiply each element by two. The end result is that $2\arg z$ contains only half the elements of $\arg z + \arg z$ as shown above.

Here is one more example of an incorrect proof. Consider eq. (14) with $z_1 = z_2 \equiv z$. Then, you might be tempted to write:

$$\arg\left(\frac{z}{z}\right) = \arg(1) = \arg z - \arg z \stackrel{?}{=} 0\,.$$

This is clearly wrong since $\arg(1) = 2\pi n$, where $n$ is the set of integers. Again, the error occurs with the step:

$$\arg z - \arg z \stackrel{?}{=} 0\,. \tag{19}$$

The fallacy of this statement is the same as above. When you subtract $\arg z$ as a set from itself, the element you choose from the first $\arg z$ need not be the same as the element you choose from the second $\arg z$.

## 3. Properties of the principal value of the argument

The properties of the principal value Arg $z$ are not as simple as those given in eqs. (13)–(15), since the range of Arg $z$ is restricted to lie within the principal range $-\pi < \text{Arg } z \leq \pi$. Instead, the following relations are satisfied, assuming $z_1,\ z_2 \neq 0$,

$$\text{Arg } (z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2\pi N_+\,, \tag{20}$$

$$\text{Arg } (z_1/z_2) = \text{Arg } z_1 - \text{Arg } z_2 + 2\pi N_-\,, \tag{21}$$

where the integers $N_\pm$ are determined as follows:

$$N_\pm = \begin{cases} -1\,, & \text{if} \quad \text{Arg } z_1 \pm \text{Arg } z_2 > \pi\,, \\ \;\;0\,, & \text{if} \quad -\pi < \text{Arg } z_1 \pm \text{Arg } z_2 \leq \pi\,, \\ \;\;1\,, & \text{if} \quad \text{Arg } z_1 \pm \text{Arg } z_2 \leq -\pi\,. \end{cases} \tag{22}$$

Eq. (22) is really two separate equations for $N_+$ and $N_-$, respectively. To obtain the equation for $N_+$, one replaces the $\pm$ sign with a plus sign wherever it appears on the right hand side of eq. (22). To obtain the equation for $N_-$, one replaces the $\pm$ sign with a minus sign wherever it appears on the right hand side of eq. (22).

If we set $z_1 = 1$ in eq. (21), we find that

$$\text{Arg}(1/z) = \text{Arg } z^* = \begin{cases} \;\;\text{Arg}\, z\,, & \text{if Im } z = 0 \;\text{ and } z \neq 0\,, \\ -\text{Arg}\, z\,, & \text{if Im } z \neq 0\,. \end{cases} \tag{23}$$

Note that for $z$ real, both $1/z$ and $z^*$ are also real so that in this case $z = z^*$ and $\text{Arg}(1/z) = \text{Arg } z^* = \text{Arg } z$.

If $n$ is an integer, then

$$\arg z^n = \arg z + \arg z + \cdots + \arg z \neq n \arg z\,, \tag{24}$$

where the final inequality above was noted in the case of $n = 2$ in eq. (16). The corresponding property of Arg $z$ is much simpler:

$$\text{Arg}(z^n) = n\text{Arg } z + 2\pi N_n\,, \quad \text{for } z \neq 0, \tag{25}$$

where the integer $N_n$ is given by:

$$N_n = \left[\frac{1}{2} - \frac{n}{2\pi}\text{Arg } z\right]\,, \tag{26}$$

and [ ] is the greatest integer bracket function introduced in eq. (4). It is straightforward to verify eqs. (20)–(23) and eq. (25). These formulae follow from the corresponding properties of $\arg z$, taking into account the requirement that Arg $z$ must lie within the principal interval, $-\pi < \theta \leq \pi$.