Chapter 6

Complex Numbers

I'm sorry. You have reached an imaginary number. Please rotate your phone 90 degrees and dial again.

-Message on answering machine of Cathy Vargas.

6.1 Complex Numbers

Shortcomings of real numbers. When you started algebra, you learned that the quadratic equation: $x^2+2ax+b=0$ has either two, one or no solutions. For example:

- $x^2-3x+2=0$ has the two solutions $x=1$ and $x=2$.
- For $x^2-2x+1=0$, $x=1$ is a solution of multiplicity two.
- $x^2+1=0$ has no solutions.

This is a little unsatisfactory. We can formally solve the general quadratic equation.

$$x^2 + 2ax + b = 0$$
$$(x + a)^2 = a^2 - b$$
$$x = -a \pm \sqrt{a^2 - b}$$

However, the solutions are defined only when the discriminant $a^2 - b$ is non-negative. This is because the square root function $\sqrt{x}$ is a bijection from $\mathbb{R}^{0+}$ to $\mathbb{R}^{0+}$. (See Figure 6.1.)

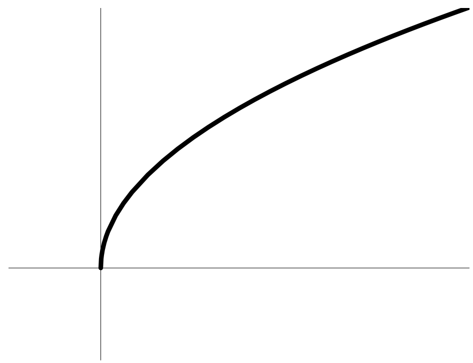


Figure 6.1: $y = \sqrt{x}$

A new mathematical constant. We cannot solve $x^2 = -1$ because the square root of -1 is not defined. To overcome this apparent shortcoming of the real number system, we create a new symbolic constant $\sqrt{-1}$. In performing arithmetic, we will treat $\sqrt{-1}$ as we would a real constant like π or a formal variable like x, i.e. $\sqrt{-1} + \sqrt{-1} = 2\sqrt{-1}$. This constant has the property: $\left(\sqrt{-1}\right)^2 = -1$. Now we can express the solutions of $x^2 = -1$ as $x = \sqrt{-1}$ and $x = -\sqrt{-1}$. These satisfy the equation since $\left(\sqrt{-1}\right)^2 = -1$ and $\left(-\sqrt{-1}\right)^2 = (-1)^2 \left(\sqrt{-1}\right)^2 = -1$. Note that we can express the square root of any negative real number in terms of $\sqrt{-1}$: $\sqrt{-r} = \sqrt{-1}\sqrt{r}$ for $r \geq 0$.

Euler's notation. Euler introduced the notation of using the letter i to denote $\sqrt{-1}$. We will use the symbol $\imath$, an i without a dot, to denote $\sqrt{-1}$. This helps us distinguish it from i used as a variable or index.[1] We call any number of the form $\imath b$, $b \in \mathbb{R}$, a *pure imaginary number*.[2] Let a and b be real numbers. The product of a real number and an imaginary number is an imaginary number: $(a)(\imath b) = \imath(ab)$. The product of two imaginary numbers is a real number: $(\imath a)(\imath b) = -ab$. However the sum of a real number and an imaginary number $a + \imath b$ is neither real nor imaginary. We call numbers of the form $a + \imath b$ *complex numbers*.[3]

The quadratic. Now we return to the quadratic with real coefficients, $x^2 + 2ax + b = 0$. It has the solutions $x = -a \pm \sqrt{a^2 - b}$. The solutions are real-valued only if $a^2 - b \geq 0$. If not, then we can define solutions as complex numbers. If the discriminant is negative, we write $x = -a \pm \imath\sqrt{b - a^2}$. Thus every quadratic polynomial with real coefficients has exactly two solutions, counting multiplicities. The fundamental theorem of algebra states that an n^{th} degree polynomial with complex coefficients has n, not necessarily distinct, complex roots. We will prove this result later using the theory of functions of a complex variable.

Component operations. Consider the complex number $z = x + \imath y$, $(x, y \in \mathbb{R})$. The *real part* of z is $\Re(z) = x$; the *imaginary part* of z is $\Im(z) = y$. Two complex numbers, $z = x + \imath y$ and $\zeta = \xi + \imath\psi$, are equal if and only if $x = \xi$ and $y = \psi$. The *complex conjugate*[4] of $z = x + \imath y$ is $\overline{z} \equiv x - \imath y$. The notation $z^* \equiv x - \imath y$ is also used.

A little arithmetic. Consider two complex numbers: $z = x + \imath y$, $\zeta = \xi + \imath\psi$. It is easy to express the sum or difference as a complex number.

$$z + \zeta = (x + \xi) + \imath(y + \psi), \quad z - \zeta = (x - \xi) + \imath(y - \psi)$$

It is also easy to form the product.

$$z\zeta = (x + \imath y)(\xi + \imath\psi) = x\xi + \imath x\psi + \imath y\xi + \imath^2 y\psi = (x\xi - y\psi) + \imath(x\psi + y\xi)$$

[1] Electrical engineering types prefer to use $\jmath$ or j to denote $\sqrt{-1}$.

[2] "Imaginary" is an unfortunate term. Real numbers are artificial; constructs of the mind. Real numbers are no more real than imaginary numbers.

[3] Here complex means "composed of two or more parts", not "hard to separate, analyze, or solve". Those who disagree have a complex number complex.

[4] Conjugate: having features in common but opposite or inverse in some particular.

The quotient is a bit more difficult. (Assume that ζ is nonzero.) How do we express $z/\zeta = (x + \imath y)/(\xi + \imath\psi)$ as the sum of a real number and an imaginary number? The trick is to multiply the numerator and denominator by the complex conjugate of ζ.

$$\frac{z}{\zeta} = \frac{x + \imath y}{\xi + \imath\psi} = \frac{x + \imath y}{\xi + \imath\psi}\frac{\xi - \imath\psi}{\xi - \imath\psi} = \frac{x\xi - \imath x\psi - \imath y\xi - \imath^2 y\psi}{\xi^2 - \imath\xi\psi + \imath\psi\xi - \imath^2\psi^2} = \frac{(x\xi + y\psi) - \imath(x\psi + y\xi)}{\xi^2 + \psi^2} = \frac{(x\xi + y\psi)}{\xi^2 + \psi^2} - \imath\frac{x\psi + y\xi}{\xi^2 + \psi^2}$$

Now we recognize it as a complex number.

Field properties. The set of complex numbers $\mathbb{C}$ form a field. That essentially means that we can do arithmetic with complex numbers. When performing arithmetic, we simply treat $\imath$ as a symbolic constant with the property that $\imath^2 = -1$. The field of complex numbers satisfy the following list of properties. Each one is easy to verify; some are proved below. (Let $z, \zeta, \omega \in \mathbb{C}$.)

1. Closure under addition and multiplication.

$$\begin{aligned} z + \zeta &= (x + \imath y) + (\xi + \imath\psi) \\ &= (x + \xi) + \imath\,(y + \psi) \in \mathbb{C} \\ z\zeta &= (x + \imath y)\,(\xi + \imath\psi) \\ &= x\xi + \imath x\psi + \imath y\xi + \imath^2 y\psi \\ &= (x\xi - y\psi) + \imath\,(x\psi + \xi y) \in \mathbb{C} \end{aligned}$$

2. Commutativity of addition and multiplication. $z + \zeta = \zeta + z$. $z\zeta = \zeta z$.

3. Associativity of addition and multiplication. $(z + \zeta) + \omega = z + (\zeta + \omega)$. $(z\zeta)\,\omega = z\,(\zeta\omega)$.

4. Distributive law. $z\,(\zeta + \omega) = z\zeta + z\omega$.

5. Identity with respect to addition and multiplication. Zero is the additive identity element, $z + 0 = z$; unity is the muliplicative identity element, $z(1) = z$.

6. Inverse with respect to addition. $z + (-z) = (x + \imath y) + (-x - \imath y) = (x - x) + \imath(y - y) = 0$.

7. Inverse with respect to multiplication for nonzero numbers. $zz^{-1} = 1$, where

$$z^{-1} = \frac{1}{z} = \frac{1}{x + \imath y} = \frac{1}{x + \imath y}\frac{x - \imath y}{x - \imath y} = \frac{x - \imath y}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \imath \frac{y}{x^2 + y^2}$$

Properties of the complex conjugate. Using the field properties of complex numbers, we can derive the following properties of the complex conjugate, $\overline{z} = x - \imath y$.

1. $\overline{(\overline{z})} = z$,
2. $\overline{z + \zeta} = \overline{z} + \overline{\zeta}$,
3. $\overline{z\zeta} = \overline{z}\,\overline{\zeta}$,
4. $\overline{\left(\frac{z}{\zeta}\right)} = \frac{(\overline{z})}{(\overline{\zeta})}$.

6.2 The Complex Plane

Complex plane. We can denote a complex number $z = x + \imath y$ as an ordered pair of real numbers (x, y). Thus we can represent a complex number as a point in $\mathbb{R}^2$ where the first component is the real part and the second component is the imaginary part of z. This is called the *complex plane* or the *Argand diagram*. (See Figure 6.2.) A complex number written as $z = x + \imath y$ is said to be in *Cartesian form*, or $a + \imath b$ *form*.

Recall that there are two ways of describing a point in the complex plane: an ordered pair of coordinates (x, y) that give the horizontal and vertical offset from the origin or the distance r from the origin and the angle θ from the positive horizontal axis. The angle θ is not unique. It is only determined up to an additive integer multiple of 2π.

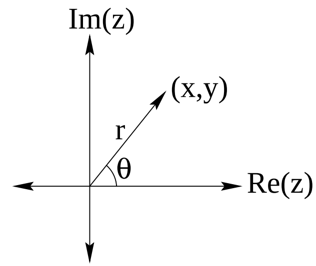


Figure 6.2: The complex plane.

Modulus. The *magnitude* or *modulus* of a complex number is the distance of the point from the origin. It is defined as $|z| = |x + \imath y| = \sqrt{x^2 + y^2}$. Note that $z\overline{z} = (x + \imath y)(x - \imath y) = x^2 + y^2 = |z|^2$. The modulus has the following properties.

1. $|z\zeta| = |z|\,|\zeta|$
2. $\left|\dfrac{z}{\zeta}\right| = \dfrac{|z|}{|\zeta|}$ for $\zeta \neq 0$.
3. $|z + \zeta| \leq |z| + |\zeta|$
4. $|z + \zeta| \geq ||z| - |\zeta||$

We could prove the first two properties by expanding in $x + \imath y$ form, but it would be fairly messy. The proofs will become simple after polar form has been introduced. The second two properties follow from the triangle inequalities in geometry. This will become apparent after the relationship between complex numbers and vectors is introduced. One can show that

$$|z_1 z_2 \cdots z_n| = |z_1|\,|z_2| \cdots |z_n|$$

and

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

with proof by induction.

Argument. The *argument* of a complex number is the angle that the vector with tail at the origin and head at $z = x + \imath y$ makes with the positive x-axis. The argument is denoted $\arg(z)$. Note that the argument is defined for all nonzero numbers and is only determined up to an additive integer multiple of 2π. That is, the argument of a complex number is the set of values: $\{\theta + 2\pi n \mid n \in \mathbb{Z}\}$. The *principal argument* of a complex number is that angle in the set $\arg(z)$ which lies in the range $(-\pi, \pi]$. The principal argument is denoted $\operatorname{Arg}(z)$. We prove the following identities in Exercise 6.10.

$$\arg(z\zeta) = \arg(z) + \arg(\zeta)$$
$$\operatorname{Arg}(z\zeta) \neq \operatorname{Arg}(z) + \operatorname{Arg}(\zeta)$$
$$\arg\left(z^2\right) = \arg(z) + \arg(z) \neq 2\arg(z)$$

Example 6.2.1 *Consider the equation $|z - 1 - \imath| = 2$. The set of points satisfying this equation is a circle of radius 2 and center at $1 + \imath$ in the complex plane. You can see this by noting that $|z - 1 - \imath|$ is the distance from the point $(1, 1)$. (See Figure 6.3.)*

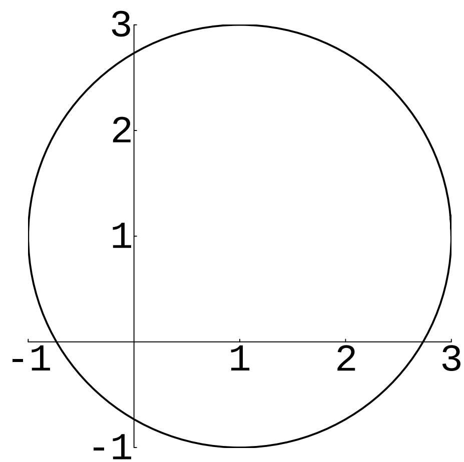


Figure 6.3: Solution of $|z - 1 - \imath| = 2$.

Another way to derive this is to substitute $z = x + \imath y$ into the equation.

$$\begin{gathered}|x + \imath y - 1 - \imath| = 2 \\ \sqrt{(x-1)^2 + (y-1)^2} = 2 \\ (x-1)^2 + (y-1)^2 = 4\end{gathered}$$

This is the analytic geometry equation for a circle of radius 2 centered about $(1, 1)$.

Example 6.2.2 *Consider the curve described by*

$$|z| + |z - 2| = 4.$$

Note that $|z|$ is the distance from the origin in the complex plane and $|z - 2|$ is the distance from $z = 2$. The equation is

$$(\text{distance from } (0, 0)) + (\text{distance from } (2, 0)) = 4.$$

From geometry, we know that this is an ellipse with foci at $(0, 0)$ and $(2, 0)$, major axis 2, and minor axis $\sqrt{3}$. (See Figure 6.4.)

We can use the substitution $z = x + \imath y$ to get the equation in algebraic form.

$$\begin{gathered}|z| + |z - 2| = 4 \\ |x + \imath y| + |x + \imath y - 2| = 4 \\ \sqrt{x^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 4 \\ x^2 + y^2 = 16 - 8\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2 \\ x - 5 = -2\sqrt{(x-2)^2 + y^2} \\ x^2 - 10x + 25 = 4x^2 - 16x + 16 + 4y^2 \\ \frac{1}{4}(x-1)^2 + \frac{1}{3}y^2 = 1\end{gathered}$$

Thus we have the standard form for an equation describing an ellipse.

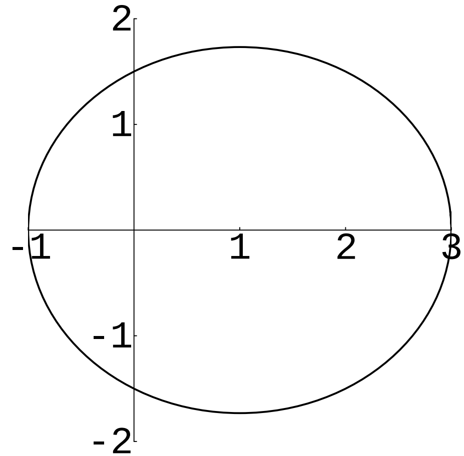


Figure 6.4: Solution of $|z| + |z - 2| = 4$.

6.3 Polar Form

Polar form. A complex number written in Cartesian form, $z = x + \imath y$, can be converted *polar form*, $z = r(\cos\theta + \imath \sin\theta)$, using trigonometry. Here $r = |z|$ is the modulus and $\theta = \arctan(x, y)$ is the argument of z. The argument is the angle between the x axis and the vector with its head at (x, y). (See Figure 6.5.) Note that θ is not unique. If $z = r(\cos\theta + \imath \sin\theta)$ then $z = r(\cos(\theta + 2n\pi) + \imath \sin(\theta + 2n\pi))$ for any $n \in \mathbb{Z}$.

The arctangent. Note that $\arctan(x, y)$ is not the same thing as the old arctangent that you learned about in trigonometry $\arctan(x, y)$ is sensitive to the quadrant of the point (x, y), while $\arctan\left(\frac{y}{x}\right)$ is not. For example,

$$\arctan(1, 1) = \frac{\pi}{4} + 2n\pi \quad \text{and} \quad \arctan(-1, -1) = \frac{-3\pi}{4} + 2n\pi,$$

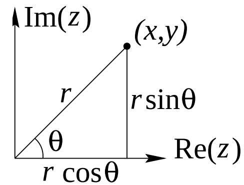


Figure 6.5: Polar form.

whereas

$$\arctan\left(\frac{-1}{-1}\right) = \arctan\left(\frac{1}{1}\right) = \arctan(1).$$

Euler's formula. *Euler's formula*, $\mathrm{e}^{\imath\theta} = \cos\theta + \imath\sin\theta$,[5] allows us to write the polar form more compactly. Expressing the polar form in terms of the exponential function of imaginary argument makes arithmetic with complex numbers much more convenient.

$$z = r(\cos\theta + \imath\sin\theta) = r\,\mathrm{e}^{\imath\theta}$$

The exponential of an imaginary argument has all the nice properties that we know from studying functions of a real variable, like $\mathrm{e}^{\imath a}\,\mathrm{e}^{\imath b} = \mathrm{e}^{\imath(a+b)}$. Later on we will introduce the exponential of a complex number.

Using Euler's Formula, we can express the cosine and sine in terms of the exponential.

$$\frac{\mathrm{e}^{\imath\theta} + \mathrm{e}^{-\imath\theta}}{2} = \frac{(\cos(\theta) + \imath\sin(\theta)) + (\cos(-\theta) + \imath\sin(-\theta))}{2} = \cos(\theta)$$
$$\frac{\mathrm{e}^{\imath\theta} - \mathrm{e}^{-\imath\theta}}{\imath 2} = \frac{(\cos(\theta) + \imath\sin(\theta)) - (\cos(-\theta) + \imath\sin(-\theta))}{\imath 2} = \sin(\theta)$$

Arithmetic with complex numbers. Note that it is convenient to add complex numbers in Cartesian form.

$$z + \zeta = (x + \imath y) + (\xi + \imath\psi) = (x + \xi) + \imath\,(y + \psi)$$

[5] See Exercise 6.17 for justification of Euler's formula.

However, it is difficult to multiply or divide them in Cartesian form.

$$
\begin{gathered}
z\zeta = (x + \imath y)(\xi + \imath\psi) = (x\xi - y\psi) + \imath(x\psi + \xi y) \\
\frac{z}{\zeta} = \frac{x + \imath y}{\xi + \imath\psi} = \frac{(x + \imath y)(\xi - \imath\psi)}{(\xi + \imath\psi)(\xi - \imath\psi)} = \frac{x\xi + y\psi}{\xi^2 + \psi^2} + \imath\frac{\xi y - x\psi}{\xi^2 + \psi^2}
\end{gathered}
$$

On the other hand, it is difficult to add complex numbers in polar form.

$$
\begin{aligned}
z + \zeta &= r\,\mathrm{e}^{\imath\theta} + \rho\,\mathrm{e}^{\imath\phi} \\
&= r(\cos\theta + \imath\sin\theta) + \rho(\cos\phi + \imath\sin\phi) \\
&= r\cos\theta + \rho\cos\phi + \imath(r\sin\theta + \rho\sin\phi) \\
&= \sqrt{(r\cos\theta + \rho\cos\phi)^2 + (r\sin\theta + \rho\sin\phi)^2} \\
&\qquad \times \mathrm{e}^{\imath\arctan(r\cos\theta + \rho\cos\phi,\, r\sin\theta + \rho\sin\phi)} \\
&= \sqrt{r^2 + \rho^2 + 2\cos(\theta - \phi)}\,\mathrm{e}^{\imath\arctan(r\cos\theta + \rho\cos\phi,\, r\sin\theta + \rho\sin\phi)}
\end{aligned}
$$

However, it is convenient to multiply and divide them in polar form.

$$
\begin{gathered}
z\zeta = r\,\mathrm{e}^{\imath\theta}\,\rho\,\mathrm{e}^{\imath\phi} = r\rho\,\mathrm{e}^{\imath(\theta+\phi)} \\
\frac{z}{\zeta} = \frac{r\,\mathrm{e}^{\imath\theta}}{\rho\,\mathrm{e}^{\imath\phi}} = \frac{r}{\rho}\,\mathrm{e}^{\imath(\theta-\phi)}
\end{gathered}
$$

Keeping this in mind will make working with complex numbers a shade or two less grungy.

Result 6.3.1 Euler's formula is

$$\mathrm{e}^{\imath\theta} = \cos\theta + \imath\sin\theta.$$

We can write the cosine and sine in terms of the exponential.

$$\cos(\theta) = \frac{\mathrm{e}^{\imath\theta} + \mathrm{e}^{-\imath\theta}}{2}, \qquad \sin(\theta) = \frac{\mathrm{e}^{\imath\theta} - \mathrm{e}^{-\imath\theta}}{\imath 2}$$

To change between Cartesian and polar form, use the identities

$$r\,\mathrm{e}^{\imath\theta} = r\cos\theta + \imath r\sin\theta,$$
$$x + \imath y = \sqrt{x^2 + y^2}\,\mathrm{e}^{\imath\arctan(x,y)}.$$

Cartesian form is convenient for addition. Polar form is convenient for multiplication and division.

Example 6.3.1 *We write $5 + \imath 7$ in polar form.*

$$5 + \imath 7 = \sqrt{74}\,\mathrm{e}^{\imath\arctan(5,7)}$$

We write $2\,\mathrm{e}^{\imath\pi/6}$ in Cartesian form.

$$\begin{aligned} 2\,\mathrm{e}^{\imath\pi/6} &= 2\cos\left(\frac{\pi}{6}\right) + 2\imath\sin\left(\frac{\pi}{6}\right) \\ &= \sqrt{3} + \imath \end{aligned}$$

Example 6.3.2 *We will prove the trigonometric identity*

$$\cos^4\theta = \frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta) + \frac{3}{8}.$$

We start by writing the cosine in terms of the exponential.

$$\begin{aligned}
\cos^4\theta &= \left(\frac{\mathrm{e}^{\imath\theta}+\mathrm{e}^{-\imath\theta}}{2}\right)^4 \\
&= \frac{1}{16}\left(\mathrm{e}^{\imath 4\theta} + 4\,\mathrm{e}^{\imath 2\theta} + 6 + 4\,\mathrm{e}^{-\imath 2\theta} + \mathrm{e}^{-\imath 4\theta}\right) \\
&= \frac{1}{8}\left(\frac{\mathrm{e}^{\imath 4\theta}+\mathrm{e}^{-\imath 4\theta}}{2}\right) + \frac{1}{2}\left(\frac{\mathrm{e}^{\imath 2\theta}+\mathrm{e}^{-\imath 2\theta}}{2}\right) + \frac{3}{8} \\
&= \frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta) + \frac{3}{8}
\end{aligned}$$

By the definition of exponentiation, we have $\mathrm{e}^{\imath n\theta} = \left(\mathrm{e}^{\imath\theta}\right)^n$ We apply Euler's formula to obtain a result which is useful in deriving trigonometric identities.

$$\cos(n\theta) + \imath\sin(n\theta) = (\cos\theta + \imath\sin\theta)^n$$

Result 6.3.2 DeMoivre's Theorem.[a]

$$\cos(n\theta) + \imath\sin(n\theta) = (\cos\theta + \imath\sin\theta)^n$$

[a]It's amazing what passes for a theorem these days. I would think that this would be a corollary at most.

Example 6.3.3 *We will express* $\cos(5\theta)$ *in terms of* $\cos\theta$ *and* $\sin(5\theta)$ *in terms of* $\sin\theta$. *We start with DeMoivre's theorem.*

$$\mathrm{e}^{\imath 5\theta} = \left(\mathrm{e}^{\imath\theta}\right)^5$$

$$
\begin{aligned}
\cos(5\theta) + \imath \sin(5\theta) &= (\cos\theta + \imath \sin\theta)^5 \\
&= \binom{5}{0}\cos^5\theta + \imath\binom{5}{1}\cos^4\theta\sin\theta - \binom{5}{2}\cos^3\theta\sin^2\theta - \imath\binom{5}{3}\cos^2\theta\sin^3\theta \\
&\quad + \binom{5}{4}\cos\theta\sin^4\theta + \imath\binom{5}{5}\sin^5\theta \\
&= \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right) + \imath\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right)
\end{aligned}
$$

Then we equate the real and imaginary parts.

$$
\begin{aligned}
\cos(5\theta) &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\
\sin(5\theta) &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta
\end{aligned}
$$

Finally we use the Pythagorean identity, $\cos^2\theta + \sin^2\theta = 1$.

$$
\cos(5\theta) = \cos^5\theta - 10\cos^3\theta\left(1 - \cos^2\theta\right) + 5\cos\theta\left(1 - \cos^2\theta\right)^2
$$

$$
\boxed{\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}
$$

$$
\sin(5\theta) = 5\left(1 - \sin^2\theta\right)^2\sin\theta - 10\left(1 - \sin^2\theta\right)\sin^3\theta + \sin^5\theta
$$

$$
\boxed{\sin(5\theta) = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta}
$$

6.4 Arithmetic and Vectors

Addition. We can represent the complex number $z = x + \imath y = r\,\mathrm{e}^{\imath\theta}$ as a vector in Cartesian space with tail at the origin and head at (x, y), or equivalently, the vector of length r and angle θ. With the vector representation, we can add complex numbers by connecting the tail of one vector to the head of the other. The vector $z + \zeta$ is the diagonal of the parallelogram defined by z and ζ. (See Figure 6.6.)

Negation. The negative of $z = x + \imath y$ is $-z = -x - \imath y$. In polar form we have $z = r\,\mathrm{e}^{\imath\theta}$ and $-z = r\,\mathrm{e}^{\imath(\theta+\pi)}$, (more generally, $z = r\,\mathrm{e}^{\imath(\theta+(2n+1)\pi)}$, $n \in \mathbb{Z}$. In terms of vectors, $-z$ has the same magnitude but opposite direction as z. (See Figure 6.6.)

Multiplication. The product of $z = r\,e^{\imath\theta}$ and $\zeta = \rho\,e^{\imath\phi}$ is $z\zeta = r\rho\,e^{\imath(\theta+\phi)}$. The length of the vector $z\zeta$ is the product of the lengths of z and ζ. The angle of $z\zeta$ is the sum of the angles of z and ζ. (See Figure 6.6.)

Note that $\arg(z\zeta) = \arg(z) + \arg(\zeta)$. Each of these arguments has an infinite number of values. If we write out the multi-valuedness explicitly, we have

$$\{\theta + \phi + 2\pi n : n \in \mathbb{Z}\} = \{\theta + 2\pi n : n \in \mathbb{Z}\} + \{\phi + 2\pi n : n \in \mathbb{Z}\}$$

The same is not true of the principal argument. In general, $\operatorname{Arg}(z\zeta) \neq \operatorname{Arg}(z) + \operatorname{Arg}(\zeta)$. Consider the case $z = \zeta = e^{\imath 3\pi/4}$. Then $\operatorname{Arg}(z) = \operatorname{Arg}(\zeta) = 3\pi/4$, however, $\operatorname{Arg}(z\zeta) = -\pi/2$.

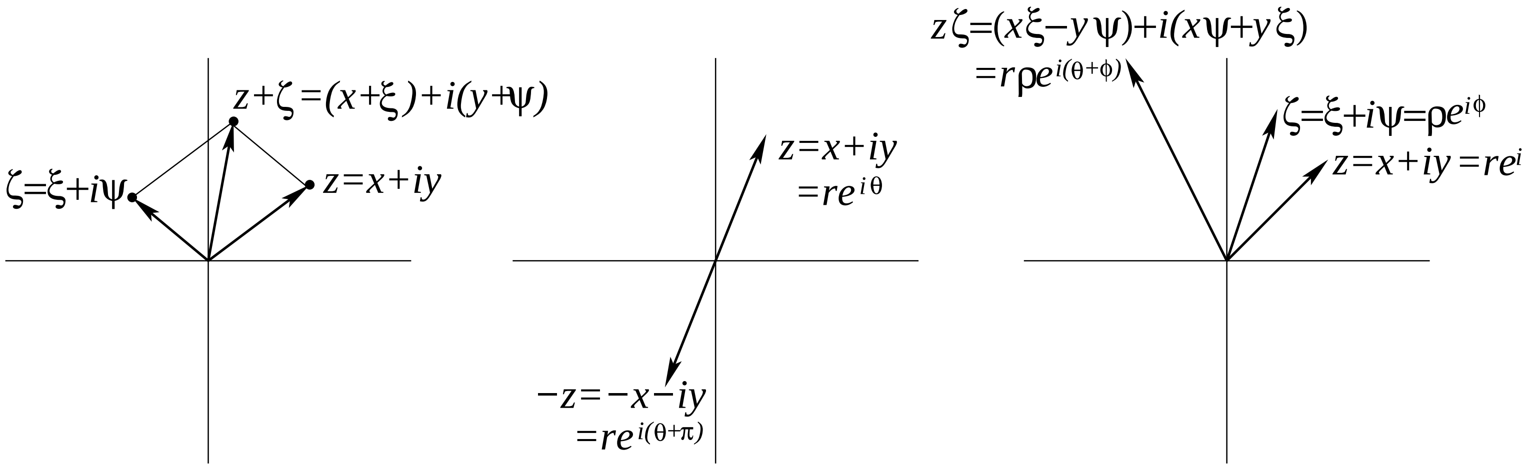


Figure 6.6: Addition, negation and multiplication.

