# Does a power series converge on its circle of convergence?

Consider a power series expansion of a complex function,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \,. \tag{1}$$

The radius of convergence can usually be found by applying the ratio test. Namely, we require that

$$\lim_{n\to\infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| = R^{-1}|z| < 1 \,, \tag{2}$$

where the radius of convergence $R$ is defined by

$$R^{-1} \equiv \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| . \tag{3}$$

Then eq. (2) implies that as a consequence of the ratio test, the power series given by eq. (1) converges if $|z| < R$.

Suppose that the radius of convergence satisfies $0 < R < \infty$. Then, the ratio test is inconclusive for values of $z$ on the circle of convergence, which satisfy $|z| = R$ (or equivalently, values of $z = Re^{i\theta}$ for $-\pi < \theta \leq \pi$). In this short note, I will state and prove a theorem that addresses the question of whether a power series converges on its circle of convergence.

Without loss of generality, one can restrict the discussion to the case of $R = 1$ as follows. If the power series given by eq. (1) has a radius of convergence $R$ (where $0 < R < \infty$), then one can introduce a new complex variable $w = z/R$. Hence, eq. (1) is equivalent to a power series with radius of convergence equal to 1,

$$f(w) = \sum_{n=0}^{\infty} b_n w^n \,,$$

where $b_n \equiv R^n a_n$.

**Theorem:** If the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ respects the following three properties:

1. There exists a nonnegative integer $n_0$ such that the coefficients $a_n$ are real and nonnegative for all $n \geq n_0$.
2. $a_n \geq a_{n+1}$ for all non-negative integers, $n = n_0, n_0 + 1, n_0 + 2, n_0 + 3, \ldots$
3. $a_n \to 0$ as $n \to \infty$

then the power series converges at all points on the circle $|z| = 1$ in the complex plane, with the possible exception of $z = 1$.

The second and third properties of the coefficients specified by the theorem imply that $R^{-1} \leq 1$ in light of eq. (3), which means that $R \geq 1$. If $R > 1$, then the power series converges for all $|z| = 1$ due to the ratio test. In contrast, the theorem is nontrivial when $R = 1$ since in this case $|z| = 1$ corresponds to the circle of convergence, where the ratio test is inconclusive.

Note that one or more properties of the coefficients specified in the statement of the theorem are not satisfied and $R = 1$, then the convergence properties of $f(z)$ for $|z| = 1$ must be determined by some other means.

**Proof of the theorem**: One can rewrite eq. (1) as,

$$f(z) = \sum_{n=0}^{N} a_n z^n + R_N(z)\,, \tag{4}$$

where $N > n_0$ and

$$R_N(z) = \sum_{n=N+1}^{\infty} a_n z^n\,. \tag{5}$$

It then follows that

$$(z-1)R_N(z) = -a_{N+1}z^{N+1} + (a_{N+1} - a_{N+2})z^{N+2} + (a_{N+2} - a_{N+3})z^{N+3} + \ldots\,, \tag{6}$$

where all the coefficients on the right hand side of eq. (6) that multiply a power of $z$, with the exception of the first term, are nonnegative.

We can now apply a generalization of the triangle inequality* to eq. (6) to obtain,

$$|z-1|\,|R_N(z)| \leq a_{N+1}|z|^{N+1} + (a_{N+1} - a_{N+2})|z|^{N+2} + (a_{N+2} - a_{N+3})|z|^{N+3} + \ldots \tag{7}$$

One can now set $|z| = 1$ in eq. (7). Due to the telescoping nature of the infinite series above [starting with the second term on the right hand side of eq. (7)], we end up with

$$|z-1|\,|R_N(z)| \leq 2a_{N+1}\,. \tag{8}$$

Assuming that $z \neq 1$, it follows that

$$|R_N(z)| \leq \frac{2a_{N+1}}{|z-1|}\,. \tag{9}$$

Since $a_N \to 0$ and $N \to \infty$, it follows that

$$\lim_{N\to\infty} R_N(z) = 0\,, \quad \text{for } |z| = 1 \text{ and } z \neq 1. \tag{10}$$

This result is equivalent to the statement that the power series for $f(z)$ converges for $|z| = 1$ and $z \neq 1$.† Thus, the theorem is proved.

*See the class handout entitled *Complex conjugation, Modulus, and Inequalities* for a discussion of the triangle inequality.

†Recall that one definition of convergence states that given a positive error bound $\epsilon$, then one can always find an $N$ such that $|R_n(z)| < \epsilon$ for any $n \geq N$.

Applying the theorem to a real power series with $R = 1$, it follows that the series for $f(-1)$ converges. In this case, the theorem is equivalent to the one underlying the alternating series test (see the class handout entitled *The Alternating Series Test*).

Three examples are instructive. First, the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \,, \tag{11}$$

converges for $|z| < 1$ and diverges for all $|z| = 1$. This is not a counterexample, since the geometric series does not satisfy the third property specified in the theorem.

Second, the principal value of the complex logarithm has the following power series expansion,

$$\text{Ln}(1-z) = -\sum_{n=0}^{\infty} \frac{z^n}{n} \,. \tag{12}$$

This series converges for $|z| \leq 1$, $z \neq 1$, as a consequence of the theorem. Of course, the power series diverges at $z = 1$ since the resulting series is the negative of the harmonic series, which is known to be divergent. Note that this result also implies that the series

$$\text{Ln}(1+z) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^n}{n} \,, \tag{13}$$

converges for $|z| \leq 1$, $z \neq -1$, since one can simply replace the complex variable $z$ with $-z$ in eq. (12).

Third, the dilogarithm function introduced in eq. (18) of the class handout, entitled *Theorems About Power Series*, can be extended to a complex function, whose principal value is given by

$$\text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \,. \tag{14}$$

This power series converges for all $|z| \leq 1$. That is, eq. (14) converges at all points on the circle of convergence. Although the theorem cannot be used to prove that eq. (14) converges at $z = 1$, we know from the $p$-series test mentioned in problem 6-15 on p. 13 of Boas that the series for $\text{Li}_2(1)$ converges. Indeed, $\text{Li}_2(1) = \frac{1}{6}\pi^2$ is a very well known result.

The last two examples confirm that the theorem does not address the convergence property at $z = 1$, since the power series given in eqs. (12) and (14) satisfy the conditions of the theorem, and yet eq. (12) diverges at $z = 1$ whereas eq. (14) converges at $z = 1$.

## Reference

The theorem discussed in these notes is known as Picard's theorem. The proof of this theorem is inspired by Mario O. González, *Classical Complex Analysis* (Marcel Dekker, Inc., New York, NY, 1992). In particular, see Theorem 8.14 on pp. 556–557.