Multiplicative inverse. Assume that z is nonzero. The multiplicative inverse of $z = r\,e^{\imath\theta}$ is $\frac{1}{z} = \frac{1}{r}\,e^{-\imath\theta}$. The length of $\frac{1}{z}$ is the multiplicative inverse of the length of z. The angle of $\frac{1}{z}$ is the negative of the angle of z. (See Figure 6.7.)

Division. Assume that ζ is nonzero. The quotient of $z = r\,\mathrm{e}^{\imath\theta}$ and $\zeta = \rho\,\mathrm{e}^{\imath\phi}$ is $\frac{z}{\zeta} = \frac{r}{\rho}\,\mathrm{e}^{\imath(\theta-\phi)}$. The length of the vector $\frac{z}{\zeta}$ is the quotient of the lengths of z and ζ. The angle of $\frac{z}{\zeta}$ is the difference of the angles of z and ζ. (See Figure 6.7.)

Complex conjugate. The complex conjugate of $z = x + \imath y = r\,\mathrm{e}^{\imath\theta}$ is $\overline{z} = x - \imath y = r\,\mathrm{e}^{-\imath\theta}$. $\overline{z}$ is the mirror image of z, reflected across the x axis. In other words, $\overline{z}$ has the same magnitude as z and the angle of $\overline{z}$ is the negative of the angle of z. (See Figure 6.7.)

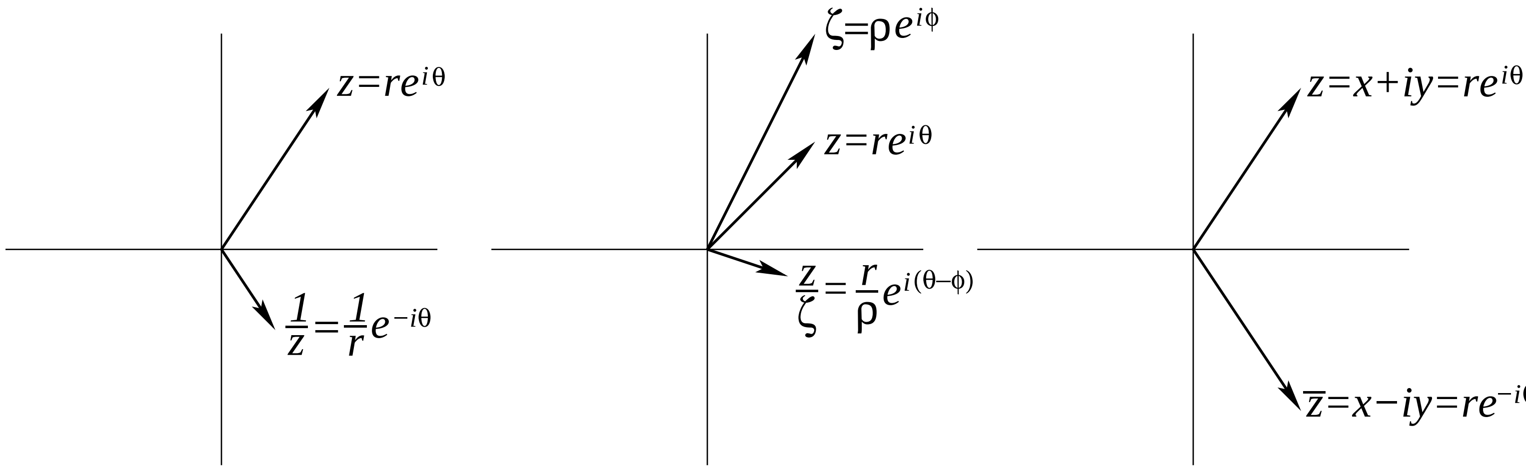


Figure 6.7: Multiplicative inverse, division and complex conjugate.

6.5 Integer Exponents

Consider the product $(a + b)^n$, $n \in \mathbb{Z}$. If we know $\arctan(a, b)$ then it will be most convenient to expand the product working in polar form. If not, we can write n in base 2 to efficiently do the multiplications.

Example 6.5.1 *Suppose that we want to write* $\left(\sqrt{3}+\imath\right)^{20}$ *in Cartesian form.*[6] *We can do the multiplication directly. Note that* 20 *is* 10100 *in base 2. That is,* $20 = 2^4 + 2^2$*. We first calculate the powers of the form* $\left(\sqrt{3}+\imath\right)^{2^n}$ *by successive squaring.*

$$\begin{aligned}
\left(\sqrt{3}+\imath\right)^2 &= 2+\imath 2\sqrt{3} \\
\left(\sqrt{3}+\imath\right)^4 &= -8+\imath 8\sqrt{3} \\
\left(\sqrt{3}+\imath\right)^8 &= -128-\imath 128\sqrt{3} \\
\left(\sqrt{3}+\imath\right)^{16} &= -32768+\imath 32768\sqrt{3}
\end{aligned}$$

Next we multiply $\left(\sqrt{3}+\imath\right)^4$ *and* $\left(\sqrt{3}+\imath\right)^{16}$ *to obtain the answer.*

$$\left(\sqrt{3}+\imath\right)^{20} = \left(-32768+\imath 32768\sqrt{3}\right)\left(-8+\imath 8\sqrt{3}\right) = -524288-\imath 524288\sqrt{3}$$

Since we know that $\arctan\left(\sqrt{3},1\right) = \pi/6$*, it is easiest to do this problem by first changing to modulus-argument form.*

$$\begin{aligned}
\left(\sqrt{3}+\imath\right)^{20} &= \left(\sqrt{\left(\sqrt{3}\right)^2+1^2}\,\mathrm{e}^{\imath\arctan(\sqrt{3},1)}\right)^{20} \\
&= \left(2\,\mathrm{e}^{\imath\pi/6}\right)^{20} \\
&= 2^{20}\,\mathrm{e}^{\imath 4\pi/3} \\
&= 1048576\left(-\frac{1}{2}-\imath\frac{\sqrt{3}}{2}\right) \\
&= -524288-\imath 524288\sqrt{3}
\end{aligned}$$

[6] No, I have no idea why we would want to do that. Just humor me. If you pretend that you're interested, I'll do the same. Believe me, expressing your real feelings here isn't going to do anyone any good.

Example 6.5.2 *Consider $(5+\imath 7)^{11}$. We will do the exponentiation in polar form and write the result in Cartesian form.*

$$
\begin{aligned}
(5+\imath 7)^{11} &= \left(\sqrt{74}\,\mathrm{e}^{\imath \arctan(5,7)}\right)^{11} \\
&= 74^5\sqrt{74}(\cos(11\arctan(5,7)) + \imath\sin(11\arctan(5,7))) \\
&= 2219006624\sqrt{74}\cos(11\arctan(5,7)) + \imath 2219006624\sqrt{74}\sin(11\arctan(5,7))
\end{aligned}
$$

The result is correct, but not very satisfying. This expression could be simplified. You could evaluate the trigonometric functions with some fairly messy trigonometric identities. This would take much more work than directly multiplying $(5+\imath 7)^{11}$.

6.6 Rational Exponents

In this section we consider complex numbers with rational exponents, $z^{p/q}$, where p/q is a rational number. First we consider unity raised to the $1/n$ power. We define $1^{1/n}$ as the set of numbers $\{z\}$ such that $z^n = 1$.

$$
1^{1/n} = \{z \mid z^n = 1\}
$$

We can find these values by writing z in modulus-argument form.

$$
\begin{gathered}
z^n = 1 \\
r^n\,\mathrm{e}^{\imath n\theta} = 1 \\
r^n = 1 \qquad n\theta = 0 \mod 2\pi \\
r = 1 \qquad \theta = 2\pi k \text{ for } k \in \mathbb{Z} \\
1^{1/n} = \left\{\mathrm{e}^{\imath 2\pi k/n} \mid k \in \mathbb{Z}\right\}
\end{gathered}
$$

There are only n distinct values as a result of the 2π periodicity of $\mathrm{e}^{\imath\theta}$. $\mathrm{e}^{\imath 2\pi} = \mathrm{e}^{\imath 0}$.

$$
1^{1/n} = \left\{\mathrm{e}^{\imath 2\pi k/n} \mid k = 0, \ldots, n-1\right\}
$$

These values are equally spaced points on the unit circle in the complex plane.

Example 6.6.1 $1^{1/6}$ *has the 6 values,*

$$\left\{ e^{\imath 0}, e^{\imath \pi/3}, e^{\imath 2\pi/3}, e^{\imath \pi}, e^{\imath 4\pi/3}, e^{\imath 5\pi/3} \right\}.$$

In Cartesian form this is

$$\left\{ 1, \frac{1 + \imath\sqrt{3}}{2}, \frac{-1 + \imath\sqrt{3}}{2}, -1, \frac{-1 - \imath\sqrt{3}}{2}, \frac{1 - \imath\sqrt{3}}{2} \right\}.$$

The sixth roots of unity are plotted in Figure 6.8.

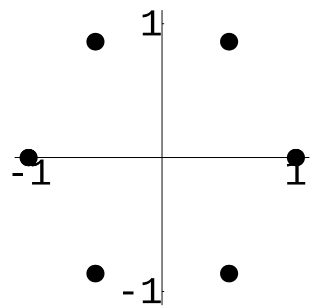


Figure 6.8: The sixth roots of unity.

The n^{th} roots of the complex number $c = \alpha\, e^{\imath\beta}$ are the set of numbers $z = r\, e^{\imath\theta}$ such that

$$z^n = c = \alpha\, e^{\imath\beta}$$
$$r^n\, e^{\imath n\theta} = \alpha\, e^{\imath\beta}$$
$$r = \sqrt[n]{\alpha} \qquad n\theta = \beta \mod 2\pi$$
$$r = \sqrt[n]{\alpha} \qquad \theta = (\beta + 2\pi k)/n \text{ for } k = 0, \ldots, n-1.$$

Thus

$$c^{1/n} = \left\{ \sqrt[n]{\alpha}\, e^{\imath(\beta+2\pi k)/n} \mid k = 0, \ldots, n-1 \right\} = \left\{ \sqrt[n]{|c|}\, e^{\imath(\operatorname{Arg}(c)+2\pi k)/n} \mid k = 0, \ldots, n-1 \right\}$$

Principal roots. The *principal* n^{th} *root* is denoted

$$\sqrt[n]{z} \equiv \sqrt[n]{z}\, \mathrm{e}^{\imath \operatorname{Arg}(z)/n}.$$

Thus the principal root has the property

$$-\pi/n < \operatorname{Arg}\left(\sqrt[n]{z}\right) \leq \pi/n.$$

This is consistent with the notation from functions of a real variable: $\sqrt[n]{x}$ denotes the positive n^{th} root of a positive real number. We adopt the convention that $z^{1/n}$ denotes the n^{th} roots of z, which is a set of n numbers and $\sqrt[n]{z}$ is the principal n^{th} root of z, which is a single number. The n^{th} roots of z are the principal n^{th} root of z times the n^{th} roots of unity.

$$z^{1/n} = \left\{ \sqrt[n]{r}\, \mathrm{e}^{\imath(\operatorname{Arg}(z)+2\pi k)/n} \mid k = 0, \ldots, n-1 \right\}$$
$$z^{1/n} = \left\{ \sqrt[n]{z}\, \mathrm{e}^{\imath 2\pi k/n} \mid k = 0, \ldots, n-1 \right\}$$
$$z^{1/n} = \sqrt[n]{z} 1^{1/n}$$

Rational exponents. We interpret $z^{p/q}$ to mean $z^{(p/q)}$. That is, we first simplify the exponent, i.e. reduce the fraction, before carrying out the exponentiation. Therefore $z^{2/4} = z^{1/2}$ and $z^{10/5} = z^2$. If p/q is a reduced fraction, (p and q are relatively prime, in other words, they have no common factors), then

$$z^{p/q} \equiv \left(z^p\right)^{1/q}.$$

Thus $z^{p/q}$ is a set of q values. Note that for an un-reduced fraction r/s,

$$\left(z^r\right)^{1/s} \neq \left(z^{1/s}\right)^r.$$

The former expression is a set of s values while the latter is a set of no more that s values. For instance, $\left(1^2\right)^{1/2} = 1^{1/2} = \pm 1$ and $\left(1^{1/2}\right)^2 = (\pm 1)^2 = 1$.

Example 6.6.2 *Consider* $2^{1/5}$, $(1+\imath)^{1/3}$ *and* $(2+\imath)^{5/6}$.

$$2^{1/5} = \sqrt[5]{2}\, \mathrm{e}^{\imath 2\pi k/5}, \quad \textit{for } k = 0, 1, 2, 3, 4$$

$$
\begin{aligned}
(1+\imath)^{1/3} &= \left(\sqrt{2}\,\mathrm{e}^{\imath\pi/4}\right)^{1/3} \\
&= \sqrt[6]{2}\,\mathrm{e}^{\imath\pi/12}\,\mathrm{e}^{\imath 2\pi k/3}, \quad \text{for } k = 0, 1, 2
\end{aligned}
$$

$$
\begin{aligned}
(2+\imath)^{5/6} &= \left(\sqrt{5}\,\mathrm{e}^{\imath \operatorname{Arctan}(2,1)}\right)^{5/6} \\
&= \left(\sqrt{5^5}\,\mathrm{e}^{\imath 5 \operatorname{Arctan}(2,1)}\right)^{1/6} \\
&= \sqrt[12]{5^5}\,\mathrm{e}^{\imath \frac{5}{6} \operatorname{Arctan}(2,1)}\,\mathrm{e}^{\imath\pi k/3}, \quad \text{for } k = 0, 1, 2, 3, 4, 5
\end{aligned}
$$

Example 6.6.3 *We find the roots of* $z^5 + 4$.

$$
\begin{aligned}
(-4)^{1/5} &= (4\,\mathrm{e}^{\imath\pi})^{1/5} \\
&= \sqrt[5]{4}\,\mathrm{e}^{\imath\pi(1+2k)/5}, \quad \text{for } k = 0, 1, 2, 3, 4
\end{aligned}
$$

6.7 Exercises

Complex Numbers

Exercise 6.1
If $z = x + \imath y$, write the following in the form $a + \imath b$:

1. $(1 + \imath 2)^7$

2. $\dfrac{1}{(\overline{zz})}$

3. $\dfrac{\imath z + \overline{z}}{(3 + \imath)^9}$

Hint, Solution

Exercise 6.2
Verify that:

1. $\dfrac{1 + \imath 2}{3 - \imath 4} + \dfrac{2 - \imath}{\imath 5} = -\dfrac{2}{5}$

2. $(1 - \imath)^4 = -4$

Hint, Solution

Exercise 6.3
Write the following complex numbers in the form $a + \imath b$.

1. $\left(1 + \imath\sqrt{3}\right)^{-10}$

2. $(11 + \imath 4)^2$

Hint, Solution

Exercise 6.4
Write the following complex numbers in the form $a + \imath b$

1. $\left(\dfrac{2+\imath}{\imath 6 - (1 - \imath 2)} \right)^2$
2. $(1 - \imath)^7$

Hint, Solution

Exercise 6.5
If $z = x + \imath y$, write the following in the form $u(x, y) + \imath v(x, y)$.

1. $\overline{\left(\dfrac{\overline{z}}{z} \right)}$
2. $\dfrac{z + \imath 2}{2 - \imath \overline{z}}$

Hint, Solution

Exercise 6.6
Quaternions are sometimes used as a generalization of complex numbers. A quaternion u may be defined as

$$u = u_0 + \imath u_1 + \jmath u_2 + k u_3$$

where u_0, u_1, u_2 and u_3 are real numbers and $\imath$, $\jmath$ and k are objects which satisfy

$$\imath^2 = \jmath^2 = k^2 = -1, \quad \imath\jmath = k, \quad \jmath\imath = -k$$

and the usual associative and distributive laws. Show that for any quaternions u, w there exists a quaternion v such that

$$uv = w$$

except for the case $u_0 = u_1 = u_2 = u_3$.
Hint, Solution

Exercise 6.7
Let $\alpha \neq 0$, $\beta \neq 0$ be two complex numbers. Show that $\alpha = t\beta$ for some real number t (i.e. the vectors defined by α and β are parallel) if and only if $\Im\left(\alpha\overline{\beta}\right) = 0$.
Hint, Solution

The Complex Plane

Exercise 6.8
Find and depict all values of

1. $(1 + \imath)^{1/3}$

2. $\imath^{1/4}$

Identify the principal root.
Hint, Solution

Exercise 6.9
Sketch the regions of the complex plane:

1. $|\Re(z)| + 2|\Im(z)| \leq 1$

2. $1 \leq |z - \imath| \leq 2$

3. $|z - \imath| \leq |z + \imath|$

Hint, Solution

Exercise 6.10
Prove the following identities.

1. $\arg(z\zeta) = \arg(z) + \arg(\zeta)$

2. $\operatorname{Arg}(z\zeta) \neq \operatorname{Arg}(z) + \operatorname{Arg}(\zeta)$

3. $\arg\left(z^2\right) = \arg(z) + \arg(z) \neq 2\arg(z)$

Hint, Solution

Exercise 6.11
Show, both by geometric and algebraic arguments, that for complex numbers z and ζ the inequalities

$$||z| - |\zeta|| \leq |z + \zeta| \leq |z| + |\zeta|$$

hold.
Hint, Solution

Exercise 6.12
Find all the values of

1. $(-1)^{-3/4}$
2. $8^{1/6}$

and show them graphically.
Hint, Solution

Exercise 6.13
Find all values of

1. $(-1)^{-1/4}$
2. $16^{1/8}$

and show them graphically.
Hint, Solution

Exercise 6.14
Sketch the regions or curves described by

1. $1 < |z - \imath 2| < 2$

2. $|\Re(z)| + 5|\Im(z)| = 1$

3. $|z - \imath| = |z + \imath|$

Hint, Solution

Exercise 6.15
Sketch the regions or curves described by

1. $|z - 1 + \imath| \leq 1$

2. $\Re(z) - \Im(z) = 5$

3. $|z - \imath| + |z + \imath| = 1$

Hint, Solution

Exercise 6.16
Solve the equation

$$|\mathrm{e}^{\imath\theta} - 1| = 2$$

for θ $(0 \leq \theta \leq \pi)$ and verify the solution geometrically.
Hint, Solution

Polar Form

Exercise 6.17
Show that Euler's formula, $\mathrm{e}^{\imath\theta} = \cos\theta + \imath\sin\theta$, is formally consistent with the standard Taylor series expansions for the real functions e^x, $\cos x$ and $\sin x$. Consider the Taylor series of e^x about $x = 0$ to be the definition of the exponential function for complex argument.
Hint, Solution

Exercise 6.18
Use de Moivre's formula to derive the trigonometric identity

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta).$$

Hint, Solution

Exercise 6.19
Establish the formula

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}, \qquad (z \neq 1),$$

for the sum of a finite geometric series; then derive the formulas

1. $1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) = \dfrac{1}{2} + \dfrac{\sin((n + 1/2))}{2\sin(\theta/2)}$

2. $\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta) = \dfrac{1}{2}\cot\dfrac{\theta}{2} - \dfrac{\cos((n + 1/2))}{2\sin(\theta/2)}$

where $0 < \theta < 2\pi$.
Hint, Solution

Arithmetic and Vectors

Exercise 6.20
Prove $|z\zeta| = |z||\zeta|$ and $\left|\frac{z}{\zeta}\right| = \frac{|z|}{|\zeta|}$ using polar form.
Hint, Solution

Exercise 6.21
Prove that

$$|z + \zeta|^2 + |z - \zeta|^2 = 2\left(|z|^2 + |\zeta|^2\right).$$

Interpret this geometrically.
Hint, Solution

Integer Exponents

Exercise 6.22
Write $(1 + \imath)^{10}$ in Cartesian form with the following two methods:

1. Just do the multiplication. If it takes you more than four multiplications, you suck.

2. Do the multiplication in polar form.

Hint, Solution

Rational Exponents

Exercise 6.23
Show that each of the numbers $z = -a + (a^2 - b)^{1/2}$ satisfies the equation $z^2 + 2az + b = 0$.
Hint, Solution

6.8 Hints

Complex Numbers

Hint 6.1

Hint 6.2

Hint 6.3

Hint 6.4

Hint 6.5

Hint 6.6

Hint 6.7

The Complex Plane

Hint 6.8

Hint 6.9

Hint 6.10
Write the multivaluedness explicitly.

Hint 6.11
Consider a triangle with vertices at 0, z and $z+\zeta$.

Hint 6.12

Hint 6.13

Hint 6.14

Hint 6.15

Hint 6.16

Polar Form

Hint 6.17
Find the Taylor series of $\mathrm{e}^{\imath\theta}$, $\cos\theta$ and $\sin\theta$. Note that $\imath^{2n}=(-1)^n$.

Hint 6.18

Hint 6.19

Arithmetic and Vectors

Hint 6.20
$|\,\mathrm{e}^{\imath\theta}\,| = 1.$

Hint 6.21
Consider the parallelogram defined by z and ζ.

Integer Exponents

Hint 6.22
For the first part,

$$(1+\imath)^{10} = \left(\left((1+\imath)^2\right)^2\right)^2 (1+\imath)^2.$$

Rational Exponents

Hint 6.23
Substitite the numbers into the equation.

6.9 Solutions

Complex Numbers

Solution 6.1

1. We can do the exponentiation by directly multiplying.

$$
\begin{aligned}
(1+\imath 2)^7 &= (1+\imath 2)(1+\imath 2)^2(1+\imath 2)^4 \\
&= (1+\imath 2)(-3+\imath 4)(-3+\imath 4)^2 \\
&= (11-\imath 2)(-7-\imath 24) \\
&= 29+\imath 278
\end{aligned}
$$

   We can also do the problem using De Moivre's Theorem.

$$
\begin{aligned}
(1+\imath 2)^7 &= \left(\sqrt{5}\,\mathrm{e}^{\imath \arctan(1,2)}\right)^7 \\
&= 125\sqrt{5}\,\mathrm{e}^{\imath 7\arctan(1,2)} \\
&= 125\sqrt{5}\cos(7\arctan(1,2)) + \imath 125\sqrt{5}\sin(7\arctan(1,2))
\end{aligned}
$$

2.

$$
\begin{aligned}
\frac{1}{(\overline{zz})} &= \frac{1}{(x-\imath y)^2} \\
&= \frac{1}{(x-\imath y)^2}\frac{(x+\imath y)^2}{(x+\imath y)^2} \\
&= \frac{(x+\imath y)^2}{(x^2+y^2)^2} \\
&= \frac{x^2-y^2}{(x^2+y^2)^2} + \imath\frac{2xy}{(x^2+y^2)^2}
\end{aligned}
$$

3. We can evaluate the expression using De Moivre's Theorem.

$$
\begin{aligned}
\frac{\imath z + \overline{z}}{(3+\imath)^9} &= (-y + \imath x + x - \imath y)(3+\imath)^{-9} \\
&= (1+\imath)(x-y)\left(\sqrt{10}\,\mathrm{e}^{\imath \arctan(3,1)}\right)^{-9} \\
&= (1+\imath)(x-y)\frac{1}{10000\sqrt{10}}\,\mathrm{e}^{-\imath 9 \arctan(3,1)} \\
&= \frac{(1+\imath)(x-y)}{10000\sqrt{10}}\left(\cos(9\arctan(3,1)) - \imath \sin(9\arctan(3,1))\right) \\
&= \frac{(x-y)}{10000\sqrt{10}}\left(\cos(9\arctan(3,1)) + \sin(9\arctan(3,1))\right) \\
&\quad + \imath \frac{(x-y)}{10000\sqrt{10}}\left(\cos(9\arctan(3,1)) - \sin(9\arctan(3,1))\right)
\end{aligned}
$$

We can also do this problem by directly multiplying but it's a little grungy.

$$
\begin{aligned}
\frac{\imath z + \overline{z}}{(3+\imath)^9} &= \frac{(-y + \imath x + x - \imath y)(3-\imath)^9}{10^9} \\
&= \frac{(1+\imath)(x-y)(3-\imath)\left(\left((3-\imath)^2\right)^2\right)^2}{10^9} \\
&= \frac{(1+\imath)(x-y)(3-\imath)\left((8-\imath 6)^2\right)^2}{10^9} \\
&= \frac{(1+\imath)(x-y)(3-\imath)(28-\imath 96)^2}{10^9} \\
&= \frac{(1+\imath)(x-y)(3-\imath)(-8432-\imath 5376)}{10^9} \\
&= \frac{(x-y)(-22976-\imath 38368)}{10^9} \\
&= \frac{359(y-x)}{15625000} + \imath \frac{1199(y-x)}{31250000}
\end{aligned}
$$

Solution 6.2

1.

$$
\begin{aligned}
\frac{1+\imath 2}{3-\imath 4} + \frac{2-\imath}{\imath 5} &= \frac{1+\imath 2}{3-\imath 4}\frac{3+\imath 4}{3+\imath 4} + \frac{2-\imath}{\imath 5}\frac{-\imath}{-\imath} \\
&= \frac{-5+\imath 10}{25} + \frac{-1-\imath 2}{5} \\
&= -\frac{2}{5}
\end{aligned}
$$

2.

$$
(1-\imath)^4 = (-\imath 2)^2 = -4
$$

Solution 6.3

1. First we do the multiplication in Cartesian form.

$$
\begin{aligned}
\left(1+\imath\sqrt{3}\right)^{-10} &= \left(\left(1+\imath\sqrt{3}\right)^{2}\left(1+\imath\sqrt{3}\right)^{8}\right)^{-1} \\
&= \left(\left(-2+\imath 2\sqrt{3}\right)\left(-2+\imath 2\sqrt{3}\right)^{4}\right)^{-1} \\
&= \left(\left(-2+\imath 2\sqrt{3}\right)\left(-8-\imath 8\sqrt{3}\right)^{2}\right)^{-1} \\
&= \left(\left(-2+\imath 2\sqrt{3}\right)\left(-128+\imath 128\sqrt{3}\right)\right)^{-1} \\
&= \left(-512-\imath 512\sqrt{3}\right)^{-1} \\
&= \frac{1}{512}\frac{-1}{1+\imath\sqrt{3}} \\
&= \frac{1}{512}\frac{-1}{1+\imath\sqrt{3}}\frac{1-\imath\sqrt{3}}{1-\imath\sqrt{3}} \\
&= -\frac{1}{2048}+\imath\frac{\sqrt{3}}{2048}
\end{aligned}
$$

Now we do the multiplication in modulus-argument, (polar), form.

$$\begin{aligned}
\left(1+\imath\sqrt{3}\right)^{-10} &= \left(2\,\mathrm{e}^{\imath\pi/3}\right)^{-10} \\
&= 2^{-10}\,\mathrm{e}^{-\imath 10\pi/3} \\
&= \frac{1}{1024}\left(\cos\left(-\frac{10\pi}{3}\right)+\imath\sin\left(-\frac{10\pi}{3}\right)\right) \\
&= \frac{1}{1024}\left(\cos\left(\frac{4\pi}{3}\right)-\imath\sin\left(\frac{4\pi}{3}\right)\right) \\
&= \frac{1}{1024}\left(-\frac{1}{2}+\imath\frac{\sqrt{3}}{2}\right) \\
&= -\frac{1}{2048}+\imath\frac{\sqrt{3}}{2048}
\end{aligned}$$

2.

$$(11+\imath 4)^2 = 105+\imath 88$$

Solution 6.4

1.

$$\begin{aligned}
\left(\frac{2+\imath}{\imath 6-(1-\imath 2)}\right)^2 &= \left(\frac{2+\imath}{-1+\imath 8}\right)^2 \\
&= \frac{3+\imath 4}{-63-\imath 16} \\
&= \frac{3+\imath 4}{-63-\imath 16}\frac{-63+\imath 16}{-63+\imath 16} \\
&= -\frac{253}{4225}-\imath\frac{204}{4225}
\end{aligned}$$

2.

$$
\begin{aligned}
(1-\imath)^7 &= \left((1-\imath)^2\right)^2 (1-\imath)^2(1-\imath) \\
&= (-\imath 2)^2(-\imath 2)(1-\imath) \\
&= (-4)(-2-\imath 2) \\
&= 8+\imath 8
\end{aligned}
$$

Solution 6.5

1.

$$
\begin{aligned}
\overline{\left(\frac{\overline{z}}{z}\right)} &= \overline{\left(\frac{\overline{x+\imath y}}{x+\imath y}\right)} \\
&= \overline{\left(\frac{x-\imath y}{x+\imath y}\right)} \\
&= \frac{x+\imath y}{x-\imath y} \\
&= \frac{x+\imath y}{x-\imath y}\frac{x+\imath y}{x+\imath y} \\
&= \frac{x^2-y^2}{x^2+y^2}+\imath\frac{2xy}{x^2+y^2}
\end{aligned}
$$

2.

$$
\begin{aligned}
\frac{z+\imath 2}{2-\imath\overline{z}} &= \frac{x+\imath y+\imath 2}{2-\imath(x-\imath y)} \\
&= \frac{x+\imath(y+2)}{2-y-\imath x} \\
&= \frac{x+\imath(y+2)}{2-y-\imath x}\frac{2-y+\imath x}{2-y+\imath x} \\
&= \frac{x(2-y)-(y+2)x}{(2-y)^2+x^2}+\imath\frac{x^2+(y+2)(2-y)}{(2-y)^2+x^2} \\
&= \frac{-2xy}{(2-y)^2+x^2}+\imath\frac{4+x^2-y^2}{(2-y)^2+x^2}
\end{aligned}
$$

Solution 6.6

Method 1. We expand the equation $uv = w$ in its components.

$$
\begin{gathered}
uv = w \\
(u_0+\imath u_1+\jmath u_2+k u_3)(v_0+\imath v_1+\jmath v_2+k v_3) = w_0+\imath w_1+\jmath w_2+k w_3
\end{gathered}
$$

$$
\begin{aligned}
(u_0v_0-u_1v_1-u_2v_2-u_3v_3)+\imath(u_1v_0+u_0v_1-u_3v_2+u_2v_3)+\jmath(u_2v_0+u_3v_1+u_0v_2-u_1v_3) \\
+k(u_3v_0-u_2v_1+u_1v_2+u_0v_3) = w_0+\imath w_1+\jmath w_2+k w_3
\end{aligned}
$$

We can write this as a matrix equation.

$$
\begin{pmatrix}
u_0 & -u_1 & -u_2 & -u_3 \\
u_1 & u_0 & -u_3 & u_2 \\
u_2 & u_3 & u_0 & -u_1 \\
u_3 & -u_2 & u_1 & u_0
\end{pmatrix}
\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}
=
\begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}
$$

This linear system of equations has a unique solution for v if and only if the determinant of the matrix is nonzero. The determinant of the matrix is $\left(u_0^2+u_1^2+u_2^2+u_3^2\right)^2$. This is zero if and only if $u_0 = u_1 = u_2 = u_3 = 0$. Thus there exists a unique v such that $uv = w$ if u is nonzero. This v is

$$v = \Big((u_0w_0 + u_1w_1 + u_2w_2 + u_3w_3) + \imath\,(-u_1w_0 + u_0w_1 + u_3w_2 - u_2w_3) + \jmath\,(-u_2w_0 - u_3w_1 + u_0w_2 + u_1w_3) \\ + k\,(-u_3w_0 + u_2w_1 - u_1w_2 + u_0w_3) \Big) / \left(u_0^2 + u_1^2 + u_2^2 + u_3^2\right)$$

Method 2. Note that $\overline{u}u$ is a real number.

$$\begin{aligned} \overline{u}u &= (u_0 - \imath u_1 - \jmath u_2 - ku_3)(u_0 + \imath u_1 + \jmath u_2 + ku_3) \\ &= \left(u_0^2 + u_1^2 + u_2^2 + u_3^2\right) + \imath\,(u_0u_1 - u_1u_0 - u_2u_3 + u_3u_2) \\ &\qquad + \jmath\,(u_0u_2 + u_1u_3 - u_2u_0 - u_3u_1) + k\,(u_0u_3 - u_1u_2 + u_2u_1 - u_3u_0) \\ &= \left(u_0^2 + u_1^2 + u_2^2 + u_3^2\right) \end{aligned}$$

$\overline{u}u = 0$ only if $u = 0$. We solve for v by multiplying by the conjugate of u and dividing by $\overline{u}u$.

$$\begin{aligned} uv &= w \\ \overline{u}uv &= \overline{u}w \\ v &= \frac{\overline{u}w}{\overline{u}u} \\ v &= \frac{(u_0 - \imath u_1 - \jmath u_2 - ku_3)(w_0 + \imath w_1 + \jmath w_2 + kw_3)}{u_0^2 + u_1^2 + u_2^2 + u_3^2} \end{aligned}$$

$$v = \Big((u_0w_0 + u_1w_1 + u_2w_2 + u_3w_3) + \imath\,(-u_1w_0 + u_0w_1 + u_3w_2 - u_2w_3) + \jmath\,(-u_2w_0 - u_3w_1 + u_0w_2 + u_1w_3) \\ + k\,(-u_3w_0 + u_2w_1 - u_1w_2 + u_0w_3) \Big) / \left(u_0^2 + u_1^2 + u_2^2 + u_3^2\right)$$

Solution 6.7

If $\alpha = t\beta$, then $\alpha\overline{\beta} = t|\beta|^2$, which is a real number. Hence $\Im\left(\alpha\overline{\beta}\right) = 0$.

Now assume that $\Im\left(\alpha\overline{\beta}\right) = 0$. This implies that $\alpha\overline{\beta} = r$ for some $r \in \mathbb{R}$. We multiply by β and simplify.

$$\begin{aligned} \alpha|\beta|^2 &= r\beta \\ \alpha &= \frac{r}{|\beta|^2}\beta \end{aligned}$$

By taking $t = \frac{r}{|\beta|^2}$ We see that $\alpha = t\beta$ for some real number t.

Solution 6.8

1.

$$\begin{aligned}
(1+\imath)^{1/3} &= \left(\sqrt{2}\,\mathrm{e}^{\imath\pi/4}\right)^{1/3} \\
&= \sqrt[6]{2}\,\mathrm{e}^{\imath\pi/12}\,1^{1/3} \\
&= \sqrt[6]{2}\,\mathrm{e}^{\imath\pi/12}\,\mathrm{e}^{\imath 2\pi k/3}, \quad k=0,1,2 \\
&= \left\{\sqrt[6]{2}\,\mathrm{e}^{\imath\pi/12}, \sqrt[6]{2}\,\mathrm{e}^{\imath 3\pi/4}, \sqrt[6]{2}\,\mathrm{e}^{\imath 17\pi/12}\right\}
\end{aligned}$$

The principal root is

$$\sqrt[3]{1+\imath} = \sqrt[6]{2}\,\mathrm{e}^{\imath\pi/12}.$$

The roots are depicted in Figure 6.9.

2.

$$\begin{aligned}
\imath^{1/4} &= \left(\mathrm{e}^{\imath\pi/2}\right)^{1/4} \\
&= \mathrm{e}^{\imath\pi/8}\,1^{1/4} \\
&= \mathrm{e}^{\imath\pi/8}\,\mathrm{e}^{\imath 2\pi k/4}, \quad k=0,1,2,3 \\
&= \left\{\mathrm{e}^{\imath\pi/8}, \mathrm{e}^{\imath 5\pi/8}, \mathrm{e}^{\imath 9\pi/8}, \mathrm{e}^{\imath 13\pi/8}\right\}
\end{aligned}$$

The principal root is

$$\sqrt[4]{\imath} = \mathrm{e}^{\imath\pi/8}.$$

The roots are depicted in Figure 6.10.

Solution 6.9

1.

$$|\Re(z)| + 2|\Im(z)| \leq 1$$
$$|x| + 2|y| \leq 1$$

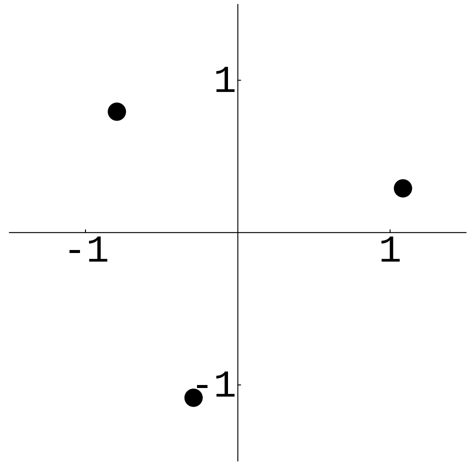


Figure 6.9: $(1+\imath)^{1/3}$

In the first quadrant, this is the triangle below the line $y = (1-x)/2$. We reflect this triangle across the coordinate axes to obtain triangles in the other quadrants. Explicitly, we have the set of points: $\{z = x + \imath y \mid -1 \leq x \leq 1 \wedge |y| \leq (1-|x|)/2\}$. See Figure 6.11.

2. $|z-\imath|$ is the distance from the point $\imath$ in the complex plane. Thus $1 < |z-\imath| < 2$ is an annulus centered at $z = \imath$ between the radii 1 and 2. See Figure 6.12.

3. The points which are closer to $z = \imath$ than $z = -\imath$ are those points in the upper half plane. See Figure 6.13.

Solution 6.10

Let $z = r\,\mathrm{e}^{\imath\theta}$ and $\zeta = \rho\,\mathrm{e}^{\imath\phi}$.

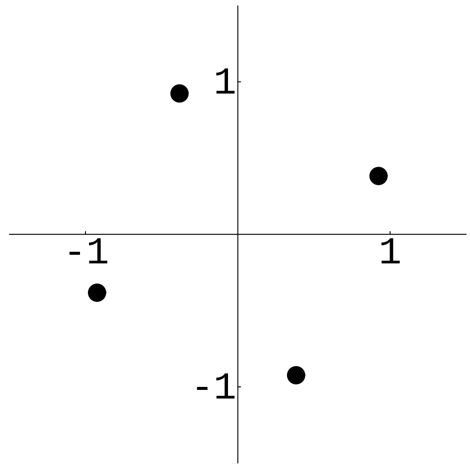


Figure 6.10: $\imath^{1/4}$

1.

$$\arg(z\zeta) = \arg(z) + \arg(\zeta)$$
$$\arg\left(r\rho\,\mathrm{e}^{\imath(\theta+\phi)}\right) = \{\theta + 2\pi m\} + \{\phi + 2\pi n\}$$
$$\{\theta + \phi + 2\pi k\} = \{\theta + \phi + 2\pi m\}$$

2.

$$\operatorname{Arg}(z\zeta) \neq \operatorname{Arg}(z) + \operatorname{Arg}(\zeta)$$

Consider $z = \zeta = -1$. $\operatorname{Arg}(z) = \operatorname{Arg}(\zeta) = \pi$, however $\operatorname{Arg}(z\zeta) = \operatorname{Arg}(1) = 0$. The identity becomes $0 \neq 2\pi$.

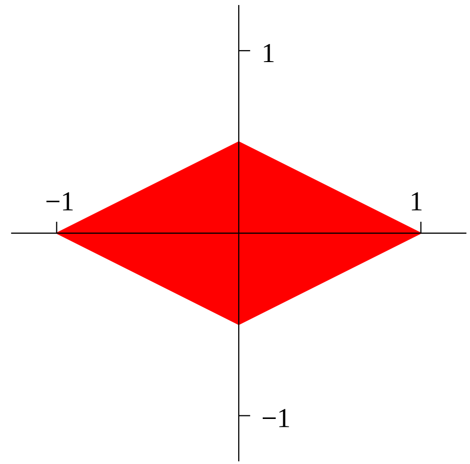


Figure 6.11: $|\Re(z)| + 2|\Im(z)| \leq 1$

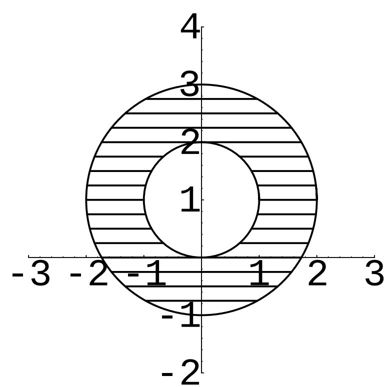


Figure 6.12: $1 < |z - \imath| < 2$

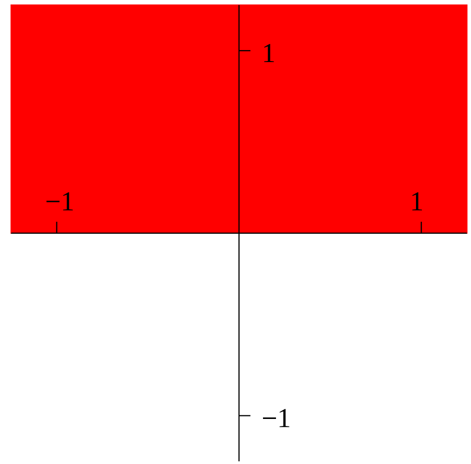


Figure 6.13: The upper half plane.

3.

$$\arg\left(z^2\right) = \arg(z) + \arg(z) \neq 2\arg(z)$$
$$\arg\left(r^2\,\mathrm{e}^{\imath 2\theta}\right) = \{\theta + 2\pi k\} + \{\theta + 2\pi m\} \neq 2\{\theta + 2\pi n\}$$
$$\{2\theta + 2\pi k\} = \{2\theta + 2\pi m\} \neq \{2\theta + 4\pi n\}$$

Solution 6.11
Consider a triangle in the complex plane with vertices at 0, z and $z + \zeta$. (See Figure 6.14.)

The lengths of the sides of the triangle are $|z|$, $|\zeta|$ and $|z + \zeta|$ The second inequality shows that one side of the triangle must be less than or equal to the sum of the other two sides.

$$|z + \zeta| \leq |z| + |\zeta|$$

The first inequality shows that the length of one side of the triangle must be greater than or equal to the difference in

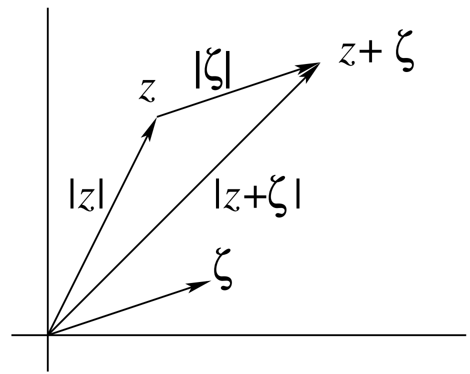


Figure 6.14: Triangle inequality.

the length of the other two sides.

$$|z+\zeta| \geq ||z|-|\zeta||$$

Now we prove the inequalities algebraically. We will reduce the inequality to an identity. Let $z = r\,\mathrm{e}^{\imath\theta}$, $\zeta = \rho\,\mathrm{e}^{\imath\phi}$.

$$\begin{gathered}
||z|-|\zeta|| \leq |z+\zeta| \leq |z|+|\zeta| \\
|r-\rho| \leq |r\,\mathrm{e}^{\imath\theta} + \rho\,\mathrm{e}^{\imath\phi}| \leq r+\rho \\
(r-\rho)^2 \leq \left(r\,\mathrm{e}^{\imath\theta} + \rho\,\mathrm{e}^{\imath\phi}\right)\left(r\,\mathrm{e}^{-\imath\theta} + \rho\,\mathrm{e}^{-\imath\phi}\right) \leq (r+\rho)^2 \\
r^2+\rho^2-2r\rho \leq r^2+\rho^2+r\rho\,\mathrm{e}^{\imath(\theta-\phi)} + r\rho\,\mathrm{e}^{\imath(-\theta+\phi)} \leq r^2+\rho^2+2r\rho \\
-2r\rho \leq 2r\rho\cos(\theta-\phi) \leq 2r\rho \\
-1 \leq \cos(\theta-\phi) \leq 1
\end{gathered}$$

Solution 6.12

1.

$$
\begin{aligned}
(-1)^{-3/4} &= \left((-1)^{-3}\right)^{1/4} \\
&= (-1)^{1/4} \\
&= (\mathrm{e}^{\imath\pi})^{1/4} \\
&= \mathrm{e}^{\imath\pi/4}\, 1^{1/4} \\
&= \mathrm{e}^{\imath\pi/4}\, \mathrm{e}^{\imath k\pi/2}, \quad k = 0, 1, 2, 3 \\
&= \left\{\mathrm{e}^{\imath\pi/4}, \mathrm{e}^{\imath 3\pi/4}, \mathrm{e}^{\imath 5\pi/4}, \mathrm{e}^{\imath 7\pi/4}\right\} \\
&= \left\{\frac{1+\imath}{\sqrt{2}}, \frac{-1+\imath}{\sqrt{2}}, \frac{-1-\imath}{\sqrt{2}}, \frac{1-\imath}{\sqrt{2}}\right\}
\end{aligned}
$$

See Figure 6.15.

2.

$$
\begin{aligned}
8^{1/6} &= \sqrt[6]{8} 1^{1/6} \\
&= \sqrt{2}\, \mathrm{e}^{\imath k\pi/3}, \quad k = 0, 1, 2, 3, 4, 5 \\
&= \left\{\sqrt{2}, \sqrt{2}\, \mathrm{e}^{\imath\pi/3}, \sqrt{2}\, \mathrm{e}^{\imath 2\pi/3}, \sqrt{2}\, \mathrm{e}^{\imath\pi}, \sqrt{2}\, \mathrm{e}^{\imath 4\pi/3}, \sqrt{2}\, \mathrm{e}^{\imath 5\pi/3}\right\} \\
&= \left\{\sqrt{2}, \frac{1+\imath\sqrt{3}}{\sqrt{2}}, \frac{-1+\imath\sqrt{3}}{\sqrt{2}}, -\sqrt{2}, \frac{-1-\imath\sqrt{3}}{\sqrt{2}}, \frac{1-\imath\sqrt{3}}{\sqrt{2}}\right\}
\end{aligned}
$$

See Figure 6.16.

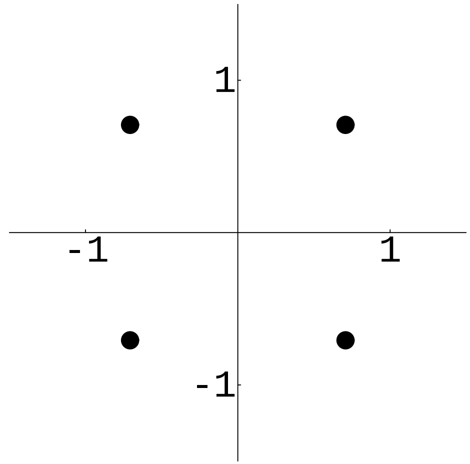


Figure 6.15: $(-1)^{-3/4}$

Solution 6.13

1.

$$
\begin{aligned}
(-1)^{-1/4} &= ((-1)^{-1})^{1/4} \\
&= (-1)^{1/4} \\
&= (\mathrm{e}^{\imath\pi})^{1/4} \\
&= \mathrm{e}^{\imath\pi/4}\, 1^{1/4} \\
&= \mathrm{e}^{\imath\pi/4}\, \mathrm{e}^{\imath k\pi/2}, \quad k = 0, 1, 2, 3 \\
&= \left\{\mathrm{e}^{\imath\pi/4}, \mathrm{e}^{\imath 3\pi/4}, \mathrm{e}^{\imath 5\pi/4}, \mathrm{e}^{\imath 7\pi/4}\right\} \\
&= \left\{\frac{1+\imath}{\sqrt{2}}, \frac{-1+\imath}{\sqrt{2}}, \frac{-1-\imath}{\sqrt{2}}, \frac{1-\imath}{\sqrt{2}}\right\}
\end{aligned}
$$

See Figure 6.17.

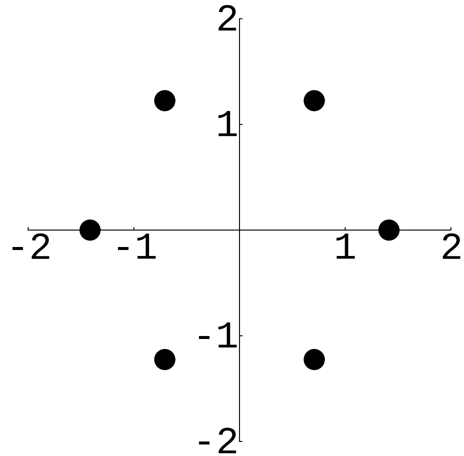


Figure 6.16: $8^{1/6}$

2.

$$
\begin{aligned}
16^{1/8} &= \sqrt[8]{16} 1^{1/8} \\
&= \sqrt{2}\, \mathrm{e}^{\imath k\pi/4}, \quad k = 0,1,2,3,4,5,6,7 \\
&= \left\{ \sqrt{2}, \sqrt{2}\, \mathrm{e}^{\imath\pi/4}, \sqrt{2}\, \mathrm{e}^{\imath\pi/2}, \sqrt{2}\, \mathrm{e}^{\imath 3\pi/4}, \sqrt{2}\, \mathrm{e}^{\imath\pi}, \sqrt{2}\, \mathrm{e}^{\imath 5\pi/4}, \sqrt{2}\, \mathrm{e}^{\imath 3\pi/2}, \sqrt{2}\, \mathrm{e}^{\imath 7\pi/4} \right\} \\
&= \left\{ \sqrt{2}, 1+\imath, \imath\sqrt{2}, -1+\imath, -\sqrt{2}, -1-\imath, -\imath\sqrt{2}, 1-\imath \right\}
\end{aligned}
$$

See Figure 6.18.

Solution 6.14

1. $|z - \imath 2|$ is the distance from the point $\imath 2$ in the complex plane. Thus $1 < |z - \imath 2| < 2$ is an annulus. See Figure 6.19.

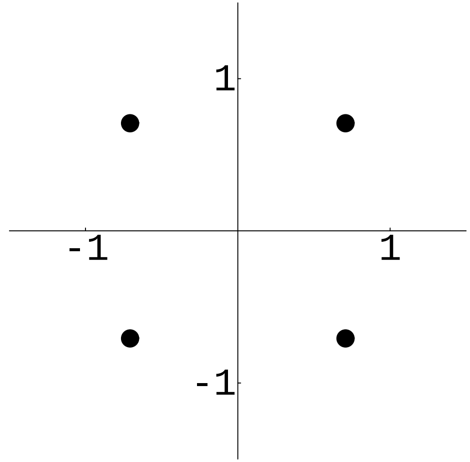


Figure 6.17: $(-1)^{-1/4}$

2.

$$|\Re(z)| + 5|\Im(z)| = 1$$
$$|x| + 5|y| = 1$$

In the first quadrant this is the line $y = (1-x)/5$. We reflect this line segment across the coordinate axes to obtain line segments in the other quadrants. Explicitly, we have the set of points: $\{z = x + \imath y \mid -1 < x < 1 \wedge y = \pm(1-|x|)/5\}$. See Figure 6.20.

3. The set of points equidistant from $\imath$ and $-\imath$ is the real axis. See Figure 6.21.

Solution 6.15

1. $|z - 1 + \imath|$ is the distance from the point $(1-\imath)$. Thus $|z - 1 + \imath| \leq 1$ is the disk of unit radius centered at $(1-\imath)$. See Figure 6.22.

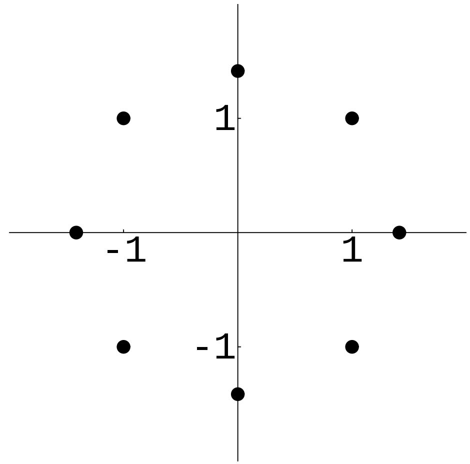


Figure 6.18: $16^{-1/8}$

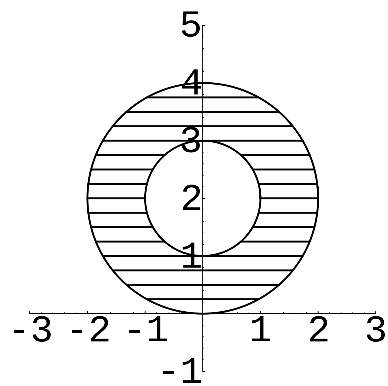


Figure 6.19: $1 < |z - \imath 2| < 2$

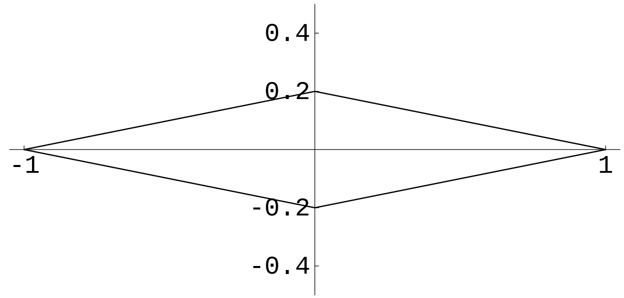


Figure 6.20: $|\Re(z)| + 5|\Im(z)| = 1$

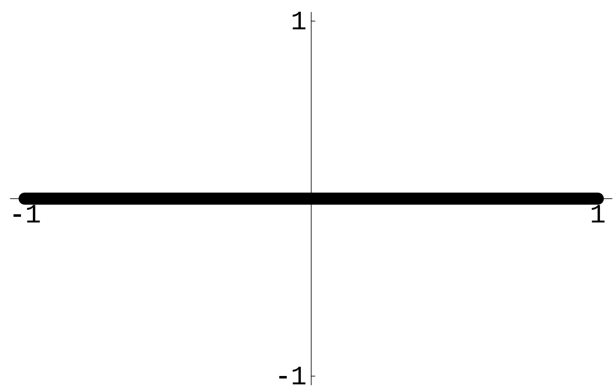


Figure 6.21: $|z - \imath| = |z + \imath|$

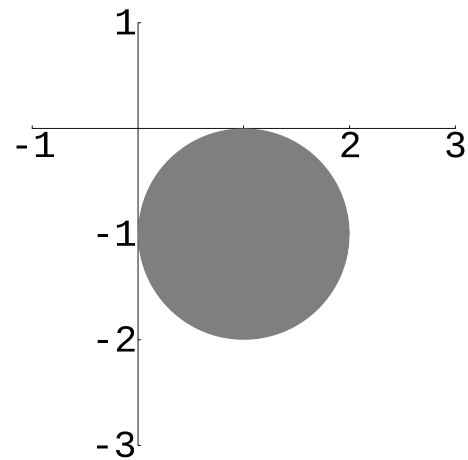


Figure 6.22: $|z - 1 + \imath| < 1$

2.

$$\begin{aligned} \Re(z) - \Im(z) &= 5 \\ x - y &= 5 \\ y &= x - 5 \end{aligned}$$

See Figure 6.23.

3. Since $|z - \imath| + |z + \imath| \geq 2$, there are no solutions of $|z - \imath| + |z + \imath| = 1$.

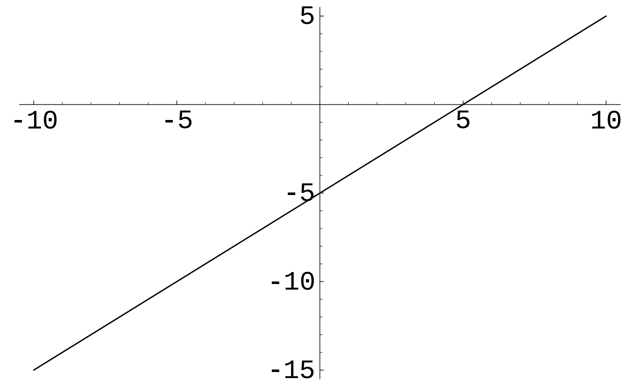


Figure 6.23: $\Re(z) - \Im(z) = 5$

Solution 6.16

$$
\begin{gathered}
|\,\mathrm{e}^{\imath\theta} - 1| = 2 \\
\left(\mathrm{e}^{\imath\theta} - 1\right)\left(\mathrm{e}^{-\imath\theta} - 1\right) = 4 \\
1 - \mathrm{e}^{\imath\theta} - \mathrm{e}^{-\imath\theta} + 1 = 4 \\
-2\cos(\theta) = 2 \\
\theta = \pi
\end{gathered}
$$

$\left\{\mathrm{e}^{\imath\theta} \mid 0 \leq \theta \leq \pi\right\}$ is a unit semi-circle in the upper half of the complex plane from 1 to -1. The only point on this semi-circle that is a distance 2 from the point 1 is the point -1, which corresponds to $\theta = \pi$.

Polar Form

Solution 6.17

We recall the Taylor series expansion of e^x about $x = 0$.

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We take this as the definition of the exponential function for complex argument.

$$\begin{aligned}
\mathrm{e}^{\imath\theta} &= \sum_{n=0}^{\infty} \frac{(\imath\theta)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\imath^n}{n!}\theta^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n} + \imath \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1}
\end{aligned}$$

We compare this expression to the Taylor series for the sine and cosine.

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n}, \qquad \sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1},$$

Thus $\mathrm{e}^{\imath\theta}$ and $\cos\theta + \imath\sin\theta$ have the same Taylor series expansions about $\theta = 0$.

$$\boxed{\mathrm{e}^{\imath\theta} = \cos\theta + \imath\sin\theta}$$

Solution 6.18

$$\begin{gathered}
\cos(3\theta) + \imath\sin(3\theta) = (\cos\theta + \imath\sin\theta)^3 \\
\cos(3\theta) + \imath\sin(3\theta) = \cos^3\theta + \imath 3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \imath\sin^3\theta
\end{gathered}$$

We equate the real parts of the equation.

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$$

Solution 6.19
Define the partial sum,

$$S_n(z) = \sum_{k=0}^{n} z^k.$$

Now consider $(1-z)S_n(z)$.

$$\begin{aligned}
(1-z)S_n(z) &= (1-z)\sum_{k=0}^{n} z^k \\
(1-z)S_n(z) &= \sum_{k=0}^{n} z^k - \sum_{k=1}^{n+1} z^k \\
(1-z)S_n(z) &= 1 - z^{n+1}
\end{aligned}$$

We divide by $1-z$. Note that $1-z$ is nonzero.

$$\begin{aligned}
S_n(z) &= \frac{1-z^{n+1}}{1-z} \\
1 + z + z^2 + \cdots + z^n &= \frac{1-z^{n+1}}{1-z}, \qquad (z \neq 1)
\end{aligned}$$

Now consider $z = \mathrm{e}^{\imath\theta}$ where $0 < \theta < 2\pi$ so that z is not unity.

$$\begin{aligned}
\sum_{k=0}^{n} \left(\mathrm{e}^{\imath\theta}\right)^k &= \frac{1-\left(\mathrm{e}^{\imath\theta}\right)^{n+1}}{1-\mathrm{e}^{\imath\theta}} \\
\sum_{k=0}^{n} \mathrm{e}^{\imath k\theta} &= \frac{1-\mathrm{e}^{\imath(n+1)\theta}}{1-\mathrm{e}^{\imath\theta}}
\end{aligned}$$

In order to get $\sin(\theta/2)$ in the denominator, we multiply top and bottom by $\mathrm{e}^{-\imath\theta/2}$.

$$\sum_{k=0}^{n}(\cos(k\theta)+\imath\sin(k\theta)) = \frac{\mathrm{e}^{-\imath\theta/2}-\mathrm{e}^{\imath(n+1/2)\theta}}{\mathrm{e}^{-\imath\theta/2}-\mathrm{e}^{\imath\theta/2}}$$

$$\sum_{k=0}^{n}\cos(k\theta)+\imath\sum_{k=0}^{n}\sin(k\theta) = \frac{\cos(\theta/2)-\imath\sin(\theta/2)-\cos((n+1/2)\theta)-\imath\sin((n+1/2)\theta)}{-2\imath\sin(\theta/2)}$$

$$\sum_{k=0}^{n}\cos(k\theta)+\imath\sum_{k=1}^{n}\sin(k\theta) = \frac{1}{2}+\frac{\sin((n+1/2)\theta)}{\sin(\theta/2)}+\imath\left(\frac{1}{2}\cot(\theta/2)-\frac{\cos((n+1/2)\theta)}{\sin(\theta/2)}\right)$$

1. We take the real and imaginary part of this to obtain the identities.

$$\boxed{\sum_{k=0}^{n}\cos(k\theta) = \frac{1}{2}+\frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}}$$

2. $$\boxed{\sum_{k=1}^{n}\sin(k\theta) = \frac{1}{2}\cot(\theta/2)-\frac{\cos((n+1/2)\theta)}{2\sin(\theta/2)}}$$

Arithmetic and Vectors

Solution 6.20

$$\begin{aligned}
|z\zeta| &= |r\,\mathrm{e}^{\imath\theta}\,\rho\,\mathrm{e}^{\imath\phi}| \\
&= |r\rho\,\mathrm{e}^{\imath(\theta+\phi)}| \\
&= |r\rho| \\
&= |r||\rho| \\
&= |z||\zeta|
\end{aligned}$$

$$
\begin{aligned}
\left|\frac{z}{\zeta}\right| &= \left|\frac{r\,\mathrm{e}^{\imath\theta}}{\rho\,\mathrm{e}^{\imath\phi}}\right| \\
&= \left|\frac{r}{\rho}\,\mathrm{e}^{\imath(\theta-\phi)}\right| \\
&= \left|\frac{r}{\rho}\right| \\
&= \frac{|r|}{|\rho|} \\
&= \frac{|z|}{|\zeta|}
\end{aligned}
$$

Solution 6.21

$$
\begin{aligned}
|z+\zeta|^2 + |z-\zeta|^2 &= (z+\zeta)\left(\overline{z}+\overline{\zeta}\right) + (z-\zeta)\left(\overline{z}-\overline{\zeta}\right) \\
&= z\overline{z} + z\overline{\zeta} + \zeta\overline{z} + \zeta\overline{\zeta} + z\overline{z} - z\overline{\zeta} - \zeta\overline{z} + \zeta\overline{\zeta} \\
&= 2\left(|z|^2 + |\zeta|^2\right)
\end{aligned}
$$

Consider the parallelogram defined by the vectors z and ζ. The lengths of the sides are z and ζ and the lengths of the diagonals are $z+\zeta$ and $z-\zeta$. We know from geometry that the sum of the squared lengths of the diagonals of a parallelogram is equal to the sum of the squared lengths of the four sides. (See Figure 6.24.)

Integer Exponents

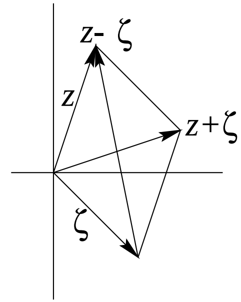


Figure 6.24: The parallelogram defined by z and ζ.

Solution 6.22

1.

$$
\begin{aligned}
(1+\imath)^{10} &= \left(\left((1+\imath)^2\right)^2\right)^2 (1+\imath)^2 \\
&= \left((\imath 2)^2\right)^2 (\imath 2) \\
&= (-4)^2 (\imath 2) \\
&= 16(\imath 2) \\
&= \imath 32
\end{aligned}
$$

2.

$$
\begin{aligned}
(1+\imath)^{10} &= \left(\sqrt{2}\,\mathrm{e}^{\imath\pi/4}\right)^{10} \\
&= \left(\sqrt{2}\right)^{10} \mathrm{e}^{\imath 10\pi/4} \\
&= 32\,\mathrm{e}^{\imath\pi/2} \\
&= \imath 32
\end{aligned}
$$

Solution 6.23

We substitite the numbers into the equation to obtain an identity.

$$z^2 + 2az + b = 0$$
$$\left(-a + \left(a^2 - b\right)^{1/2}\right)^2 + 2a\left(-a + \left(a^2 - b\right)^{1/2}\right) + b = 0$$
$$a^2 - 2a\left(a^2 - b\right)^{1/2} + a^2 - b - 2a^2 + 2a\left(a^2 - b\right)^{1/2} + b = 0$$
$$0 = 0$$

Chapter 7

Functions of a Complex Variable

If brute force isn't working, you're not using enough of it.

-Tim Mauch

In this chapter we introduce the algebra of functions of a complex variable. We will cover the trigonometric and inverse trigonometric functions. The properties of trigonometric functions carry over directly from real-variable theory. However, because of multi-valuedness, the inverse trigonometric functions are significantly trickier than their real-variable counterparts.

7.1 Curves and Regions

In this section we introduce curves and regions in the complex plane. This material is necessary for the study of branch points in this chapter and later for contour integration.

Curves. Consider two continuous functions $x(t)$ and $y(t)$ defined on the interval $t \in [t_0..t_1]$. The set of points in the complex plane,

$$\{z(t) = x(t) + \imath y(t) \mid t \in [t_0 \dots t_1]\},$$

defines a *continuous curve* or simply a *curve*. If the endpoints coincide ($z(t_0) = z(t_1)$) it is a *closed curve*. (We assume that $t_0 \neq t_1$.) If the curve does not intersect itself, then it is said to be a *simple curve*.

If $x(t)$ and $y(t)$ have continuous derivatives and the derivatives do not both vanish at any point, then it is a *smooth curve*.[1] This essentially means that the curve does not have any corners or other nastiness.

A continuous curve which is composed of a finite number of smooth curves is called a *piecewise smooth curve*. We will use the word *contour* as a synonym for a piecewise smooth curve.

See Figure 7.1 for a smooth curve, a piecewise smooth curve, a simple closed curve and a non-simple closed curve.

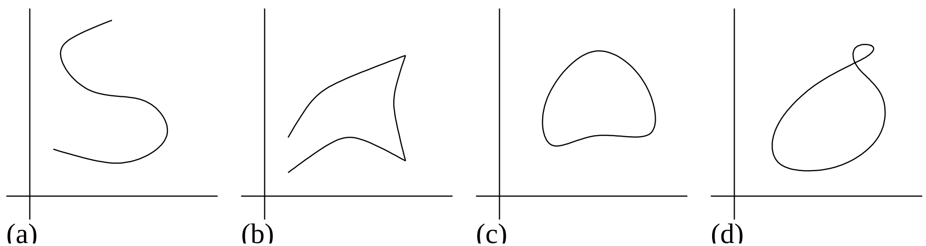


Figure 7.1: (a) Smooth curve. (b) Piecewise smooth curve. (c) Simple closed curve. (d) Non-simple closed curve.

Regions. A region R is *connected* if any two points in R can be connected by a curve which lies entirely in R. A region is *simply-connected* if every closed curve in R can be continuously shrunk to a point without leaving R. A region which is not simply-connected is said to be *multiply-connected region*. Another way of defining simply-connected is that a path connecting two points in R can be continuously deformed into any other path that connects those points. Figure 7.2 shows a simply-connected region with two paths which can be continuously deformed into one another and two multiply-connected regions with paths which cannot be deformed into one another.

Jordan curve theorem. A continuous, simple, closed curve is known as a *Jordan curve*. The Jordan Curve Theorem, which seems intuitively obvious but is difficult to prove, states that a Jordan curve divides the plane into

[1] Why is it necessary that the derivatives do not both vanish?

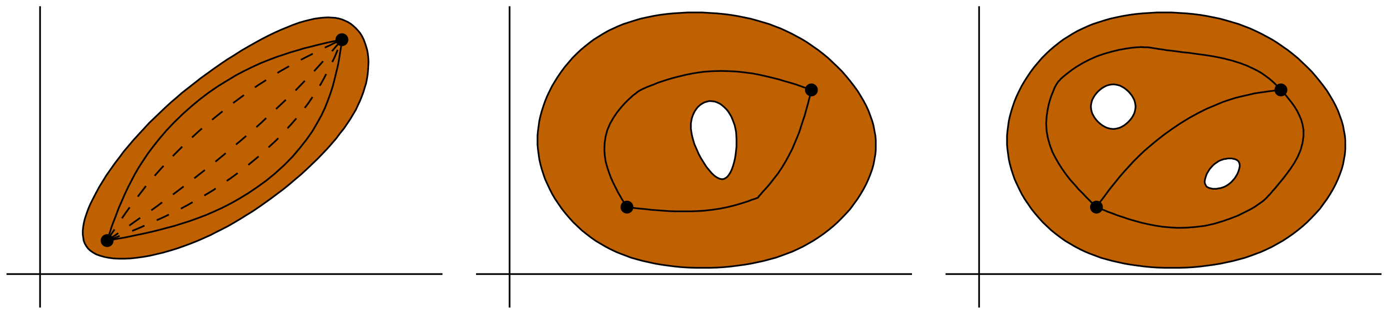


Figure 7.2: A simply-connected and two multiply-connected regions.

a simply-connected, bounded region and an unbounded region. These two regions are called the interior and exterior regions, respectively. The two regions share the curve as a boundary. Points in the interior are said to be inside the curve; points in the exterior are said to be outside the curve.

Traversal of a contour. Consider a Jordan curve. If you traverse the curve in the *positive* direction, then the inside is to your left. If you traverse the curve in the opposite direction, then the outside will be to your left and you will go around the curve in the negative direction. For circles, the positive direction is the *counter-clockwise* direction. The positive direction is consistent with the way angles are measured in a right-handed coordinate system, i.e. for a circle centered on the origin, the positive direction is the direction of increasing angle. For an oriented contour C, we denote the contour with opposite orientation as $-C$.

Boundary of a region. Consider a simply-connected region. The boundary of the region is traversed in the positive direction if the region is to the left as you walk along the contour. For multiply-connected regions, the boundary may be a set of contours. In this case the boundary is traversed in the positive direction if each of the contours is traversed in the positive direction. When we refer to the boundary of a region we will assume it is given the positive orientation. In Figure 7.3 the boundaries of three regions are traversed in the positive direction.

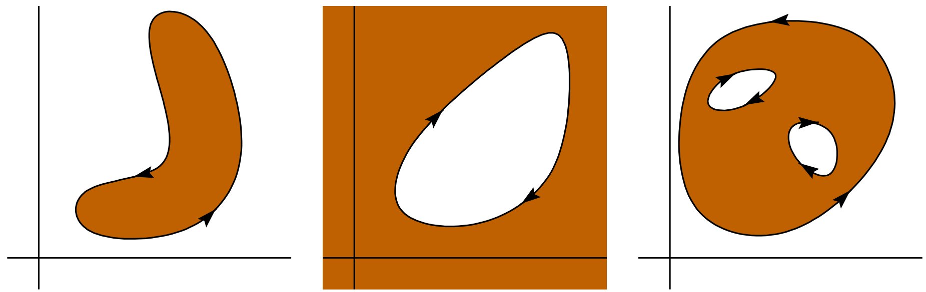


Figure 7.3: Traversing the boundary in the positive direction.

Two interpretations of a curve. Consider a simple closed curve as depicted in Figure 7.4a. By giving it an orientation, we can make a contour that either encloses the bounded domain Figure 7.4b or the unbounded domain Figure 7.4c. Thus a curve has two interpretations. It can be thought of as enclosing either the points which are "inside" or the points which are "outside".[2]

7.2 The Point at Infinity and the Stereographic Projection

Complex infinity. In real variables, there are only two ways to get to infinity. We can either go up or down the number line. Thus signed infinity makes sense. By going up or down we respectively approach $+\infty$ and $-\infty$. In the complex plane there are an infinite number of ways to approach infinity. We stand at the origin, point ourselves in any direction and go straight. We could walk along the positive real axis and approach infinity via positive real numbers. We could walk along the positive imaginary axis and approach infinity via pure imaginary numbers. We could generalize the real variable notion of signed infinity to a complex variable notion of directional infinity, but this will not be useful

[2] A farmer wanted to know the most efficient way to build a pen to enclose his sheep, so he consulted an engineer, a physicist and a mathematician. The engineer suggested that he build a circular pen to get the maximum area for any given perimeter. The physicist suggested that he build a fence at infinity and then shrink it to fit the sheep. The mathematician constructed a little fence around himself and then defined himself to be outside.

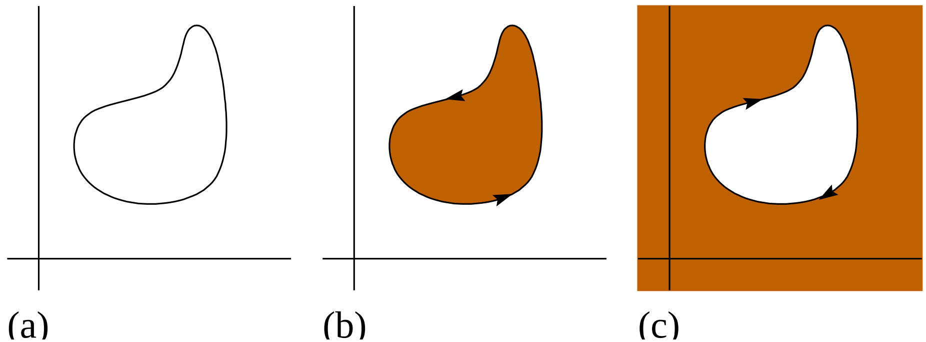


Figure 7.4: Two interpretations of a curve.

for our purposes. Instead, we introduce *complex infinity* or the *point at infinity* as the limit of going infinitely far along any direction in the complex plane. The complex plane together with the point at infinity form the *extended complex plane*.

Stereographic projection. We can visualize the point at infinity with the *stereographic projection*. We place a unit sphere on top of the complex plane so that the south pole of the sphere is at the origin. Consider a line passing through the north pole and a point $z = x + \imath y$ in the complex plane. In the stereographic projection, the point point z is mapped to the point where the line intersects the sphere. (See Figure 7.5.) Each point $z = x + \imath y$ in the complex plane is mapped to a unique point (X, Y, Z) on the sphere.

$$X = \frac{4x}{|z|^2 + 4}, \quad Y = \frac{4y}{|z|^2 + 4}, \quad Z = \frac{2|z|^2}{|z|^2 + 4}$$

The origin is mapped to the south pole. The point at infinity, $|z| = \infty$, is mapped to the north pole.

In the stereographic projection, circles in the complex plane are mapped to circles on the unit sphere. Figure 7.6 shows circles along the real and imaginary axes under the mapping. Lines in the complex plane are also mapped to circles on the unit sphere. The right diagram in Figure 7.6 shows lines emanating from the origin under the mapping.

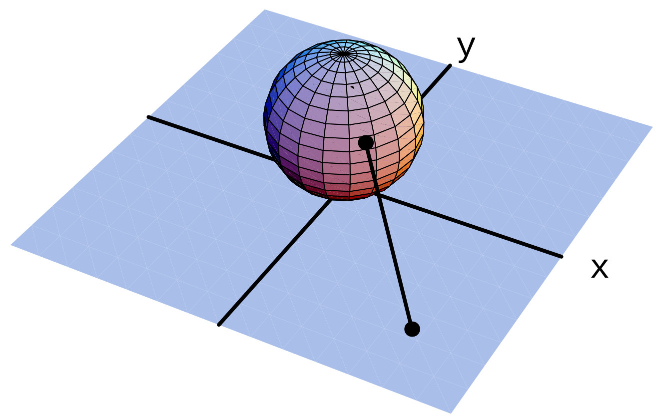


Figure 7.5: The stereographic projection.

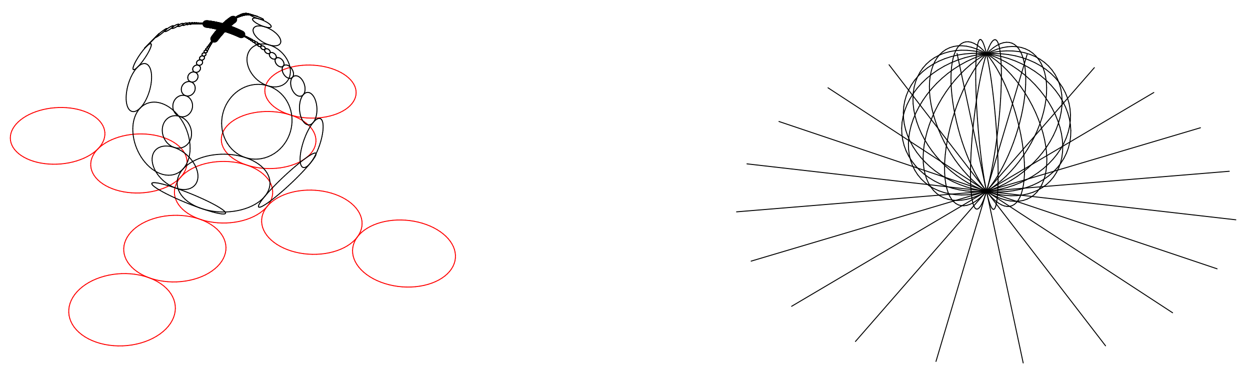


Figure 7.6: The stereographic projection of circles and lines.

The stereographic projection helps us reason about the point at infinity. When we consider the complex plane by itself, the point at infinity is an abstract notion. We can't draw a picture of the point at infinity. It may be hard to accept the notion of a jordan curve enclosing the point at infinity. However, in the stereographic projection, the point at infinity is just an ordinary point (namely the north pole of the sphere).

7.3 A Gentle Introduction to Branch Points

In this section we will introduce the concepts of *branches*, *branch points* and *branch cuts*. These concepts (which are notoriously difficult to understand for beginners) are typically defined in terms functions of a complex variable. Here we will develop these ideas as they relate to the arctangent function $\arctan(x, y)$. Hopefully this simple example will make the treatment in Section 7.9 more palateable.

First we review some properties of the arctangent. It is a mapping from $\mathbb{R}^2$ to $\mathbb{R}$. It measures the angle around the origin from the positive x axis. Thus it is a multi-valued function. For a fixed point in the domain, the function values differ by integer multiples of 2π. The arctangent is not defined at the origin nor at the point at infinity; it is singular at these two points. If we plot some of the values of the arctangent, it looks like a corkscrew with axis through the origin. A portion of this function is plotted in Figure 7.7.

Most of the tools we have for analyzing functions (continuity, differentiability, etc.) depend on the fact that the function is single-valued. In order to work with the arctangent we need to select a portion to obtain a single-valued function. Consider the domain $(-1..2) \times (1..4)$. On this domain we select the value of the arctangent that is between 0 and π. The domain and a plot of the selected values of the arctangent are shown in Figure 7.8.

CONTINUE.

7.4 Cartesian and Modulus-Argument Form

We can write a function of a complex variable z as a function of x and y or as a function of r and θ with the substitutions $z = x + \imath y$ and $z = r\,\mathrm{e}^{\imath\theta}$, respectively. Then we can separate the real and imaginary components or write the function in modulus-argument form,

$$f(z) = u(x, y) + \imath v(x, y), \quad \text{or} \quad f(z) = u(r, \theta) + \imath v(r, \theta),$$

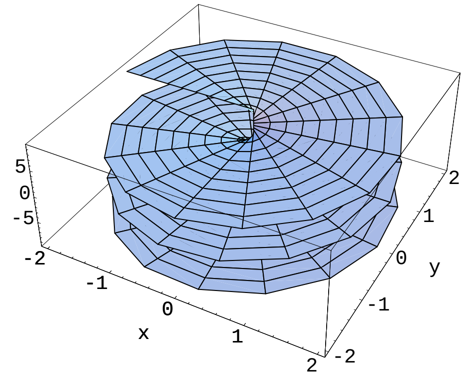


Figure 7.7: Plots of $\Re(\log z)$ and a portion of $\Im(\log z)$.

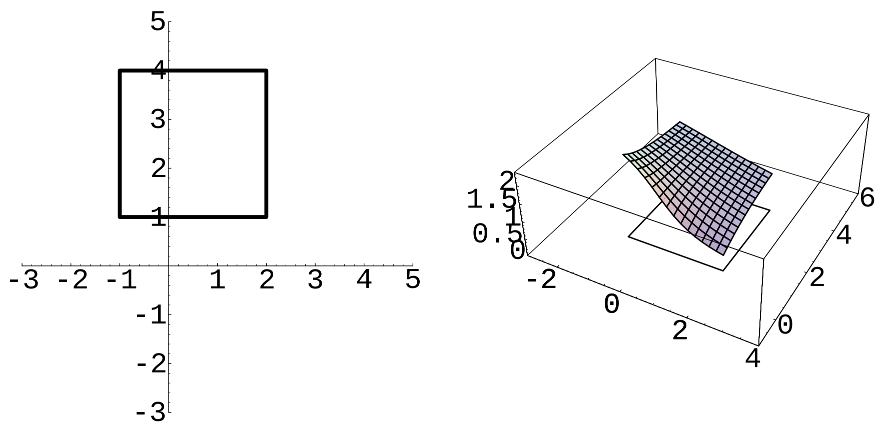


Figure 7.8: A domain and a selected value of the arctangent for the points in the domain.

$$f(z) = \rho(x, y)\,\mathrm{e}^{\imath\phi(x,y)}, \quad \text{or} \quad f(z) = \rho(r, \theta)\,\mathrm{e}^{\imath\phi(r,\theta)}.$$

Example 7.4.1 *Consider the functions $f(z) = z$, $f(z) = z^3$ and $f(z) = \frac{1}{1-z}$. We write the functions in terms of x and y and separate them into their real and imaginary components.*

$$\begin{aligned} f(z) &= z \\ &= x + \imath y \end{aligned}$$

$$\begin{aligned} f(z) &= z^3 \\ &= (x + \imath y)^3 \\ &= x^3 + \imath x^2 y - x y^2 - \imath y^3 \\ &= \left(x^3 - x y^2\right) + \imath \left(x^2 y - y^3\right) \end{aligned}$$

$$\begin{aligned} f(z) &= \frac{1}{1-z} \\ &= \frac{1}{1 - x - \imath y} \\ &= \frac{1}{1 - x - \imath y} \frac{1 - x + \imath y}{1 - x + \imath y} \\ &= \frac{1 - x}{(1-x)^2 + y^2} + \imath \frac{y}{(1-x)^2 + y^2} \end{aligned}$$

Example 7.4.2 *Consider the functions $f(z) = z$, $f(z) = z^3$ and $f(z) = \frac{1}{1-z}$. We write the functions in terms of r and θ and write them in modulus-argument form.*

$$\begin{aligned} f(z) &= z \\ &= r\,\mathrm{e}^{\imath\theta} \end{aligned}$$

$$\begin{aligned} f(z) &= z^3 \\ &= \left(r\,\mathrm{e}^{\imath\theta}\right)^3 \\ &= r^3\,\mathrm{e}^{\imath 3\theta} \end{aligned}$$

$$\begin{aligned} f(z) &= \frac{1}{1-z} \\ &= \frac{1}{1-r\,\mathrm{e}^{\imath\theta}} \\ &= \frac{1}{1-r\,\mathrm{e}^{\imath\theta}}\frac{1}{1-r\,\mathrm{e}^{-\imath\theta}} \\ &= \frac{1-r\,\mathrm{e}^{-\imath\theta}}{1-r\,\mathrm{e}^{\imath\theta}-r\,\mathrm{e}^{-\imath\theta}+r^2} \\ &= \frac{1-r\cos\theta+\imath r\sin\theta}{1-2r\cos\theta+r^2} \end{aligned}$$

Note that the denominator is real and non-negative.

$$\begin{aligned} &= \frac{1}{1-2r\cos\theta+r^2}\left|1-r\cos\theta+\imath r\sin\theta\right|\mathrm{e}^{\imath\arctan(1-r\cos\theta,r\sin\theta)} \\ &= \frac{1}{1-2r\cos\theta+r^2}\sqrt{(1-r\cos\theta)^2+r^2\sin^2\theta}\,\mathrm{e}^{\imath\arctan(1-r\cos\theta,r\sin\theta)} \\ &= \frac{1}{1-2r\cos\theta+r^2}\sqrt{1-2r\cos\theta+r^2\cos^2\theta+r^2\sin^2\theta}\,\mathrm{e}^{\imath\arctan(1-r\cos\theta,r\sin\theta)} \\ &= \frac{1}{\sqrt{1-2r\cos\theta+r^2}}\,\mathrm{e}^{\imath\arctan(1-r\cos\theta,r\sin\theta)} \end{aligned}$$

7.5 Graphing Functions of a Complex Variable

We cannot directly graph functions of a complex variable as they are mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. To do so would require four dimensions. However, we can can use a surface plot to graph the real part, the imaginary part, the modulus or the argument of a function of a complex variable. Each of these are scalar fields, mappings from $\mathbb{R}^2$ to $\mathbb{R}$.

Example 7.5.1 *Consider the identity function, $f(z) = z$. In Cartesian coordinates and Cartesian form, the function is $f(z) = x + \imath y$. The real and imaginary components are $u(x, y) = x$ and $v(x, y) = y$. (See Figure 7.9.) In modulus*

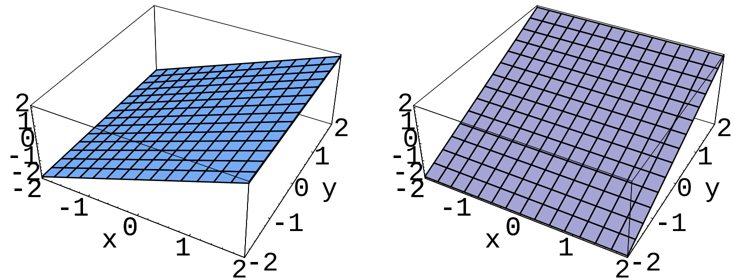


Figure 7.9: The real and imaginary parts of $f(z) = z = x + \imath y$.

argument form the function is

$$f(z) = z = r\,\mathrm{e}^{\imath\theta} = \sqrt{x^2 + y^2}\,\mathrm{e}^{\imath \arctan(x,y)}.$$

The modulus of $f(z)$ is a single-valued function which is the distance from the origin. The argument of $f(z)$ is a multi-valued function. Recall that $\arctan(x, y)$ has an infinite number of values each of which differ by an integer multiple of 2π. A few branches of $\arg(f(z))$ are plotted in Figure 7.10. The modulus and principal argument of $f(z) = z$ are plotted in Figure 7.11.

Example 7.5.2 *Consider the function $f(z) = z^2$. In Cartesian coordinates and separated into its real and imaginary components the function is*

$$f(z) = z^2 = (x + \imath y)^2 = \left(x^2 - y^2\right) + \imath 2xy.$$

Figure 7.12 shows surface plots of the real and imaginary parts of z^2. The magnitude of z^2 is

$$|z^2| = \sqrt{z^2\overline{z^2}} = z\overline{z} = (x + \imath y)(x - \imath y) = x^2 + y^2.$$

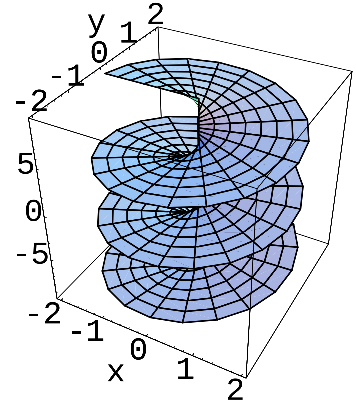


Figure 7.10: A few branches of $\arg(z)$.

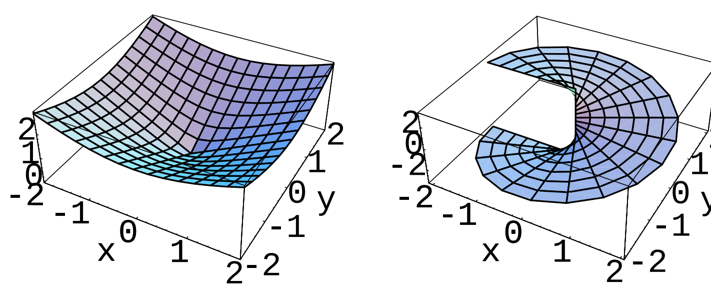
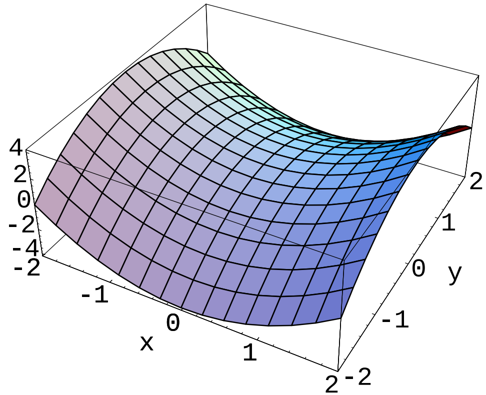


Figure 7.11: Plots of $|z|$ and $\operatorname{Arg}(z)$.



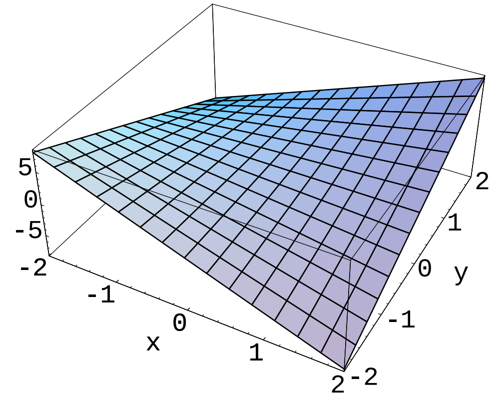


Figure 7.12: Plots of $\Re\left(z^2\right)$ and $\Im\left(z^2\right)$.

Note that

$$z^2 = \left(r\,\mathrm{e}^{\imath\theta}\right)^2 = r^2\,\mathrm{e}^{\imath 2\theta}.$$

In Figure 7.13 are plots of $\left|z^2\right|$ *and a branch of* $\arg\left(z^2\right)$.

7.6 Trigonometric Functions

The exponential function. Consider the exponential function e^z. We can use Euler's formula to write $\mathrm{e}^z = \mathrm{e}^{x+\imath y}$ in terms of its real and imaginary parts.

$$\mathrm{e}^z = \mathrm{e}^{x+\imath y} = \mathrm{e}^x\,\mathrm{e}^{\imath y} = \mathrm{e}^x \cos y + \imath\,\mathrm{e}^x \sin y$$

From this we see that the exponential function is $\imath 2\pi$ periodic: $\mathrm{e}^{z+\imath 2\pi} = \mathrm{e}^z$, and $\imath\pi$ odd periodic: $\mathrm{e}^{z+\imath\pi} = -\,\mathrm{e}^z$. Figure 7.14 has surface plots of the real and imaginary parts of e^z which show this periodicity.

The modulus of e^z is a function of x alone.

$$\left|\mathrm{e}^z\right| = \left|\mathrm{e}^{x+\imath y}\right| = \mathrm{e}^x$$

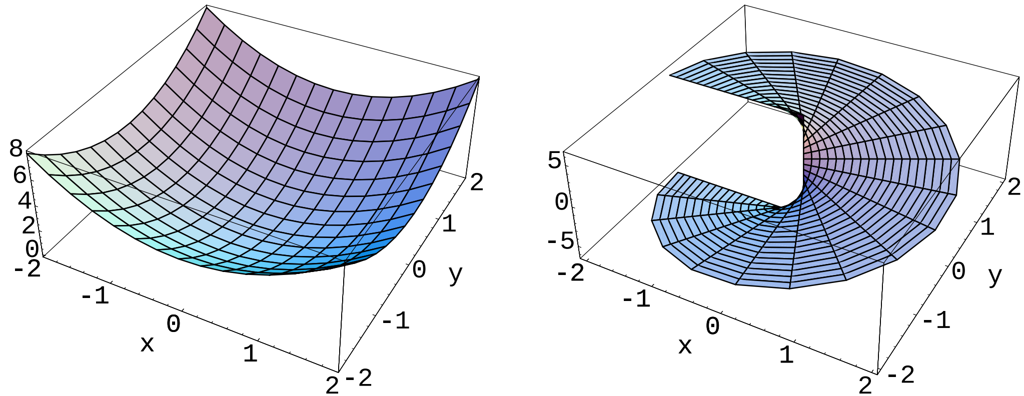


Figure 7.13: Plots of $|z^2|$ and a branch of $\arg(z^2)$.

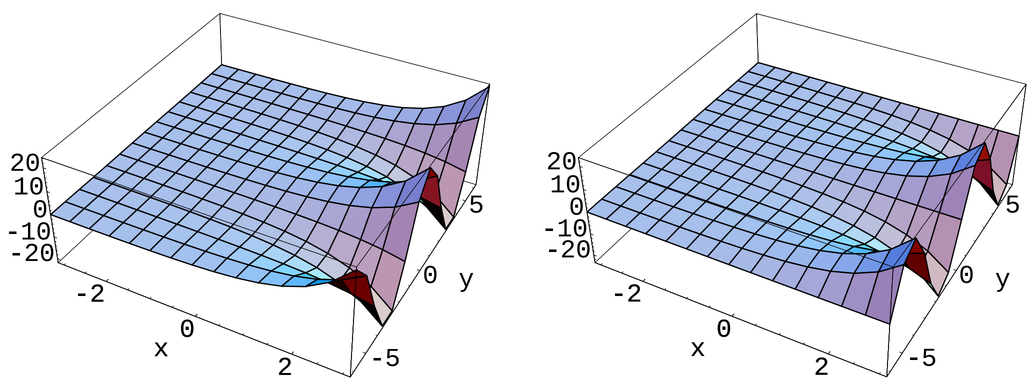


Figure 7.14: Plots of $\Re(\mathrm{e}^z)$ and $\Im(\mathrm{e}^z)$.

The argument of e^z is a function of y alone.

$$\arg\left(\mathrm{e}^z\right) = \arg\left(\mathrm{e}^{x+\imath y}\right) = \{y + 2\pi n \mid n \in \mathbb{Z}\}$$

In Figure 7.15 are plots of $|\,\mathrm{e}^z\,|$ and a branch of $\arg\left(\mathrm{e}^z\right)$.

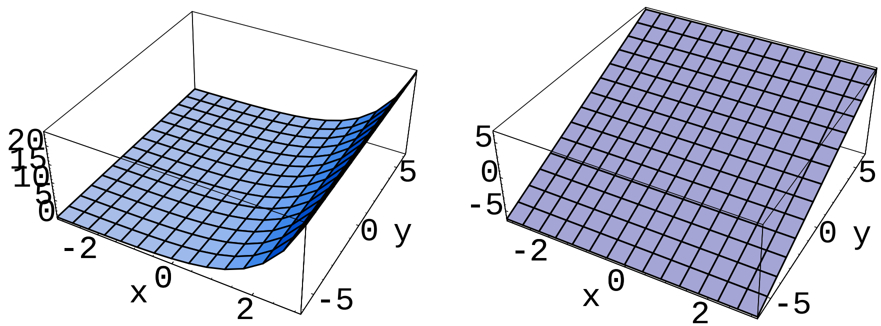


Figure 7.15: Plots of $|\,\mathrm{e}^z\,|$ and a branch of $\arg\left(\mathrm{e}^z\right)$.

Example 7.6.1 *Show that the transformation $w = \mathrm{e}^z$ maps the infinite strip, $-\infty < x < \infty$, $0 < y < \pi$, onto the upper half-plane.*

Method 1. *Consider the line $z = x + \imath c$, $-\infty < x < \infty$. Under the transformation, this is mapped to*

$$w = \mathrm{e}^{x+\imath c} = \mathrm{e}^{\imath c}\,\mathrm{e}^{x}, \quad -\infty < x < \infty.$$

This is a ray from the origin to infinity in the direction of $\mathrm{e}^{\imath c}$. Thus we see that $z = x$ is mapped to the positive, real w axis, $z = x + \imath\pi$ is mapped to the negative, real axis, and $z = x + \imath c$, $0 < c < \pi$ is mapped to a ray with angle c in the upper half-plane. Thus the strip is mapped to the upper half-plane. See Figure 7.16.

Method 2. *Consider the line $z = c + \imath y$, $0 < y < \pi$. Under the transformation, this is mapped to*

$$w = \mathrm{e}^{c+\imath y} + \mathrm{e}^{c}\,\mathrm{e}^{\imath y}, \quad 0 < y < \pi.$$

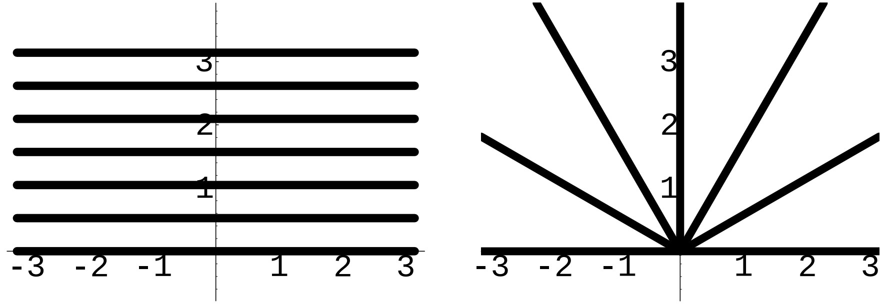


Figure 7.16: e^z maps horizontal lines to rays.

This is a semi-circle in the upper half-plane of radius e^c*. As* $c \to -\infty$*, the radius goes to zero. As* $c \to \infty$*, the radius goes to infinity. Thus the strip is mapped to the upper half-plane. See Figure 7.17.*

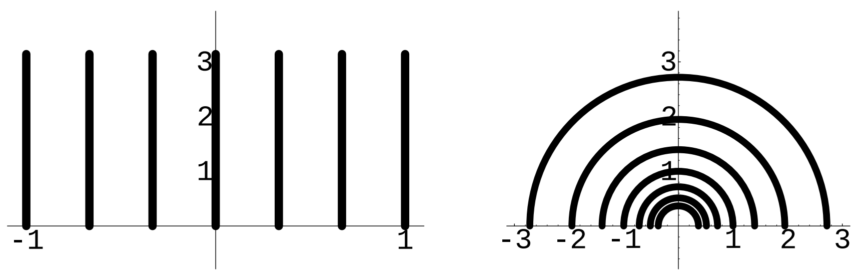


Figure 7.17: e^z maps vertical lines to circular arcs.

The sine and cosine. We can write the sine and cosine in terms of the exponential function.

$$\begin{aligned}
\frac{\mathrm{e}^{\imath z} + \mathrm{e}^{-\imath z}}{2} &= \frac{\cos(z) + \imath \sin(z) + \cos(-z) + \imath \sin(-z)}{2} \\
&= \frac{\cos(z) + \imath \sin(z) + \cos(z) - \imath \sin(z)}{2} \\
&= \cos z
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{e}^{\imath z} - \mathrm{e}^{-\imath z}}{\imath 2} &= \frac{\cos(z) + \imath \sin(z) - \cos(-z) - \imath \sin(-z)}{2} \\
&= \frac{\cos(z) + \imath \sin(z) - \cos(z) + \imath \sin(z)}{2} \\
&= \sin z
\end{aligned}$$

We separate the sine and cosine into their real and imaginary parts.

$$\cos z = \cos x \cosh y - \imath \sin x \sinh y \qquad \sin z = \sin x \cosh y + \imath \cos x \sinh y$$

For fixed y, the sine and cosine are oscillatory in x. The amplitude of the oscillations grows with increasing $|y|$. See Figure 7.18 and Figure 7.19 for plots of the real and imaginary parts of the cosine and sine, respectively. Figure 7.20 shows the modulus of the cosine and the sine.

The hyperbolic sine and cosine. The hyperbolic sine and cosine have the familiar definitions in terms of the exponential function. Thus not surprisingly, we can write the sine in terms of the hyperbolic sine and write the cosine in terms of the hyperbolic cosine. Below is a collection of trigonometric identities.

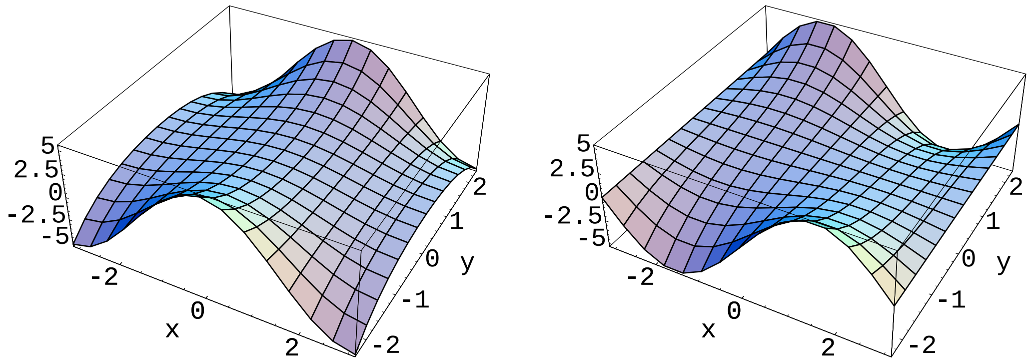


Figure 7.18: Plots of $\Re(\cos(z))$ and $\Im(\cos(z))$.

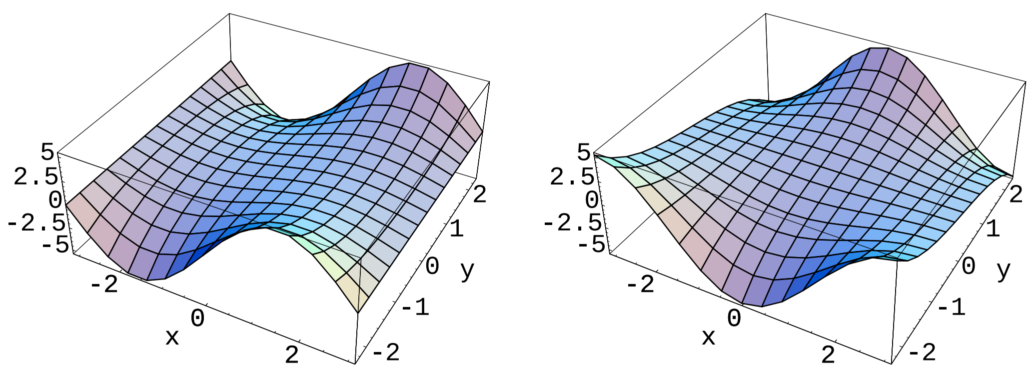


Figure 7.19: Plots of $\Re(\sin(z))$ and $\Im(\sin(z))$.

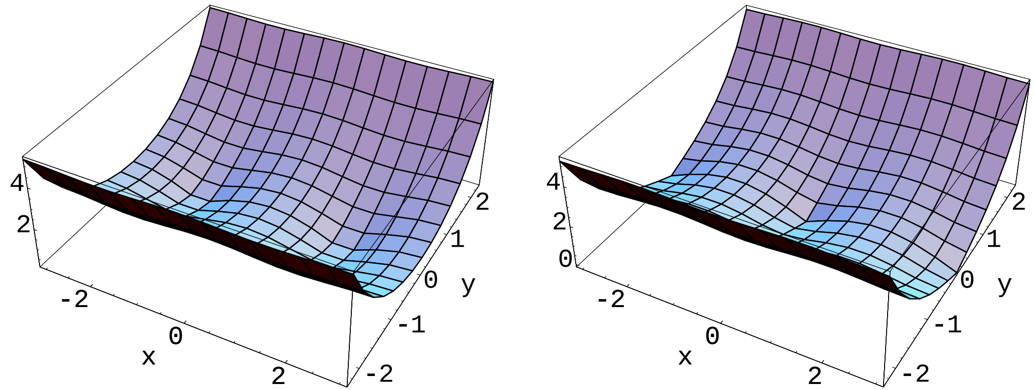


Figure 7.20: Plots of $|\cos(z)|$ and $|\sin(z)|$.

Result 7.6.1

$$\mathrm{e}^z = \mathrm{e}^x(\cos y + \imath \sin y)$$

$$\cos z = \frac{\mathrm{e}^{\imath z} + \mathrm{e}^{-\imath z}}{2} \qquad \sin z = \frac{\mathrm{e}^{\imath z} - \mathrm{e}^{-\imath z}}{\imath 2}$$

$$\cos z = \cos x \cosh y - \imath \sin x \sinh y \qquad \sin z = \sin x \cosh y + \imath \cos x \sinh y$$

$$\cosh z = \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2} \qquad \sinh z = \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}$$

$$\cosh z = \cosh x \cos y + \imath \sinh x \sin y \qquad \sinh z = \sinh x \cos y + \imath \cosh x \sin y$$

$$\sin(\imath z) = \imath \sinh z \qquad \sinh(\imath z) = \imath \sin z$$

$$\cos(\imath z) = \cosh z \qquad \cosh(\imath z) = \cos z$$

$$\log z = \ln |z| + \imath \arg(z) = \ln |z| + \imath \operatorname{Arg}(z) + \imath 2\pi n, \quad n \in \mathbb{Z}$$

7.7 Inverse Trigonometric Functions

The logarithm. The logarithm, $\log(z)$, is defined as the inverse of the exponential function e^z. The exponential function is many-to-one and thus has a multi-valued inverse. From what we know of many-to-one functions, we conclude that

$$\mathrm{e}^{\log z} = z, \quad \text{but} \quad \log\left(\mathrm{e}^z\right) \neq z.$$

This is because $\mathrm{e}^{\log z}$ is single-valued but $\log\left(\mathrm{e}^z\right)$ is not. Because e^z is $\imath 2\pi$ periodic, the logarithm of a number is a set of numbers which differ by integer multiples of $\imath 2\pi$. For instance, $\mathrm{e}^{\imath 2\pi n} = 1$ so that $\log(1) = \{\imath 2\pi n : n \in \mathbb{Z}\}$. The logarithmic function has an infinite number of branches. The value of the function on the branches differs by integer multiples of $\imath 2\pi$. It has singularities at zero and infinity. $|\log(z)| \to \infty$ as either $z \to 0$ or $z \to \infty$.

We will derive the formula for the complex variable logarithm. For now, let $\ln(x)$ denote the real variable logarithm that is defined for positive real numbers. Consider $w = \log z$. This means that $\mathrm{e}^w = z$. We write $w = u + \imath v$ in Cartesian form and $z = r\,\mathrm{e}^{\imath\theta}$ in polar form.

$$\mathrm{e}^{u+\imath v} = r\,\mathrm{e}^{\imath\theta}$$

We equate the modulus and argument of this expression.

$$\begin{aligned} \mathrm{e}^u &= r \qquad & v &= \theta + 2\pi n \\ u &= \ln r \qquad & v &= \theta + 2\pi n \end{aligned}$$

With $\log z = u + \imath v$, we have a formula for the logarithm.

$$\boxed{\log z = \ln|z| + \imath \arg(z)}$$

If we write out the multi-valuedness of the argument function we note that this has the form that we expected.

$$\log z = \ln|z| + \imath(\operatorname{Arg}(z) + 2\pi n), \quad n \in \mathbb{Z}$$

We check that our formula is correct by showing that $\mathrm{e}^{\log z} = z$

$$\mathrm{e}^{\log z} = \mathrm{e}^{\ln|z| + \imath \arg(z)} = \mathrm{e}^{\ln r + \imath\theta + \imath 2\pi n} = r\,\mathrm{e}^{\imath\theta} = z$$

Note again that $\log(\mathrm{e}^z) \neq z$.

$$\log(\mathrm{e}^z) = \ln|\,\mathrm{e}^z\,| + \imath \arg(\mathrm{e}^z) = \ln(\mathrm{e}^x) + \imath \arg\left(\mathrm{e}^{x+\imath y}\right) = x + \imath(y + 2\pi n) = z + \imath 2n\pi \neq z$$

The real part of the logarithm is the single-valued $\ln r$; the imaginary part is the multi-valued $\arg(z)$. We define the principal branch of the logarithm $\operatorname{Log} z$ to be the branch that satisfies $-\pi < \Im(\operatorname{Log} z) \leq \pi$. For positive, real numbers the principal branch, $\operatorname{Log} x$ is real-valued. We can write $\operatorname{Log} z$ in terms of the principal argument, $\operatorname{Arg} z$.

$$\operatorname{Log} z = \ln|z| + \imath \operatorname{Arg}(z)$$

See Figure 7.21 for plots of the real and imaginary part of $\operatorname{Log} z$.

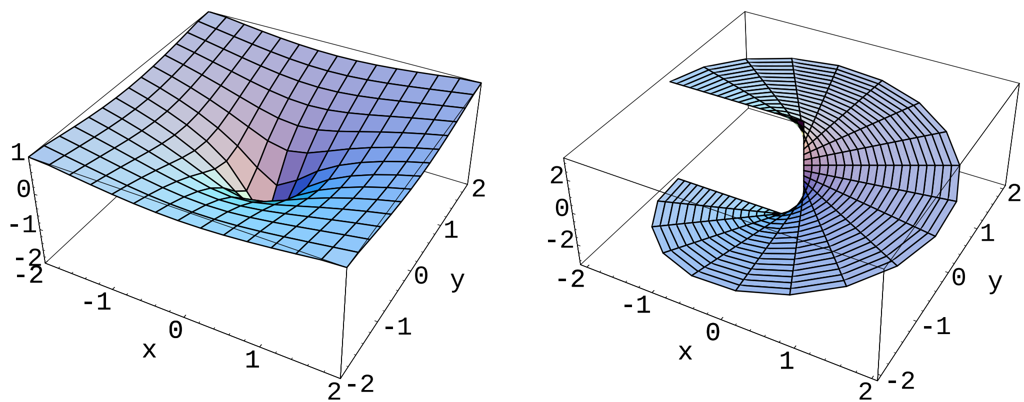


Figure 7.21: Plots of $\Re(\operatorname{Log} z)$ and $\Im(\operatorname{Log} z)$.

The form: $\mathbf{a^b}$. Consider a^b where a and b are complex and a is nonzero. We define this expression in terms of the exponential and the logarithm as

$$a^b = \mathrm{e}^{b \log a}.$$

Note that the multi-valuedness of the logarithm may make a^b multi-valued. First consider the case that the exponent is an integer.

$$a^m = \mathrm{e}^{m \log a} = \mathrm{e}^{m(\operatorname{Log} a + \imath 2n\pi)} = \mathrm{e}^{m \operatorname{Log} a} \, \mathrm{e}^{\imath 2mn\pi} = \mathrm{e}^{m \operatorname{Log} a}$$

Thus we see that a^m has a single value where m is an integer.

Now consider the case that the exponent is a rational number. Let p/q be a rational number in reduced form.

$$a^{p/q} = \mathrm{e}^{\frac{p}{q} \log a} = \mathrm{e}^{\frac{p}{q}(\operatorname{Log} a + \imath 2n\pi)} = \mathrm{e}^{\frac{p}{q} \operatorname{Log} a} \, \mathrm{e}^{\imath 2np\pi/q}.$$

This expression has q distinct values as

$$\mathrm{e}^{\imath 2np\pi/q} = \mathrm{e}^{\imath 2mp\pi/q} \quad \text{if and only if} \quad n = m \mod q.$$

Finally consider the case that the exponent b is an irrational number.

$$a^b = \mathrm{e}^{b \log a} = \mathrm{e}^{b(\operatorname{Log} a + \imath 2n\pi)} = \mathrm{e}^{b \operatorname{Log} a} \, \mathrm{e}^{\imath 2bn\pi}$$

Note that $\mathrm{e}^{\imath 2bn\pi}$ and $\mathrm{e}^{\imath 2bm\pi}$ are equal if and only if $\imath 2bn\pi$ and $\imath 2bm\pi$ differ by an integer multiple of $\imath 2\pi$, which means that bn and bm differ by an integer. This occurs only when $n = m$. Thus $\mathrm{e}^{\imath 2bn\pi}$ has a distinct value for each different integer n. We conclude that a^b has an infinite number of values.

You may have noticed something a little fishy. If b is not an integer and a is any non-zero complex number, then a^b is multi-valued. Then why have we been treating e^b as single-valued, when it is merely the case $a = e$? The answer is that in the realm of functions of a complex variable, e^z is an abuse of notation. We write e^z when we mean $\exp(z)$, the single-valued exponential function. Thus when we write e^z we do not mean "the number e raised to the z power", we mean "the exponential function of z". We denote the former scenario as $(e)^z$, which is multi-valued.

Logarithmic identities. Back in high school trigonometry when you thought that the logarithm was only defined for positive real numbers you learned the identity $\log x^a = a \log x$. This identity doesn't hold when the logarithm is defined for nonzero complex numbers. Consider the logarithm of z^a.

$$\log z^a = \operatorname{Log} z^a + \imath 2\pi n$$

$$a \log z = a(\operatorname{Log} z + \imath 2\pi n) = a \operatorname{Log} z + \imath 2 a \pi n$$

Note that

$$\log z^a \neq a \log z$$

Furthermore, since

$$\operatorname{Log} z^a = \ln |z^a| + \imath \operatorname{Arg}\left(z^a\right), \quad a \operatorname{Log} z = a \ln |z| + \imath a \operatorname{Arg}(z)$$

and $\operatorname{Arg}\left(z^a\right)$ is not necessarily the same as $a \operatorname{Arg}(z)$ we see that

$$\operatorname{Log} z^a \neq a \operatorname{Log} z.$$

Consider the logarithm of a product.

$$\begin{aligned}
\log(ab) &= \ln |ab| + \imath \arg(ab) \\
&= \ln |a| + \ln |b| + \imath \arg(a) + \imath \arg(b) \\
&= \log a + \log b
\end{aligned}$$

There is not an analogous identity for the principal branch of the logarithm since $\operatorname{Arg}(ab)$ is not in general the same as $\operatorname{Arg}(a) + \operatorname{Arg}(b)$.

Using $\log(ab) = \log(a) + \log(b)$ we can deduce that $\log\left(a^n\right) = \sum_{k=1}^{n} \log a = n \log a$, where n is a positive integer. This result is simple, straightforward and wrong. I have led you down the merry path to damnation.[3] In fact, $\log\left(a^2\right) \neq 2 \log a$. Just write the multi-valuedness explicitly,

$$\log\left(a^2\right) = \operatorname{Log}\left(a^2\right) + \imath 2 n \pi, \qquad 2 \log a = 2(\operatorname{Log} a + \imath 2 n \pi) = 2 \operatorname{Log} a + \imath 4 n \pi.$$

You can verify that

$$\log\left(\frac{1}{a}\right) = -\log a.$$

We can use this and the product identity to expand the logarithm of a quotient.

$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

[3] Don't feel bad if you fell for it. The logarithm is a tricky bastard.

For general values of a, $\log z^a \neq a \log z$. However, for some values of a, equality holds. We already know that $a = 1$ and $a = -1$ work. To determine if equality holds for other values of a, we explicitly write the multi-valuedness.

$$
\begin{aligned}
\log z^a = \log\left(\mathrm{e}^{a \log z}\right) = a \log z + \imath 2\pi k, \quad k \in \mathbb{Z} \\
a \log z = a \ln|z| + \imath a \operatorname{Arg} z + \imath a 2\pi m, \quad m \in \mathbb{Z}
\end{aligned}
$$

We see that $\log z^a = a \log z$ if and only if

$$
\{am \mid m \in \mathbb{Z}\} = \{am + k \mid k, m \in \mathbb{Z}\}.
$$

The sets are equal if and only if $a = 1/n$, $n \in \mathbb{Z}^{\pm}$. Thus we have the identity:

$$
\log\left(z^{1/n}\right) = \frac{1}{n} \log z, \quad n \in \mathbb{Z}^{\pm}
$$

Result 7.7.1 Logarithmic Identities.

$$
\begin{aligned}
a^b &= \mathrm{e}^{b \log a} \\
\mathrm{e}^{\log z} &= \mathrm{e}^{\operatorname{Log} z} = z \\
\log(ab) &= \log a + \log b \\
\log(1/a) &= -\log a \\
\log(a/b) &= \log a - \log b \\
\log\left(z^{1/n}\right) &= \frac{1}{n} \log z, \quad n \in \mathbb{Z}^{\pm}
\end{aligned}
$$

Logarithmic Inequalities.

$$
\begin{aligned}
\operatorname{Log}(uv) &\neq \operatorname{Log}(u) + \operatorname{Log}(v) \\
\log z^a &\neq a \log z \\
\operatorname{Log} z^a &\neq a \operatorname{Log} z \\
\log \mathrm{e}^z &\neq z
\end{aligned}
$$

Example 7.7.1 *Consider 1^π. We apply the definition $a^b = \mathrm{e}^{b \log a}$.*

$$
\begin{aligned}
1^\pi &= \mathrm{e}^{\pi \log(1)} \\
&= \mathrm{e}^{\pi(\ln(1) + \imath 2 n \pi)} \\
&= \mathrm{e}^{\imath 2 n \pi^2}
\end{aligned}
$$

Thus we see that 1^π has an infinite number of values, all of which lie on the unit circle $|z| = 1$ in the complex plane. However, the set 1^π is not equal to the set $|z| = 1$. There are points in the latter which are not in the former. This is analogous to the fact that the rational numbers are dense in the real numbers, but are a subset of the real numbers.

Example 7.7.2 *We find the zeros of* $\sin z$*.*

$$
\begin{aligned}
\sin z = \frac{\mathrm{e}^{\imath z} - \mathrm{e}^{-\imath z}}{\imath 2} &= 0 \\
\mathrm{e}^{\imath z} &= \mathrm{e}^{-\imath z} \\
\mathrm{e}^{\imath 2z} &= 1 \\
2z \mod 2\pi &= 0 \\
\boxed{z = n\pi, \quad n \in \mathbb{Z}}
\end{aligned}
$$

Equivalently, we could use the identity

$$\sin z = \sin x \cosh y + \imath \cos x \sinh y = 0.$$

This becomes the two equations (for the real and imaginary parts)

$$\sin x \cosh y = 0 \qquad \textit{and} \qquad \cos x \sinh y = 0.$$

Since $\cosh$ *is real-valued and positive for real argument, the first equation dictates that* $x = n\pi$*,* $n \in \mathbb{Z}$*. Since* $\cos(n\pi) = (-1)^n$ *for* $n \in \mathbb{Z}$*, the second equation implies that* $\sinh y = 0$*. For real argument,* $\sinh y$ *is only zero at* $y = 0$*. Thus the zeros are*

$$\boxed{z = n\pi, \quad n \in \mathbb{Z}}$$

Example 7.7.3 *Since we can express* $\sin z$ *in terms of the exponential function, one would expect that we could express* *the* $\sin^{-1} z$ *in terms of the logarithm.*

$$
\begin{aligned}
w &= \sin^{-1} z \\
z &= \sin w \\
z &= \frac{e^{\imath w} - e^{-\imath w}}{\imath 2} \\
e^{\imath 2w} - \imath 2z\, e^{\imath w} - 1 &= 0 \\
e^{\imath w} &= \imath z \pm \sqrt{1 - z^2} \\
w &= -\imath \log\left(\imath z \pm \sqrt{1 - z^2}\right)
\end{aligned}
$$

Thus we see how the multi-valued $\sin^{-1}$ *is related to the logarithm.*

$$
\boxed{\sin^{-1} z = -\imath \log\left(\imath z \pm \sqrt{1 - z^2}\right)}
$$

Example 7.7.4 *Consider the equation* $\sin^3 z = 1$*.*

$$
\begin{aligned}
\sin^3 z &= 1 \\
\sin z &= 1^{1/3} \\
\frac{e^{\imath z} - e^{-\imath z}}{\imath 2} &= 1^{1/3} \\
e^{\imath z} - \imath 2 (1)^{1/3} - e^{-\imath z} &= 0 \\
e^{\imath 2z} - \imath 2 (1)^{1/3} e^{\imath z} - 1 &= 0 \\
e^{\imath z} &= \frac{\imath 2 (1)^{1/3} \pm \sqrt{-4 (1)^{2/3} + 4}}{2} \\
e^{\imath z} &= \imath (1)^{1/3} \pm \sqrt{1 - (1)^{2/3}}
\end{aligned}
$$

$$
\boxed{z = -\imath \log\left(\imath (1)^{1/3} \pm \sqrt{1 - 1^{2/3}}\right)}
$$

Note that there are three sources of multi-valuedness in the expression for z. The two values of the square root are shown explicitly. There are three cube roots of unity. Finally, the logarithm has an infinite number of branches. To show this multi-valuedness explicitly, we could write

$$\boxed{z = -\imath \operatorname{Log}\left(\imath\, \mathrm{e}^{\imath 2m\pi/3} \pm \sqrt{1 - \mathrm{e}^{\imath 4m\pi/3}}\right) + 2\pi n, \qquad m = 0, 1, 2, \quad n = \ldots, -1, 0, 1, \ldots}$$

Example 7.7.5 *Consider the harmless looking equation, $\imath^z = 1$.*

Before we start with the algebra, note that the right side of the equation is a single number. $\imath^z$ is single-valued only when z is an integer. Thus we know that if there are solutions for z, they are integers. We now proceed to solve the equation.

$$\imath^z = 1$$
$$\left(\mathrm{e}^{\imath\pi/2}\right)^z = 1$$

Use the fact that z is an integer.

$$\mathrm{e}^{\imath\pi z/2} = 1$$
$$\imath\pi z/2 = \imath 2n\pi, \quad \text{for some } n \in \mathbb{Z}$$
$$\boxed{z = 4n, \quad n \in \mathbb{Z}}$$

Here is a different approach. We write down the multi-valued form of $\imath^z$. We solve the equation by requiring that all the values of $\imath^z$ are 1.

$$\imath^z = 1$$
$$\mathrm{e}^{z \log \imath} = 1$$
$$z \log \imath = \imath 2\pi n, \quad \text{for some } n \in \mathbb{Z}$$
$$z\left(\imath\frac{\pi}{2} + \imath 2\pi m\right) = \imath 2\pi n, \quad \forall m \in \mathbb{Z}, \quad \text{for some } n \in \mathbb{Z}$$
$$\imath\frac{\pi}{2} z + \imath 2\pi m z = \imath 2\pi n, \quad \forall m \in \mathbb{Z}, \quad \text{for some } n \in \mathbb{Z}$$

The only solutions that satisfy the above equation are

$$\boxed{z = 4k, \quad k \in \mathbb{Z}.}$$

Now let's consider a slightly different problem: $1 \in \imath^z$. For what values of z does $\imath^z$ have 1 as one of its values.

$$1 \in \imath^z$$
$$1 \in \mathrm{e}^{z \log \imath}$$
$$1 \in \{\mathrm{e}^{z(\imath\pi/2 + \imath 2\pi n)} \mid n \in \mathbb{Z}\}$$
$$z(\imath\pi/2 + \imath 2\pi n) = \imath 2\pi m, \quad m, n \in \mathbb{Z}$$
$$\boxed{z = \frac{4m}{1 + 4n}, \quad m, n \in \mathbb{Z}}$$

There are an infinite set of rational numbers for which $\imath^z$ has 1 as one of its values. For example,

$$\imath^{4/5} = 1^{1/5} = \left\{1, \mathrm{e}^{\imath 2\pi/5}, \mathrm{e}^{\imath 4\pi/5}, \mathrm{e}^{\imath 6\pi/5}, \mathrm{e}^{\imath 8\pi/5}\right\}$$

7.8 Riemann Surfaces

Consider the mapping $w = \log(z)$. Each nonzero point in the z-plane is mapped to an infinite number of points in the w plane.

$$w = \{\ln|z| + \imath \arg(z)\} = \{\ln|z| + \imath(\operatorname{Arg}(z) + 2\pi n) \mid n \in \mathbb{Z}\}$$

This multi-valuedness makes it hard to work with the logarithm. We would like to select one of the branches of the logarithm. One way of doing this is to decompose the z-plane into an infinite number of sheets. The sheets lie above one another and are labeled with the integers, $n \in \mathbb{Z}$. (See Figure 7.22.) We label the point z on the n^{th} sheet as (z, n). Now each point (z, n) maps to a single point in the w-plane. For instance, we can make the zeroth sheet map to the principal branch of the logarithm. This would give us the following mapping.

$$\log(z, n) = \operatorname{Log} z + \imath 2\pi n$$

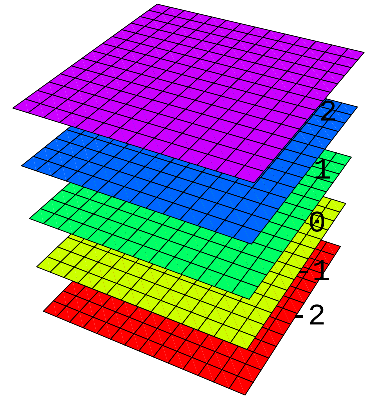


Figure 7.22: The z-plane decomposed into flat sheets.

This is a nice idea, but it has some problems. The mappings are not continuous. Consider the mapping on the zeroth sheet. As we approach the negative real axis from above z is mapped to $\ln|z| + \imath\pi$ as we approach from below it is mapped to $\ln|z| - \imath\pi$. (Recall Figure 7.21.) The mapping is not continuous across the negative real axis.

Let's go back to the regular z-plane for a moment. We start at the point $z = 1$ and selecting the branch of the logarithm that maps to zero. $(\log(1) = \imath 2\pi n)$. We make the logarithm vary continuously as we walk around the origin once in the positive direction and return to the point $z = 1$. Since the argument of z has increased by 2π, the value of the logarithm has changed to $\imath 2\pi$. If we walk around the origin again we will have $\log(1) = \imath 4\pi$. Our flat sheet decomposition of the z-plane does not reflect this property. We need a decomposition with a geometry that makes the mapping continuous and connects the various branches of the logarithm.

Drawing inspiration from the plot of $\arg(z)$, Figure 7.10, we decompose the z-plane into an infinite corkscrew with axis at the origin. (See Figure 7.23.) We define the mapping so that the logarithm varies continuously on this surface. Consider a point z on one of the sheets. The value of the logarithm at that same point on the sheet directly above it is $\imath 2\pi$ more than the original value. We call this surface, the *Riemann surface* for the logarithm. The mapping from the Riemann surface to the w-plane is continuous and one-to-one.

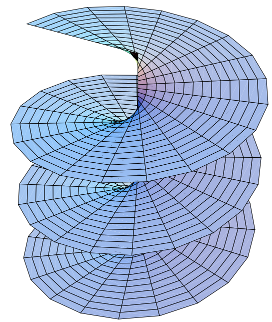


Figure 7.23: The Riemann surface for the logarithm.

7.9 Branch Points

Example 7.9.1 *Consider the function $z^{1/2}$. For each value of z, there are two values of $z^{1/2}$. We write $z^{1/2}$ in modulus-argument and Cartesian form.*

$$z^{1/2} = \sqrt{|z|}\, \mathrm{e}^{\imath \arg(z)/2}$$
$$z^{1/2} = \sqrt{|z|}\cos(\arg(z)/2) + \imath\sqrt{|z|}\sin(\arg(z)/2)$$

Figure 7.24 shows the real and imaginary parts of $z^{1/2}$ from three different viewpoints. The second and third views are looking down the x axis and y axis, respectively. Consider $\Re\left(z^{1/2}\right)$. This is a double layered sheet which intersects itself on the negative real axis. ($\Im(z^{1/2})$ has a similar structure, but intersects itself on the positive real axis.) Let's start at a point on the positive real axis on the lower sheet. If we walk around the origin once and return to the positive real axis, we will be on the upper sheet. If we do this again, we will return to the lower sheet.

Suppose we are at a point in the complex plane. We pick one of the two values of $z^{1/2}$. If the function varies continuously as we walk around the origin and back to our starting point, the value of $z^{1/2}$ will have changed. We will *be on the other branch. Because walking around the point $z = 0$ takes us to a different branch of the function, we refer to $z = 0$ as a branch point.*

Now consider the modulus-argument form of $z^{1/2}$:

$$z^{1/2} = \sqrt{|z|}\, \mathrm{e}^{\imath \arg(z)/2}.$$

Figure 7.25 shows the modulus and the principal argument of $z^{1/2}$. We see that each time we walk around the origin, the argument of $z^{1/2}$ changes by π. This means that the value of the function changes by the factor $\mathrm{e}^{\imath\pi} = -1$, i.e. the function changes sign. If we walk around the origin twice, the argument changes by 2π, so that the value of the function does not change, $\mathrm{e}^{\imath 2\pi} = 1$.

$z^{1/2}$ is a continuous function except at $z = 0$. Suppose we start at $z = 1 = \mathrm{e}^{\imath 0}$ and the function value $(\mathrm{e}^{\imath 0})^{1/2} = 1$. If we follow the first path in Figure 7.26, the argument of z varies from up to about $\frac{\pi}{4}$, down to about $-\frac{\pi}{4}$ and back to 0. The value of the function is still $(\mathrm{e}^{\imath 0})^{1/2}$.

Now suppose we follow a circular path around the origin in the positive, counter-clockwise, direction. (See the second path in Figure 7.26.) The argument of z increases by 2π. The value of the function at half turns on the path is

$$\begin{aligned}
\left(\mathrm{e}^{\imath 0}\right)^{1/2} &= 1, \\
\left(\mathrm{e}^{\imath \pi}\right)^{1/2} &= \mathrm{e}^{\imath\pi/2} = \imath, \\
\left(\mathrm{e}^{\imath 2\pi}\right)^{1/2} &= \mathrm{e}^{\imath\pi} = -1
\end{aligned}$$

As we return to the point $z = 1$, the argument of the function has changed by π and the value of the function has changed from 1 to -1. If we were to walk along the circular path again, the argument of z would increase by another 2π. The argument of the function would increase by another π and the value of the function would return to 1.

$$\left(\mathrm{e}^{\imath 4\pi}\right)^{1/2} = \mathrm{e}^{\imath 2\pi} = 1$$

In general, any time we walk around the origin, the value of $z^{1/2}$ changes by the factor -1. We call $z = 0$ a branch point. If we want a single-valued square root, we need something to prevent us from walking around the origin. We achieve this by introducing a branch cut. Suppose we have the complex plane drawn on an infinite sheet of paper. With a scissors we cut the paper from the origin to $-\infty$ along the real axis. Then if we start at $z = \mathrm{e}^{\imath 0}$, and draw a

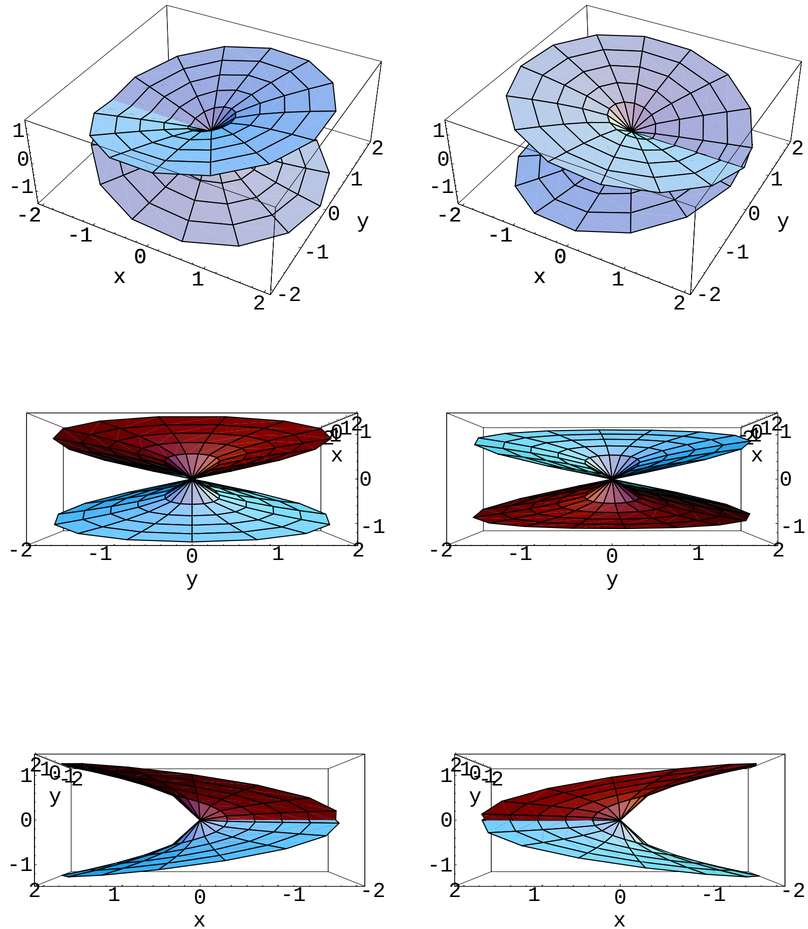


Figure 7.24: Plots of $\Re\left(z^{1/2}\right)$ (left) and $\Im\left(z^{1/2}\right)$ (right) from three viewpoints.

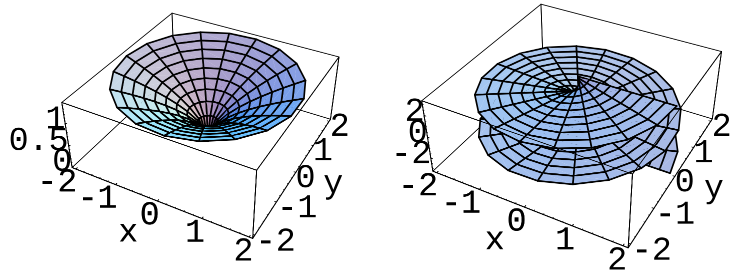


Figure 7.25: Plots of $|z^{1/2}|$ and $\mathrm{Arg}\left(z^{1/2}\right)$.

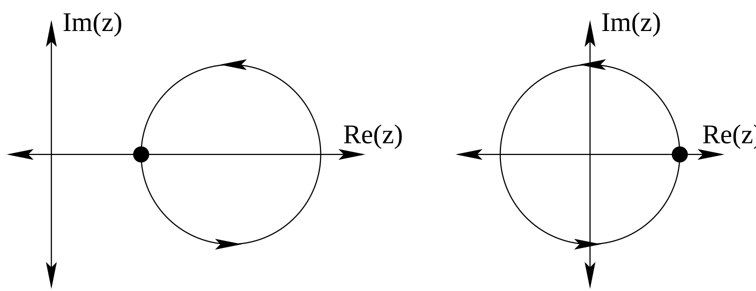


Figure 7.26: A path that does not encircle the origin and a path around the origin.

continuous line without leaving the paper, the argument of z will always be in the range $-\pi < \arg z < \pi$. This means that $-\frac{\pi}{2} < \arg\left(z^{1/2}\right) < \frac{\pi}{2}$. No matter what path we follow in this cut plane, $z = 1$ has argument zero and $(1)^{1/2} = 1$. By never crossing the negative real axis, we have constructed a single valued **branch** of the square root function. We call the cut along the negative real axis a **branch cut**.

Example 7.9.2 Consider the logarithmic function $\log z$. For each value of z, there are an infinite number of values of $\log z$. We write $\log z$ in Cartesian form.

$$\log z = \ln |z| + \imath \arg z$$

Figure 7.27 shows the real and imaginary parts of the logarithm. The real part is single-valued. The imaginary part is multi-valued and has an infinite number of branches. The values of the logarithm form an infinite-layered sheet. If we start on one of the sheets and walk around the origin once in the positive direction, then the value of the logarithm increases by $\imath 2\pi$ and we move to the next branch. $z = 0$ is a branch point of the logarithm.

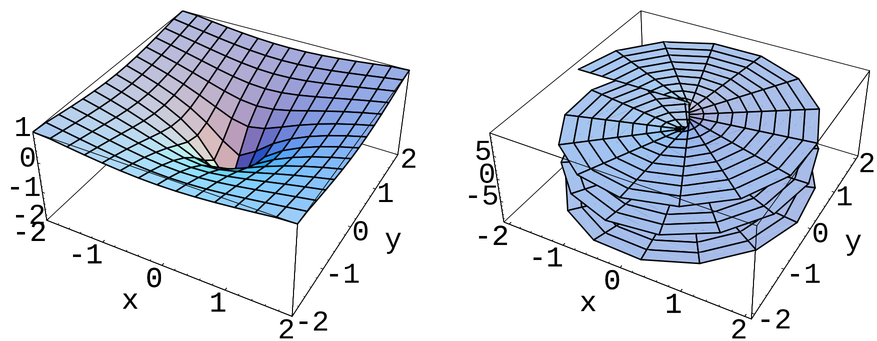


Figure 7.27: Plots of $\Re(\log z)$ and a portion of $\Im(\log z)$.

The logarithm is a continuous function except at $z = 0$. Suppose we start at $z = 1 = \mathrm{e}^{\imath 0}$ and the function value $\log\left(\mathrm{e}^{\imath 0}\right) = \ln(1) + \imath 0 = 0$. If we follow the first path in Figure 7.26, the argument of z and thus the imaginary part of the logarithm varies from up to about $\frac{\pi}{4}$, down to about $-\frac{\pi}{4}$ and back to 0. The value of the logarithm is still 0.

Now suppose we follow a circular path around the origin in the positive direction. (See the second path in Figure 7.26.) The argument of z increases by 2π. The value of the logarithm at half turns on the path is

$$\log\left(e^{\imath 0}\right) = 0,$$
$$\log\left(e^{\imath \pi}\right) = \imath\pi,$$
$$\log\left(e^{\imath 2\pi}\right) = \imath 2\pi$$

As we return to the point $z = 1$, the value of the logarithm has changed by $\imath 2\pi$. If we were to walk along the circular path again, the argument of z would increase by another 2π and the value of the logarithm would increase by another $\imath 2\pi$.

> **Result 7.9.1** A point z_0 is a **branch point** of a function $f(z)$ if the function changes value when you walk around the point on any path that encloses no singularities other than the one at $z = z_0$.

Branch points at infinity : mapping infinity to the origin. Up to this point we have considered only branch points in the finite plane. Now we consider the possibility of a branch point at infinity. As a first method of approaching this problem we map the point at infinity to the origin with the transformation $\zeta = 1/z$ and examine the point $\zeta = 0$.

Example 7.9.3 *Again consider the function $z^{1/2}$. Mapping the point at infinity to the origin, we have $f(\zeta) = (1/\zeta)^{1/2} = \zeta^{-1/2}$. For each value of ζ, there are two values of $\zeta^{-1/2}$. We write $\zeta^{-1/2}$ in modulus-argument form.*

$$\zeta^{-1/2} = \frac{1}{\sqrt{|\zeta|}} e^{-\imath \arg(\zeta)/2}$$

Like $z^{1/2}$, $\zeta^{-1/2}$ has a double-layered sheet of values. Figure 7.28 shows the modulus and the principal argument of $\zeta^{-1/2}$. We see that each time we walk around the origin, the argument of $\zeta^{-1/2}$ changes by $-\pi$. This means that the value of the function changes by the factor $e^{-\imath\pi} = -1$, i.e. the function changes sign. If we walk around the origin twice, the argument changes by -2π, so that the value of the function does not change, $e^{-\imath 2\pi} = 1$.

Since $\zeta^{-1/2}$ has a branch point at zero, we conclude that $z^{1/2}$ has a branch point at infinity.

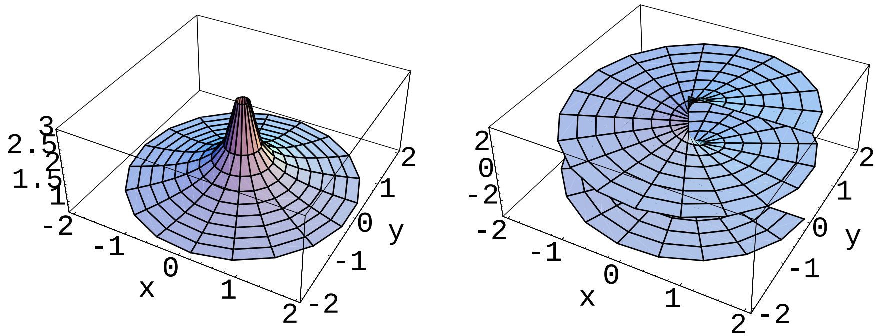


Figure 7.28: Plots of $|\zeta^{-1/2}|$ and $\operatorname{Arg}\left(\zeta^{-1/2}\right)$.

Example 7.9.4 *Again consider the logarithmic function* $\log z$*. Mapping the point at infinity to the origin, we have* $f(\zeta) = \log(1/\zeta) = -\log(\zeta)$*. From Example 7.9.2 we known that* $-\log(\zeta)$ *has a branch point at* $\zeta = 0$*. Thus* $\log z$ *has a branch point at infinity.*

Branch points at infinity : paths around infinity. We can also check for a branch point at infinity by following a path that encloses the point at infinity and no other singularities. Just draw a simple closed curve that separates the complex plane into a bounded component that contains all the singularities of the function in the finite plane. Then, depending on orientation, the curve is a contour enclosing all the finite singularities, or the point at infinity and no other singularities.

Example 7.9.5 *Once again consider the function* $z^{1/2}$*. We know that the function changes value on a curve that goes once around the origin. Such a curve can be considered to be either a path around the origin or a path around infinity. In either case the path encloses one singularity. There are branch points at the origin and at infinity. Now consider a curve that does not go around the origin. Such a curve can be considered to be either a path around neither of the branch points or both of them. Thus we see that* $z^{1/2}$ *does not change value when we follow a path that encloses neither or both of its branch points.*

Example 7.9.6 *Consider* $f(z) = \left(z^2 - 1\right)^{1/2}$*. We factor the function.*

$$f(z) = (z-1)^{1/2}(z+1)^{1/2}$$

There are branch points at $z = \pm 1$*. Now consider the point at infinity.*

$$f\left(\zeta^{-1}\right) = \left(\zeta^{-2} - 1\right)^{1/2} = \pm\zeta^{-1}\left(1 - \zeta^2\right)^{1/2}$$

Since $f\left(\zeta^{-1}\right)$ *does not have a branch point at* $\zeta = 0$*,* $f(z)$ *does not have a branch point at infinity. We could reach the same conclusion by considering a path around infinity. Consider a path that circles the branch points at* $z = \pm 1$ *once in the positive direction. Such a path circles the point at infinity once in the negative direction. In traversing this path, the value of* $f(z)$ *is multiplied by the factor* $\left(\mathrm{e}^{\imath 2\pi}\right)^{1/2}\left(\mathrm{e}^{\imath 2\pi}\right)^{1/2} = \mathrm{e}^{\imath 2\pi} = 1$*. Thus the value of the function does not change. There is no branch point at infinity.*

Diagnosing branch points. We have the definition of a branch point, but we do not have a convenient criterion for determining if a particular function has a branch point. We have seen that $\log z$ and z^α for non-integer α have branch points at zero and infinity. The inverse trigonometric functions like the arcsine also have branch points, but they can be written in terms of the logarithm and the square root. In fact all the elementary functions with branch points can be written in terms of the functions $\log z$ and z^α. Furthermore, note that the multi-valuedness of z^α comes from the logarithm, $z^\alpha = \mathrm{e}^{\alpha \log z}$. This gives us a way of quickly determining if and where a function may have branch points.

> **Result 7.9.2** Let $f(z)$ be a single-valued function. Then $\log(f(z))$ and $(f(z))^\alpha$ may have branch points only where $f(z)$ is zero or singular.

Example 7.9.7 *Consider the functions,*

1. $\left(z^2\right)^{1/2}$
2. $\left(z^{1/2}\right)^2$
3. $\left(z^{1/2}\right)^3$

Are they multi-valued? Do they have branch points?

1. $$\left(z^2\right)^{1/2} = \pm\sqrt{z^2} = \pm z$$

   Because of the $(\cdot)^{1/2}$, *the function is multi-valued. The only possible branch points are at zero and infinity. If* $\left(\left(e^{\imath 0}\right)^2\right)^{1/2} = 1$, *then* $\left(\left(e^{\imath 2\pi}\right)^2\right)^{1/2} = \left(e^{\imath 4\pi}\right)^{1/2} = e^{\imath 2\pi} = 1$. *Thus we see that the function does not change value when we walk around the origin. We can also consider this to be a path around infinity. This function is multi-valued, but has no branch points.*

2. $$\left(z^{1/2}\right)^2 = \left(\pm\sqrt{z}\right)^2 = z$$

   This function is single-valued.

3. $$\left(z^{1/2}\right)^3 = \left(\pm\sqrt{z}\right)^3 = \pm\left(\sqrt{z}\right)^3$$

   This function is multi-valued. We consider the possible branch point at $z = 0$. *If* $\left(\left(e^{0}\right)^{1/2}\right)^3 = 1$, *then* $\left(\left(e^{\imath 2\pi}\right)^{1/2}\right)^3 = \left(e^{\imath\pi}\right)^3 = e^{\imath 3\pi} = -1$. *Since the function changes value when we walk around the origin, it has a branch point at* $z = 0$. *Since this is also a path around infinity, there is a branch point there.*

Example 7.9.8 *Consider the function* $f(z) = \log\left(\frac{1}{z-1}\right)$. *Since* $\frac{1}{z-1}$ *is only zero at infinity and its only singularity is at* $z = 1$, *the only possibilities for branch points are at* $z = 1$ *and* $z = \infty$. *Since*

$$\log\left(\frac{1}{z-1}\right) = -\log(z-1)$$

and $\log w$ *has branch points at zero and infinity, we see that* $f(z)$ *has branch points at* $z = 1$ *and* $z = \infty$.

Example 7.9.9 *Consider the functions,*

1. $\mathrm{e}^{\log z}$

2. $\log \mathrm{e}^z$.

Are they multi-valued? Do they have branch points?

1.
$$\mathrm{e}^{\log z} = \exp(\operatorname{Log} z + \imath 2\pi n) = \mathrm{e}^{\operatorname{Log} z} \mathrm{e}^{\imath 2\pi n} = z$$

This function is single-valued.

2.
$$\log \mathrm{e}^z = \operatorname{Log} \mathrm{e}^z + \imath 2\pi n = z + \imath 2\pi m$$

This function is multi-valued. It may have branch points only where e^z is zero or infinite. This only occurs at $z = \infty$. Thus there are no branch points in the finite plane. The function does not change when traversing a simple closed path. Since this path can be considered to enclose infinity, there is no branch point at infinity.

Consider $(f(z))^\alpha$ where $f(z)$ is single-valued and $f(z)$ has either a zero or a singularity at $z = z_0$. $(f(z))^\alpha$ may have a branch point at $z = z_0$. If $f(z)$ is not a power of z, then it may be difficult to tell if $(f(z))^\alpha$ changes value when we walk around z_0. Factor $f(z)$ into $f(z) = g(z)h(z)$ where $h(z)$ is nonzero and finite at z_0. Then $g(z)$ captures the important behavior of $f(z)$ at the z_0. $g(z)$ tells us how fast $f(z)$ vanishes or blows up. Since $(f(z))^\alpha = (g(z))^\alpha (h(z))^\alpha$ and $(h(z))^\alpha$ does not have a branch point at z_0, $(f(z))^\alpha$ has a branch point at z_0 if and only if $(g(z))^\alpha$ has a branch point there.

Similarly, we can decompose

$$\log(f(z)) = \log(g(z)h(z)) = \log(g(z)) + \log(h(z))$$

to see that $\log(f(z))$ has a branch point at z_0 if and only if $\log(g(z))$ has a branch point there.

Result 7.9.3 Consider a single-valued function $f(z)$ that has either a zero or a singularity at $z = z_0$. Let $f(z) = g(z)h(z)$ where $h(z)$ is nonzero and finite. $(f(z))^\alpha$ has a branch point at $z = z_0$ if and only if $(g(z))^\alpha$ has a branch point there. $\log(f(z))$ has a branch point at $z = z_0$ if and only if $\log(g(z))$ has a branch point there.

Example 7.9.10 *Consider the functions,*

1. $\sin z^{1/2}$
2. $(\sin z)^{1/2}$
3. $z^{1/2} \sin z^{1/2}$
4. $(\sin z^2)^{1/2}$

Find the branch points and the number of branches.

1.
$$\sin z^{1/2} = \sin\left(\pm\sqrt{z}\right) = \pm \sin\sqrt{z}$$
$\sin z^{1/2}$ *is multi-valued. It has two branches. There may be branch points at zero and infinity. Consider the unit circle which is a path around the origin or infinity. If* $\sin\left(\left(\mathrm{e}^{\imath 0}\right)^{1/2}\right) = \sin(1)$, *then* $\sin\left(\left(\mathrm{e}^{\imath 2\pi}\right)^{1/2}\right) = \sin\left(\mathrm{e}^{\imath\pi}\right) = \sin(-1) = -\sin(1)$. *There are branch points at the origin and infinity.*

2.
$$(\sin z)^{1/2} = \pm\sqrt{\sin z}$$
The function is multi-valued with two branches. The sine vanishes at $z = n\pi$ *and is singular at infinity. There could be branch points at these locations. Consider the point* $z = n\pi$. *We can write*
$$\sin z = (z - n\pi)\frac{\sin z}{z - n\pi}$$
Note that $\frac{\sin z}{z-n\pi}$ *is nonzero and has a removable singularity at* $z = n\pi$.
$$\lim_{z\to n\pi}\frac{\sin z}{z - n\pi} = \lim_{z\to n\pi}\frac{\cos z}{1} = (-1)^n$$
Since $(z - n\pi)^{1/2}$ *has a branch point at* $z = n\pi$, $(\sin z)^{1/2}$ *has branch points at* $z = n\pi$.

Since the branch points at $z = n\pi$ go all the way out to infinity. It is not possible to make a path that encloses infinity and no other singularities. The point at infinity is a non-isolated singularity. A point can be a branch point only if it is an isolated singularity.

3.

$$
\begin{aligned}
z^{1/2} \sin z^{1/2} &= \pm\sqrt{z} \sin\left(\pm\sqrt{z}\right) \\
&= \pm\sqrt{z}\left(\pm \sin\sqrt{z}\right) \\
&= \sqrt{z} \sin\sqrt{z}
\end{aligned}
$$

The function is single-valued. Thus there could be no branch points.

4.

$$
\left(\sin z^2\right)^{1/2} = \pm\sqrt{\sin z^2}
$$

This function is multi-valued. Since $\sin z^2 = 0$ at $z = (n\pi)^{1/2}$, there may be branch points there. First consider the point $z = 0$. We can write

$$
\sin z^2 = z^2 \frac{\sin z^2}{z^2}
$$

where $\sin\left(z^2\right)/z^2$ is nonzero and has a removable singularity at $z = 0$.

$$
\lim_{z \to 0} \frac{\sin z^2}{z^2} = \lim_{z \to 0} \frac{2z \cos z^2}{2z} = 1.
$$

Since $\left(z^2\right)^{1/2}$ does not have a branch point at $z = 0$, $\left(\sin z^2\right)^{1/2}$ does not have a branch point there either. Now consider the point $z = \sqrt{n\pi}$.

$$
\sin z^2 = \left(z - \sqrt{n\pi}\right) \frac{\sin z^2}{z - \sqrt{n\pi}}
$$

$\sin\left(z^2\right)/\left(z - \sqrt{n\pi}\right)$ *in nonzero and has a removable singularity at $z = \sqrt{n\pi}$.*

$$
\lim_{z \to \sqrt{n\pi}} \frac{\sin z^2}{z - \sqrt{n\pi}} = \lim_{z \to \sqrt{n\pi}} \frac{2z \cos z^2}{1} = 2\sqrt{n\pi}(-1)^n
$$

Since $(z-\sqrt{n\pi})^{1/2}$ has a branch point at $z=\sqrt{n\pi}$, $(\sin z^2)^{1/2}$ also has a branch point there.

Thus we see that $(\sin z^2)^{1/2}$ has branch points at $z=(n\pi)^{1/2}$ for $n \in \mathbb{Z} \setminus \{0\}$. This is the set of numbers: $\{\pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots, \pm\imath\sqrt{\pi}, \pm\imath\sqrt{2\pi}, \ldots\}$. The point at infinity is a non-isolated singularity.

Example 7.9.11 *Find the branch points of*

$$f(z) = \left(z^3 - z\right)^{1/3}.$$

Introduce branch cuts. If $f(2) = \sqrt[3]{6}$ then what is $f(-2)$?

We expand $f(z)$.

$$f(z) = z^{1/3}(z-1)^{1/3}(z+1)^{1/3}.$$

There are branch points at $z=-1,0,1$. We consider the point at infinity.

$$\begin{aligned}
f\left(\frac{1}{\zeta}\right) &= \left(\frac{1}{\zeta}\right)^{1/3}\left(\frac{1}{\zeta}-1\right)^{1/3}\left(\frac{1}{\zeta}+1\right)^{1/3} \\
&= \frac{1}{\zeta}\left(1-\zeta\right)^{1/3}\left(1+\zeta\right)^{1/3}
\end{aligned}$$

Since $f(1/\zeta)$ does not have a branch point at $\zeta=0$, $f(z)$ does not have a branch point at infinity. Consider the three possible branch cuts in Figure 7.29.

The first and the third branch cuts will make the function single valued, the second will not. It is clear that the first set makes the function single valued since it is not possible to walk around any of the branch points.

The second set of branch cuts would allow you to walk around the branch points at $z=\pm 1$. If you walked around these two once in the positive direction, the value of the function would change by the factor $\mathrm{e}^{\imath 4\pi/3}$.

The third set of branch cuts would allow you to walk around all three branch points together. You can verify that if you walk around the three branch points, the value of the function will not change ($\mathrm{e}^{\imath 6\pi/3} = \mathrm{e}^{\imath 2\pi} = 1$).

Suppose we introduce the third set of branch cuts and are on the branch with $f(2) = \sqrt[3]{6}$.

$$f(2) = \left(2\,\mathrm{e}^{\imath 0}\right)^{1/3}\left(1\,\mathrm{e}^{\imath 0}\right)^{1/3}\left(3\,\mathrm{e}^{\imath 0}\right)^{1/3} = \sqrt[3]{6}$$

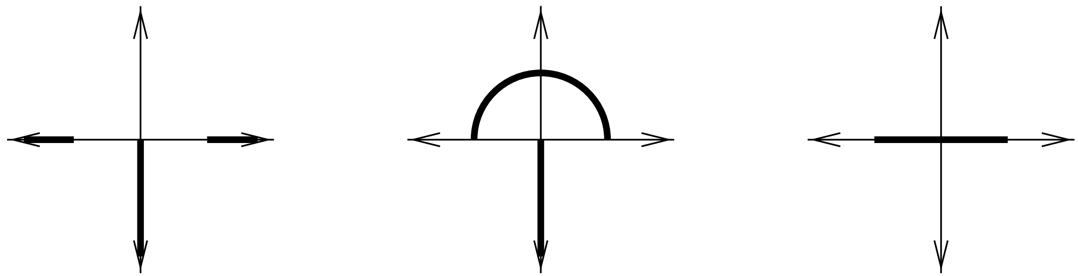


Figure 7.29: Three Possible Branch Cuts for $f(z) = (z^3 - z)^{1/3}$.

The value of $f(-2)$ is

$$
\begin{aligned}
f(-2) &= (2\,\mathrm{e}^{\imath\pi})^{1/3}\,(3\,\mathrm{e}^{\imath\pi})^{1/3}\,(1\,\mathrm{e}^{\imath\pi})^{1/3} \\
&= \sqrt[3]{2}\,\mathrm{e}^{\imath\pi/3}\,\sqrt[3]{3}\,\mathrm{e}^{\imath\pi/3}\,\sqrt[3]{1}\,\mathrm{e}^{\imath\pi/3} \\
&= \sqrt[3]{6}\,\mathrm{e}^{\imath\pi} \\
&= -\sqrt[3]{6}.
\end{aligned}
$$

Example 7.9.12 *Find the branch points and number of branches for*

$$f(z) = z^{z^2}.$$

$$z^{z^2} = \exp\left(z^2 \log z\right)$$

There may be branch points at the origin and infinity due to the logarithm. Consider walking around a circle of radius r centered at the origin in the positive direction. Since the logarithm changes by $\imath 2\pi$, the value of $f(z)$ changes by the factor $\mathrm{e}^{\imath 2\pi r^2}$. There are branch points at the origin and infinity. The function has an infinite number of branches.

Example 7.9.13 *Construct a branch of*

$$f(z) = \left(z^2 + 1\right)^{1/3}$$

such that

$$f(0) = \frac{1}{2}\left(-1 + \imath\sqrt{3}\right).$$

First we factor $f(z)$.

$$f(z) = (z - \imath)^{1/3}(z + \imath)^{1/3}$$

There are branch points at $z = \pm\imath$. Figure 7.30 shows one way to introduce branch cuts.

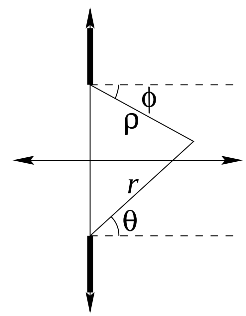


Figure 7.30: Branch Cuts for $f(z) = \left(z^2 + 1\right)^{1/3}$.

Since it is not possible to walk around any branch point, these cuts make the function single valued. We introduce the coordinates:

$$z - \imath = \rho\,\mathrm{e}^{\imath\phi}, \quad z + \imath = r\,\mathrm{e}^{\imath\theta}.$$

$$\begin{aligned} f(z) &= \left(\rho\,\mathrm{e}^{\imath\phi}\right)^{1/3}\left(r\,\mathrm{e}^{\imath\theta}\right)^{1/3} \\ &= \sqrt[3]{\rho r}\,\mathrm{e}^{\imath(\phi+\theta)/3} \end{aligned}$$

The condition

$$f(0) = \frac{1}{2}\left(-1 + \imath\sqrt{3}\right) = \mathrm{e}^{\imath(2\pi/3 + 2\pi n)}$$

can be stated

$$\sqrt[3]{1}\, e^{\imath(\phi+\theta)/3} = e^{\imath(2\pi/3+2\pi n)}$$
$$\phi + \theta = 2\pi + 6\pi n$$

The angles must be defined to satisfy this relation. One choice is

$$\boxed{\frac{\pi}{2} < \phi < \frac{5\pi}{2}, \quad -\frac{\pi}{2} < \theta < \frac{3\pi}{2}.}$$

Principal branches. We construct the principal branch of the logarithm by putting a branch cut on the negative real axis choose $z = r\, e^{\imath\theta}$, $\theta \in (-\pi, \pi)$. Thus the principal branch of the logarithm is

$$\operatorname{Log} z = \ln r + \imath\theta, \qquad -\pi < \theta < \pi.$$

Note that the if x is a negative real number, (and thus lies on the branch cut), then $\operatorname{Log} x$ is undefined.

The principal branch of z^α is

$$z^\alpha = e^{\alpha \operatorname{Log} z}.$$

Note that there is a branch cut on the negative real axis.

$$-\alpha\pi < \arg\left(e^{\alpha \operatorname{Log} z}\right) < \alpha\pi$$

The principal branch of the $z^{1/2}$ is denoted $\sqrt{z}$. The principal branch of $z^{1/n}$ is denoted $\sqrt[n]{z}$.

Example 7.9.14 *Construct $\sqrt{1-z^2}$, the principal branch of $(1-z^2)^{1/2}$.*

First note that since $(1-z^2)^{1/2} = (1-z)^{1/2}(1+z)^{1/2}$ there are branch points at $z = 1$ and $z = -1$. The principal branch of the square root has a branch cut on the negative real axis. $1 - z^2$ is a negative real number for $z \in (-\infty \ldots -1) \cup (1 \ldots \infty)$. Thus we put branch cuts on $(-\infty \ldots -1]$ and $[1 \ldots \infty)$.

7.10 Exercises

Cartesian and Modulus-Argument Form

Exercise 7.1
Find the image of the strip $2 < x < 3$ under the mapping $w = f(z) = z^2$. Does the image constitute a domain?
Hint, Solution

Exercise 7.2
For a given real number ϕ, $0 \leq \phi < 2\pi$, find the image of the sector $0 \leq \arg(z) < \phi$ under the transformation $w = z^4$. How large should ϕ be so that the w plane is covered exactly once?
Hint, Solution

Trigonometric Functions

Exercise 7.3
In Cartesian coordinates, $z = x + \imath y$, write $\sin(z)$ in Cartesian and modulus-argument form.
Hint, Solution

Exercise 7.4
Show that e^z is nonzero for all finite z.
Hint, Solution

Exercise 7.5
Show that

$$\left|\mathrm{e}^{z^2}\right| \leq \mathrm{e}^{|z|^2}.$$

When does equality hold?
Hint, Solution

Exercise 7.6
Solve $\coth(z) = 1$.
Hint, Solution

Exercise 7.7
Solve $2 \in 2^z$. That is, for what values of z is 2 one of the values of 2^z? Derive this result then verify your answer by evaluating 2^z for the solutions that your find.
Hint, Solution

Exercise 7.8
Solve $1 \in 1^z$. That is, for what values of z is 1 one of the values of 1^z? Derive this result then verify your answer by evaluating 1^z for the solutions that your find.
Hint, Solution

Logarithmic Identities

Exercise 7.9
Show that if $\Re(z_1) > 0$ and $\Re(z_2) > 0$ then

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$$

and illustrate that this relationship does not hold in general.
Hint, Solution

Exercise 7.10
Find the fallacy in the following arguments:

1. $\log(-1) = \log\left(\frac{1}{-1}\right) = \log(1) - \log(-1) = -\log(-1)$, therefore, $\log(-1) = 0$.
2. $1 = 1^{1/2} = ((-1)(-1))^{1/2} = (-1)^{1/2}(-1)^{1/2} = \imath\imath = -1$, therefore, $1 = -1$.

Hint, Solution

Exercise 7.11
Write the following expressions in modulus-argument or Cartesian form. Denote any multi-valuedness explicitly.

$$2^{2/5}, \quad 3^{1+\imath}, \quad \left(\sqrt{3} - \imath\right)^{1/4}, \quad 1^{\imath/4}.$$

Hint, Solution

Exercise 7.12
Solve $\cos z = 69$.
Hint, Solution

Exercise 7.13
Solve $\cot z = \imath 47$.
Hint, Solution

Exercise 7.14
Determine all values of

1. $\log(-\imath)$
2. $(-\imath)^{-\imath}$
3. 3^{π}
4. $\log(\log(\imath))$

and plot them in the complex plane.
Hint, Solution

Exercise 7.15
Evaluate and plot the following in the complex plane:

1. $(\cosh(\imath\pi))^{\imath 2}$
2. $\log\left(\frac{1}{1+\imath}\right)$
3. $\arctan(\imath 3)$

Hint, Solution

Exercise 7.16
Determine all values of $\imath^\imath$ and $\log\left((1+\imath)^{\imath\pi}\right)$ and plot them in the complex plane.
Hint, Solution

Exercise 7.17
Find all z for which

1. $\mathrm{e}^z = \imath$
2. $\cos z = \sin z$
3. $\tan^2 z = -1$

Hint, Solution

Exercise 7.18
Prove the following identities and identify the branch points of the functions in the extended complex plane.

1. $\arctan(z) = \frac{\imath}{2}\log\left(\frac{\imath+z}{\imath-z}\right)$
2. $\operatorname{arctanh}(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$
3. $\operatorname{arccosh}(z) = \log\left(z + \left(z^2-1\right)^{1/2}\right)$

Hint, Solution

Branch Points and Branch Cuts

Exercise 7.19
Identify the branch points of the function

$$f(z) = \log\left(\frac{z(z+1)}{z-1}\right)$$

and introduce appropriate branch cuts to ensure that the function is single-valued.
Hint, Solution

Exercise 7.20
Identify all the branch points of the function

$$w = f(z) = \left(z^3 + z^2 - 6z\right)^{1/2}$$

in the extended complex plane. Give a polar description of $f(z)$ and specify branch cuts so that your choice of angles gives a single-valued function that is continuous at $z = -1$ with $f(-1) = -\sqrt{6}$. Sketch the branch cuts in the stereographic projection.
Hint, Solution

Exercise 7.21
Consider the mapping $w = f(z) = z^{1/3}$ and the inverse mapping $z = g(w) = w^3$.

1. Describe the multiple-valuedness of $f(z)$.
2. Describe a region of the w-plane that $g(w)$ maps one-to-one to the whole z-plane.
3. Describe and attempt to draw a Riemann surface on which $f(z)$ is single-valued and to which $g(w)$ maps one-to-one. Comment on the misleading nature of your picture.
4. Identify the branch points of $f(z)$ and introduce a branch cut to make $f(z)$ single-valued.

Hint, Solution

Exercise 7.22
Determine the branch points of the function

$$f(z) = \left(z^3 - 1\right)^{1/2}.$$

Construct cuts and define a branch so that $z = 0$ and $z = -1$ do not lie on a cut, and such that $f(0) = -\imath$. What is $f(-1)$ for this branch?
Hint, Solution

Exercise 7.23
Determine the branch points of the function

$$w(z) = ((z-1)(z-6)(z+2))^{1/2}$$

Construct cuts and define a branch so that $z = 4$ does not lie on a cut, and such that $w = \imath 6$ when $z = 4$.
Hint, Solution

Exercise 7.24
Give the number of branches and locations of the branch points for the functions

1. $\cos\left(z^{1/2}\right)$

2. $(z+\imath)^{-z}$

Hint, Solution

Exercise 7.25
Find the branch points of the following functions in the extended complex plane, (the complex plane including the point at infinity).

1. $\left(z^2+1\right)^{1/2}$

2. $\left(z^3-z\right)^{1/2}$

3. $\log\left(z^2-1\right)$

4. $\log\left(\frac{z+1}{z-1}\right)$

Introduce branch cuts to make the functions single valued.
Hint, Solution

Exercise 7.26
Find all branch points and introduce cuts to make the following functions single-valued: For the first function, choose cuts so that there is no cut within the disk $|z| < 2$.

1. $f(z) = \left(z^3 + 8\right)^{1/2}$
2. $f(z) = \log\left(5 + \left(\dfrac{z+1}{z-1}\right)^{1/2}\right)$
3. $f(z) = (z + \imath 3)^{1/2}$

Hint, Solution

Exercise 7.27
Let $f(z)$ have branch points at $z = 0$ and $z = \pm\imath$, but nowhere else in the extended complex plane. How does the value and argument of $f(z)$ change while traversing the contour in Figure 7.31? Does the branch cut in Figure 7.31 make the function single-valued?

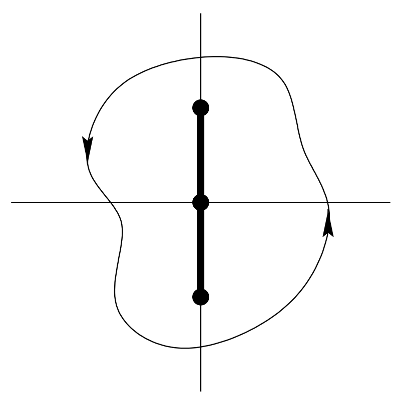


Figure 7.31: Contour around the branch points and the branch cut.

Hint, Solution

Exercise 7.28
Let $f(z)$ be analytic except for no more than a countably infinite number of singularities. Suppose that $f(z)$ has only one branch point in the finite complex plane. Does $f(z)$ have a branch point at infinity? Now suppose that $f(z)$ has two or more branch points in the finite complex plane. Does $f(z)$ have a branch point at infinity?
Hint, Solution

Exercise 7.29
Find all branch points of $(z^4+1)^{1/4}$ in the extended complex plane. Which of the branch cuts in Figure 7.32 make the function single-valued.

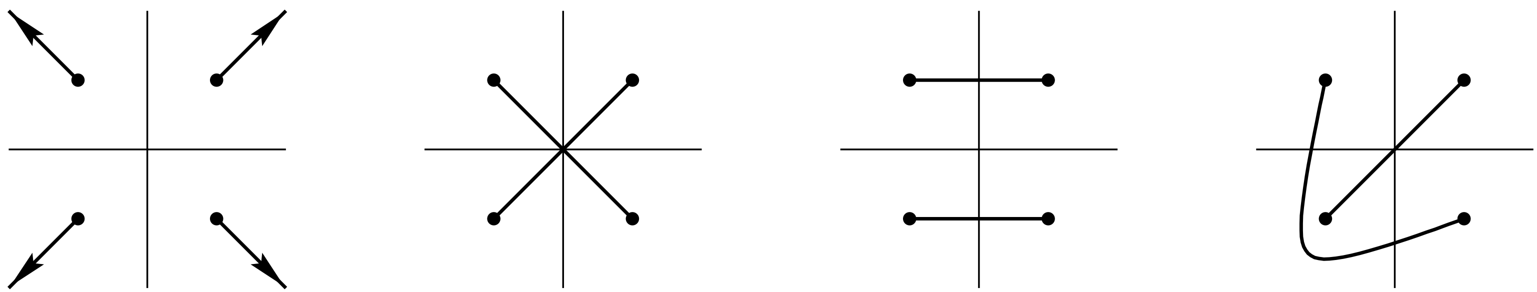


Figure 7.32: Four candidate sets of branch cuts for $(z^4+1)^{1/4}$.

Hint, Solution

Exercise 7.30
Find the branch points of

$$f(z) = \left(\frac{z}{z^2+1}\right)^{1/3}$$

in the extended complex plane. Introduce branch cuts that make the function single-valued and such that the function

is defined on the positive real axis. Define a branch such that $f(1) = 1/\sqrt[3]{2}$. Write down an explicit formula for the value of the branch. What is $f(1+\imath)$? What is the value of $f(z)$ on either side of the branch cuts?
Hint, Solution

Exercise 7.31
Find all branch points of

$$f(z) = ((z-1)(z-2)(z-3))^{1/2}$$

in the extended complex plane. Which of the branch cuts in Figure 7.33 will make the function single-valued. Using the first set of branch cuts in this figure define a branch on which $f(0) = \imath\sqrt{6}$. Write out an explicit formula for the value of the function on this branch.

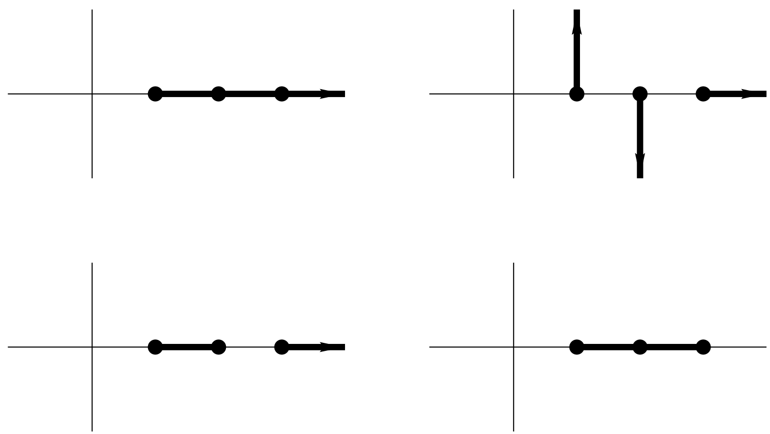


Figure 7.33: Four candidate sets of branch cuts for $((z-1)(z-2)(z-3))^{1/2}$.

Hint, Solution

Exercise 7.32
Determine the branch points of the function

$$w = \left(\left(z^2 - 2\right)(z+2)\right)^{1/3}.$$

Construct and define a branch so that the resulting cut is one line of finite extent and $w(2) = 2$. What is $w(-3)$ for this branch? What are the limiting values of w on either side of the branch cut?
Hint, Solution

Exercise 7.33
Construct the principal branch of $\arccos(z)$. ($\mathrm{Arccos}(z)$ has the property that if $x \in [-1, 1]$ then $\mathrm{Arccos}(x) \in [0, \pi]$. In particular, $\mathrm{Arccos}(0) = \frac{\pi}{2}$).
Hint, Solution

Exercise 7.34
Find the branch points of $\left(z^{1/2} - 1\right)^{1/2}$ in the finite complex plane. Introduce branch cuts to make the function single-valued.
Hint, Solution

Exercise 7.35
For the linkage illustrated in Figure 7.34, use complex variables to outline a scheme for expressing the angular position, velocity and acceleration of arm c in terms of those of arm a. (You needn't work out the equations.)
Hint, Solution

Exercise 7.36
Find the image of the strip $|\Re(z)| < 1$ and of the strip $1 < \Im(z) < 2$ under the transformations:

1. $w = 2z^2$
2. $w = \frac{z+1}{z-1}$

Hint, Solution

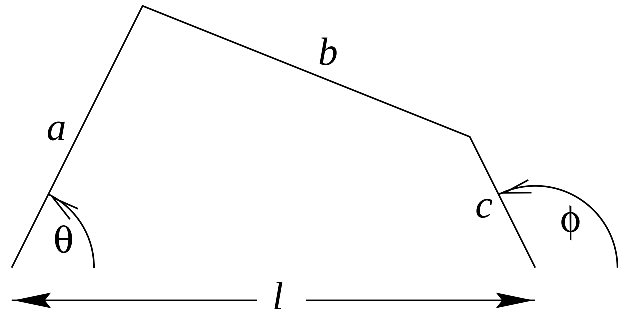


Figure 7.34: A linkage.

Exercise 7.37
Locate and classify all the singularities of the following functions:

1. $\dfrac{(z+1)^{1/2}}{z+2}$
2. $\cos\left(\dfrac{1}{1+z}\right)$
3. $\dfrac{1}{(1-\mathrm{e}^z)^2}$

In each case discuss the possibility of a singularity at the point ∞.
Hint, Solution

Exercise 7.38
Describe how the mapping $w = \sinh(z)$ transforms the infinite strip $-\infty < x < \infty$, $0 < y < \pi$ into the w-plane. Find cuts in the w-plane which make the mapping continuous both ways. What are the images of the lines (a) $y = \pi/4$; (b) $x = 1$?
Hint, Solution

7.11 Hints

Cartesian and Modulus-Argument Form

Hint 7.1

Hint 7.2

Trigonometric Functions

Hint 7.3
Recall that $\sin(z) = \frac{1}{\imath 2}\left(e^{\imath z} - e^{-\imath z}\right)$. Use Result 6.3.1 to convert between Cartesian and modulus-argument form.

Hint 7.4
Write e^z in polar form.

Hint 7.5
The exponential is an increasing function for real variables.

Hint 7.6
Write the hyperbolic cotangent in terms of exponentials.

Hint 7.7
Write out the multi-valuedness of 2^z. There is a doubly-infinite set of solutions to this problem.

Hint 7.8
Write out the multi-valuedness of 1^z.

Logarithmic Identities

Hint 7.9

Hint 7.10
Write out the multi-valuedness of the expressions.

Hint 7.11
Do the exponentiations in polar form.

Hint 7.12
Write the cosine in terms of exponentials. Multiply by $\mathrm{e}^{\imath z}$ to get a quadratic equation for $\mathrm{e}^{\imath z}$.

Hint 7.13
Write the cotangent in terms of exponentials. Get a quadratic equation for $\mathrm{e}^{\imath z}$.

Hint 7.14

Hint 7.15

Hint 7.16
$\imath^\imath$ has an infinite number of real, positive values. $\imath^\imath = \mathrm{e}^{\imath \log \imath}$. $\log\left((1+\imath)^{\imath\pi}\right)$ has a doubly infinite set of values. $\log\left((1+\imath)^{\imath\pi}\right) = \log(\exp(\imath\pi\log(1+\imath)))$.

Hint 7.17

Hint 7.18

Branch Points and Branch Cuts

Hint 7.19

Hint 7.20

Hint 7.21

Hint 7.22

Hint 7.23

Hint 7.24

Hint 7.25

1. $(z^2+1)^{1/2} = (z-\imath)^{1/2}(z+\imath)^{1/2}$

2. $(z^3-z)^{1/2} = z^{1/2}(z-1)^{1/2}(z+1)^{1/2}$

3. $\log(z^2-1) = \log(z-1) + \log(z+1)$

4. $\log\left(\frac{z+1}{z-1}\right) = \log(z+1) - \log(z-1)$

Hint 7.26

Hint 7.27

Reverse the orientation of the contour so that it encircles infinity and does not contain any branch points.

Hint 7.28
Consider a contour that encircles all the branch points in the finite complex plane. Reverse the orientation of the contour so that it contains the point at infinity and does not contain any branch points in the finite complex plane.

Hint 7.29
Factor the polynomial. The argument of $z^{1/4}$ changes by $\pi/2$ on a contour that goes around the origin once in the positive direction.

Hint 7.30

Hint 7.31
To define the branch, define angles from each of the branch points in the finite complex plane.

Hint 7.32

Hint 7.33

Hint 7.34

Hint 7.35

Hint 7.36

Hint 7.37

Hint 7.38

7.12 Solutions

Cartesian and Modulus-Argument Form

Solution 7.1

Let $w = u + \imath v$. We consider the strip $2 < x < 3$ as composed of vertical lines. Consider the vertical line: $z = c + \imath y$, $y \in \mathbb{R}$ for constant c. We find the image of this line under the mapping.

$$w = (c + \imath y)^2$$
$$w = c^2 - y^2 + \imath 2cy$$
$$u = c^2 - y^2, \quad v = 2cy$$

This is a parabola that opens to the left. We can parameterize the curve in terms of v.

$$u = c^2 - \frac{1}{4c^2} v^2, \quad v \in \mathbb{R}$$

The boundaries of the region, $x = 2$ and $x = 3$, are respectively mapped to the parabolas:

$$u = 4 - \frac{1}{16} v^2, \quad v \in \mathbb{R} \quad \text{and} \quad u = 9 - \frac{1}{36} v^2, \quad v \in \mathbb{R}$$

We write the image of the mapping in set notation.

$$\boxed{\left\{ w = u + \imath v : v \in \mathbb{R} \text{ and } 4 - \frac{1}{16} v^2 < u < 9 - \frac{1}{36} v^2 \right\}.}$$

See Figure 7.35 for depictions of the strip and its image under the mapping. The mapping is one-to-one. Since the image of the strip is open and connected, it is a domain.

Solution 7.2

We write the mapping $w = z^4$ in polar coordinates.

$$w = z^4 = \left(r \, \mathrm{e}^{\imath\theta} \right)^4 = r^4 \, \mathrm{e}^{\imath 4\theta}$$

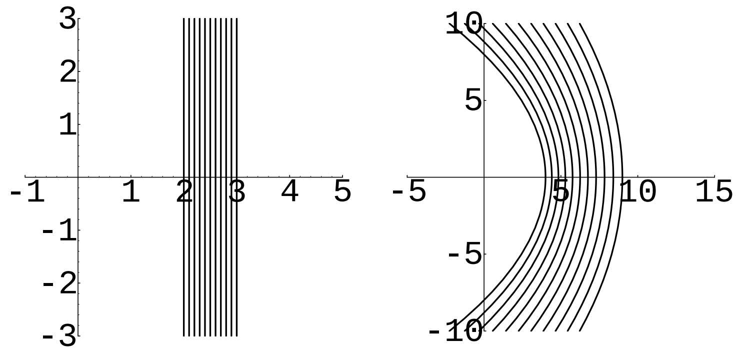


Figure 7.35: The domain $2 < x < 3$ and its image under the mapping $w = z^2$.

Thus we see that

$$w : \{r\,\mathrm{e}^{\imath\theta} \mid r \geq 0, 0 \leq \theta < \phi\} \to \{r^4\,\mathrm{e}^{\imath 4\theta} \mid r \geq 0, 0 \leq \theta < \phi\} = \{r\,\mathrm{e}^{\imath\theta} \mid r \geq 0, 0 \leq \theta < 4\phi\}.$$

We can state this in terms of the argument.

$$\boxed{w : \{z \mid 0 \leq \arg(z) < \phi\} \to \{z \mid 0 \leq \arg(z) < 4\phi\}}$$

If $\phi = \pi/2$, the sector will be mapped exactly to the whole complex plane.

Trigonometric Functions

Solution 7.3

$$
\begin{aligned}
\sin z &= \frac{1}{\imath 2}\left(\mathrm{e}^{\imath z} - \mathrm{e}^{-\imath z}\right) \\
&= \frac{1}{\imath 2}\left(\mathrm{e}^{-y+\imath x} - \mathrm{e}^{y-\imath x}\right) \\
&= \frac{1}{\imath 2}\left(\mathrm{e}^{-y}(\cos x + \imath \sin x) - \mathrm{e}^{y}(\cos x - \imath \sin x)\right) \\
&= \frac{1}{2}\left(\mathrm{e}^{-y}(\sin x - \imath \cos x) + \mathrm{e}^{y}(\sin x + \imath \cos x)\right) \\
&= \sin x \cosh y + \imath \cos x \sinh y
\end{aligned}
$$

$$
\begin{aligned}
\sin z &= \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}\, \exp(\imath \arctan(\sin x \cosh y, \cos x \sinh y)) \\
&= \sqrt{\cosh^2 y - \cos^2 x}\, \exp(\imath \arctan(\sin x \cosh y, \cos x \sinh y)) \\
&= \sqrt{\frac{1}{2}\left(\cosh(2y) - \cos(2x)\right)}\, \exp(\imath \arctan(\sin x \cosh y, \cos x \sinh y))
\end{aligned}
$$

Solution 7.4
In order that e^z be zero, the modulus, e^x must be zero. Since e^x has no finite solutions, $\mathrm{e}^z = 0$ has no finite solutions.

Solution 7.5
We write the expressions in terms of Cartesian coordinates.

$$
\begin{aligned}
\left|\mathrm{e}^{z^2}\right| &= \left|\mathrm{e}^{(x+\imath y)^2}\right| \\
&= \left|\mathrm{e}^{x^2-y^2+\imath 2xy}\right| \\
&= \mathrm{e}^{x^2-y^2}
\end{aligned}
$$

$$e^{|z|^2} = e^{|x+\imath y|^2} = e^{x^2+y^2}$$

The exponential function is an increasing function for real variables. Since $x^2 - y^2 \leq x^2 + y^2$, $e^{x^2-y^2} \leq e^{x^2+y^2}$.

$$\left| e^{z^2} \right| \leq e^{|z|^2}$$

Equality holds only when $y = 0$.

Solution 7.6

$$\coth(z) = 1$$
$$\frac{(e^z + e^{-z})/2}{(e^z - e^{-z})/2} = 1$$
$$e^z + e^{-z} = e^z - e^{-z}$$
$$e^{-z} = 0$$

There are no solutions.

Solution 7.7

We write out the multi-valuedness of 2^z.

$$2 \in 2^z$$
$$e^{\ln 2} \in e^{z \log(2)}$$
$$e^{\ln 2} \in \{e^{z(\ln(2)+\imath 2\pi n)} \mid n \in \mathbb{Z}\}$$
$$\ln 2 \in z\{\ln 2 + \imath 2\pi n + \imath 2\pi m \mid m, n \in \mathbb{Z}\}$$
$$\boxed{z = \left\{ \frac{\ln(2) + \imath 2\pi m}{\ln(2) + \imath 2\pi n} \mid m, n \in \mathbb{Z} \right\}}$$

We verify this solution. Consider m and n to be fixed integers. We express the multi-valuedness in terms of k.

$$\begin{aligned} 2^{(\ln(2)+\imath 2\pi m)/(\ln(2)+\imath 2\pi n)} &= e^{(\ln(2)+\imath 2\pi m)/(\ln(2)+\imath 2\pi n) \log(2)} \\ &= e^{(\ln(2)+\imath 2\pi m)/(\ln(2)+\imath 2\pi n)(\ln(2)+\imath 2\pi k)} \end{aligned}$$

For $k = n$, this has the value, $\mathrm{e}^{\ln(2)+\imath 2\pi m} = \mathrm{e}^{\ln(2)} = 2$.

Solution 7.8
We write out the multi-valuedness of 1^z.

$$1 \in 1^z$$
$$1 \in \mathrm{e}^{z \log(1)}$$
$$1 \in \{\mathrm{e}^{\imath z 2\pi n} \mid n \in \mathbb{Z}\}$$

The element corresponding to $n = 0$ is $\mathrm{e}^0 = 1$. Thus $1 \in 1^z$ has the solutions,

$$\boxed{z \in \mathbb{C}.}$$

That is, z may be any complex number. We verify this solution.

$$1^z = \mathrm{e}^{z \log(1)} = \mathrm{e}^{\imath z 2\pi n}$$

For $n = 0$, this has the value 1.

Logarithmic Identities

Solution 7.9
We write the relationship in terms of the natural logarithm and the principal argument.

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$$
$$\ln|z_1 z_2| + \imath \operatorname{Arg}(z_1 z_2) = \ln|z_1| + \imath \operatorname{Arg}(z_1) + \ln|z_2| + \imath \operatorname{Arg}(z_2)$$
$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$$

$\Re(z_k) > 0$ implies that $\operatorname{Arg}(z_k) \in (-\pi/2 \ldots \pi/2)$. Thus $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) \in (-\pi \ldots \pi)$. In this case the relationship holds.

The relationship does not hold in general because $\operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ is not necessarily in the interval $(-\pi \ldots \pi]$. Consider $z_1 = z_2 = -1$.

$$\operatorname{Arg}((-1)(-1)) = \operatorname{Arg}(1) = 0, \qquad \operatorname{Arg}(-1) + \operatorname{Arg}(-1) = 2\pi$$
$$\operatorname{Log}((-1)(-1)) = \operatorname{Log}(1) = 0, \qquad \operatorname{Log}(-1) + \operatorname{Log}(-1) = \imath 2\pi$$

Solution 7.10

1. The algebraic manipulations are fine. We write out the multi-valuedness of the logarithms.

$$\log(-1) = \log\left(\frac{1}{-1}\right) = \log(1) - \log(-1) = -\log(-1)$$

$$\begin{aligned}\{\imath\pi + \imath 2\pi n : n \in \mathbb{Z}\} &= \{\imath\pi + \imath 2\pi n : n \in \mathbb{Z}\} \\ &= \{\imath 2\pi n : n \in \mathbb{Z}\} - \{\imath\pi + \imath 2\pi n : n \in \mathbb{Z}\} = \{-\imath\pi - \imath 2\pi n : n \in \mathbb{Z}\}\end{aligned}$$

Thus $\log(-1) = -\log(-1)$. However this does not imply that $\log(-1) = 0$. This is because the logarithm is a set-valued function $\log(-1) = -\log(-1)$ is really saying:

$$\{\imath\pi + \imath 2\pi n : n \in \mathbb{Z}\} = \{-\imath\pi - \imath 2\pi n : n \in \mathbb{Z}\}$$

2. We consider

$$1 = 1^{1/2} = ((-1)(-1))^{1/2} = (-1)^{1/2}(-1)^{1/2} = \imath\imath = -1.$$

There are three multi-valued expressions above.

$$\begin{aligned}1^{1/2} &= \pm 1 \\ ((-1)(-1))^{1/2} &= \pm 1 \\ (-1)^{1/2}(-1)^{1/2} &= (\pm\imath)(\pm\imath) = \pm 1\end{aligned}$$

Thus we see that the first and fourth equalities are incorrect.

$$1 \neq 1^{1/2}, \quad (-1)^{1/2}(-1)^{1/2} \neq \imath\imath$$

Solution 7.11

$$\begin{aligned}2^{2/5} &= 4^{1/5} \\ &= \sqrt[5]{4}\,1^{1/5} \\ &= \sqrt[5]{4}\,\mathrm{e}^{\imath 2n\pi/5}, \quad n = 0, 1, 2, 3, 4\end{aligned}$$

$$
\begin{aligned}
3^{1+\imath} &= \mathrm{e}^{(1+\imath)\log 3} \\
&= \mathrm{e}^{(1+\imath)(\ln 3+\imath 2\pi n)} \\
&= \mathrm{e}^{\ln 3-2\pi n}\,\mathrm{e}^{\imath(\ln 3+2\pi n)}, \quad n \in \mathbb{Z}
\end{aligned}
$$

$$
\begin{aligned}
\left(\sqrt{3}-\imath\right)^{1/4} &= \left(2\,\mathrm{e}^{-\imath\pi/6}\right)^{1/4} \\
&= \sqrt[4]{2}\,\mathrm{e}^{-\imath\pi/24}\,1^{1/4} \\
&= \sqrt[4]{2}\,\mathrm{e}^{\imath(\pi n/2-\pi/24)}, \quad n = 0, 1, 2, 3
\end{aligned}
$$

$$
\begin{aligned}
1^{\imath/4} &= \mathrm{e}^{(\imath/4)\log 1} \\
&= \mathrm{e}^{(\imath/4)(\imath 2\pi n)} \\
&= \mathrm{e}^{-\pi n/2}, \quad n \in \mathbb{Z}
\end{aligned}
$$

Solution 7.12

$$\cos z = 69$$
$$\frac{e^{\imath z} + e^{-\imath z}}{2} = 69$$
$$e^{\imath 2z} - 138\, e^{\imath z} + 1 = 0$$
$$e^{\imath z} = \frac{1}{2}\left(138 \pm \sqrt{138^2 - 4}\right)$$
$$z = -\imath \log\left(69 \pm 2\sqrt{1190}\right)$$
$$z = -\imath\left(\ln\left(69 \pm 2\sqrt{1190}\right) + \imath 2\pi n\right)$$
$$\boxed{z = 2\pi n - \imath \ln\left(69 \pm 2\sqrt{1190}\right), \quad n \in \mathbb{Z}}$$

Solution 7.13

$$
\begin{gathered}
\cot z = \imath 47 \\
\frac{\left(\mathrm{e}^{\imath z} + \mathrm{e}^{-\imath z}\right)/2}{\left(\mathrm{e}^{\imath z} - \mathrm{e}^{-\imath z}\right)/(\imath 2)} = \imath 47 \\
\mathrm{e}^{\imath z} + \mathrm{e}^{-\imath z} = 47\left(\mathrm{e}^{\imath z} - \mathrm{e}^{-\imath z}\right) \\
46\,\mathrm{e}^{\imath 2z} - 48 = 0 \\
\imath 2z = \log\frac{24}{23} \\
z = -\frac{\imath}{2}\log\frac{24}{23} \\
z = -\frac{\imath}{2}\left(\ln\frac{24}{23} + \imath 2\pi n\right), \quad n \in \mathbb{Z} \\
\boxed{z = \pi n - \frac{\imath}{2}\ln\frac{24}{23}, \quad n \in \mathbb{Z}}
\end{gathered}
$$

Solution 7.14

1.

$$
\begin{aligned}
\log(-\imath) &= \ln|-\imath| + \imath\arg(-\imath) \\
&= \ln(1) + \imath\left(-\frac{\pi}{2} + 2\pi n\right), \quad n \in \mathbb{Z}
\end{aligned}
$$

$$
\boxed{\log(-\imath) = -\imath\frac{\pi}{2} + \imath 2\pi n, \quad n \in \mathbb{Z}}
$$

These are equally spaced points in the imaginary axis. See Figure 7.36.

2.

$$
\begin{aligned}
(-\imath)^{-\imath} &= \mathrm{e}^{-\imath\log(-\imath)} \\
&= \mathrm{e}^{-\imath(-\imath\pi/2 + \imath 2\pi n)}, \quad n \in \mathbb{Z}
\end{aligned}
$$

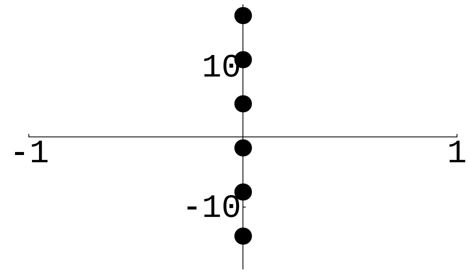


Figure 7.36: The values of $\log(-\imath)$.

$$\boxed{(-\imath)^{-\imath} = \mathrm{e}^{-\pi/2+2\pi n}, \quad n \in \mathbb{Z}}$$

These are points on the positive real axis with an accumulation point at the origin. See Figure 7.37.

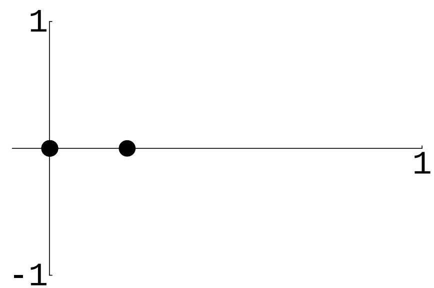


Figure 7.37: The values of $(-\imath)^{-\imath}$.

3.

$$
\begin{aligned}
3^{\pi} &= \mathrm{e}^{\pi \log(3)} \\
&= \mathrm{e}^{\pi(\ln(3)+\imath \arg(3))}
\end{aligned}
$$

$$\boxed{3^\pi = e^{\pi(\ln(3)+\imath 2\pi n)}, \quad n \in \mathbb{Z}}$$

These points all lie on the circle of radius $|e^\pi|$ centered about the origin in the complex plane. See Figure 7.38.

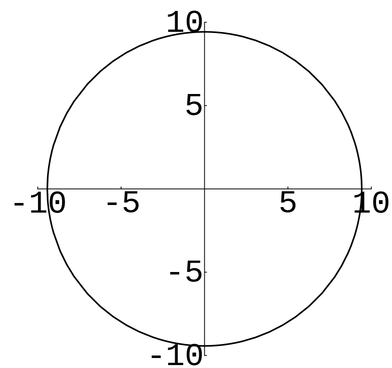


Figure 7.38: The values of 3^π.

4.

$$\begin{aligned}
\log(\log(\imath)) &= \log\left(\imath\left(\frac{\pi}{2} + 2\pi m\right)\right), \quad m \in \mathbb{Z} \\
&= \ln\left|\frac{\pi}{2} + 2\pi m\right| + \imath \operatorname{Arg}\left(\imath\left(\frac{\pi}{2} + 2\pi m\right)\right) + \imath 2\pi n, \quad m, n \in \mathbb{Z} \\
&= \ln\left|\frac{\pi}{2} + 2\pi m\right| + \imath \operatorname{sign}(1 + 4m)\frac{\pi}{2} + \imath 2\pi n, \quad m, n \in \mathbb{Z}
\end{aligned}$$

These points all lie in the right half-plane. See Figure 7.39.

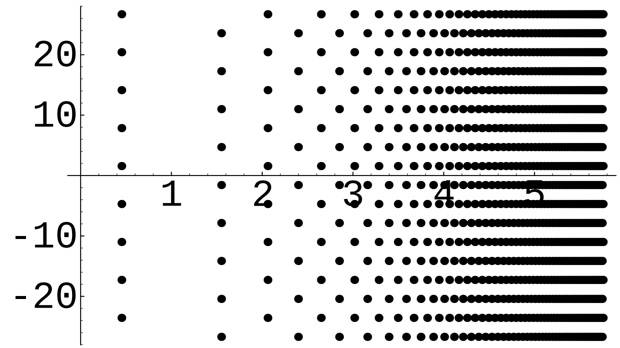


Figure 7.39: The values of $\log(\log(\imath))$.

Solution 7.15

1.

$$
\begin{aligned}
(\cosh(\imath\pi))^{\imath 2} &= \left(\frac{\mathrm{e}^{\imath\pi} + \mathrm{e}^{-\imath\pi}}{2}\right)^{\imath 2} \\
&= (-1)^{\imath 2} \\
&= \mathrm{e}^{\imath 2 \log(-1)} \\
&= \mathrm{e}^{\imath 2(\ln(1) + \imath\pi + \imath 2\pi n)}, \quad n \in \mathbb{Z} \\
&= \mathrm{e}^{-2\pi(1+2n)}, \quad n \in \mathbb{Z}
\end{aligned}
$$

These are points on the positive real axis with an accumulation point at the origin. See Figure 7.40.

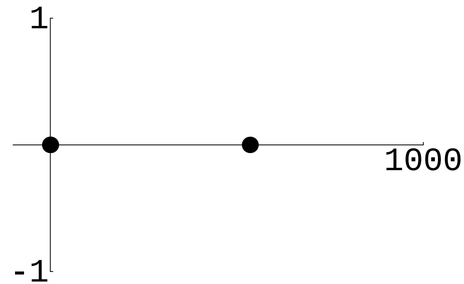


Figure 7.40: The values of $(\cosh(\imath\pi))^{\imath 2}$.

2.

$$
\begin{aligned}
\log\left(\frac{1}{1+\imath}\right) &= -\log(1+\imath) \\
&= -\log\left(\sqrt{2}\,\mathrm{e}^{\imath\pi/4}\right) \\
&= -\frac{1}{2}\ln(2) - \log\left(\mathrm{e}^{\imath\pi/4}\right) \\
&= -\frac{1}{2}\ln(2) - \imath\pi/4 + \imath 2\pi n, \quad n \in \mathbb{Z}
\end{aligned}
$$

These are points on a vertical line in the complex plane. See Figure 7.41.

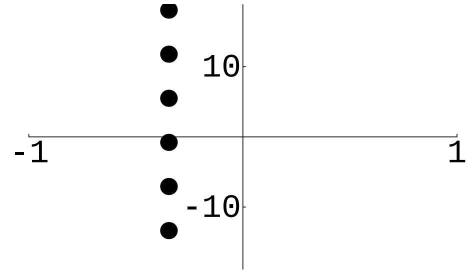


Figure 7.41: The values of $\log\left(\frac{1}{1+\imath}\right)$.

3.

$$
\begin{aligned}
\arctan(\imath 3) &= \frac{1}{\imath 2} \log\left(\frac{\imath - \imath 3}{\imath + \imath 3}\right) \\
&= \frac{1}{\imath 2} \log\left(-\frac{1}{2}\right) \\
&= \frac{1}{\imath 2} \left(\ln\left(\frac{1}{2}\right) + \imath\pi + \imath 2\pi n\right), \quad n \in \mathbb{Z} \\
&= \frac{\pi}{2} + \pi n + \frac{\imath}{2} \ln(2)
\end{aligned}
$$

These are points on a horizontal line in the complex plane. See Figure 7.42.

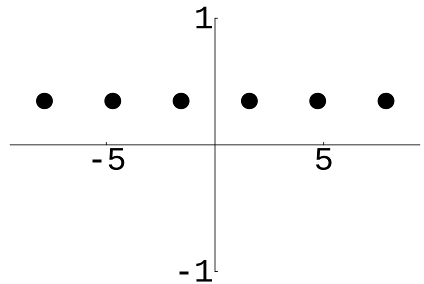


Figure 7.42: The values of $\arctan(\imath 3)$.

Solution 7.16

$$
\begin{aligned}
\imath^\imath &= \mathrm{e}^{\imath \log(\imath)} \\
&= \mathrm{e}^{\imath(\ln|\imath| + \imath \operatorname{Arg}(\imath) + \imath 2\pi n)}, \quad n \in \mathbb{Z} \\
&= \mathrm{e}^{\imath(\imath\pi/2 + \imath 2\pi n)}, \quad n \in \mathbb{Z} \\
&= \mathrm{e}^{-\pi(1/2 + 2n)}, \quad n \in \mathbb{Z}
\end{aligned}
$$

These are points on the positive real axis. There is an accumulation point at $z = 0$. See Figure 7.43.

$$
\begin{aligned}
\log\left((1+\imath)^{\imath\pi}\right) &= \log\left(\mathrm{e}^{\imath\pi \log(1+\imath)}\right) \\
&= \imath\pi \log(1+\imath) + \imath 2\pi n, \quad n \in \mathbb{Z} \\
&= \imath\pi\left(\ln|1+\imath| + \imath \operatorname{Arg}(1+\imath) + \imath 2\pi m\right) + \imath 2\pi n, \quad m, n \in \mathbb{Z} \\
&= \imath\pi\left(\frac{1}{2}\ln 2 + \imath\frac{\pi}{4} + \imath 2\pi m\right) + \imath 2\pi n, \quad m, n \in \mathbb{Z} \\
&= -\pi^2\left(\frac{1}{4} + 2m\right) + \imath\pi\left(\frac{1}{2}\ln 2 + 2n\right), \quad m, n \in \mathbb{Z}
\end{aligned}
$$

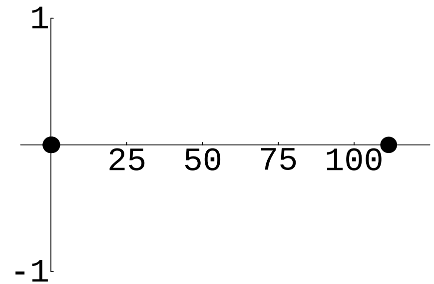


Figure 7.43: The values of $\imath^\imath$.

See Figure 7.44 for a plot.

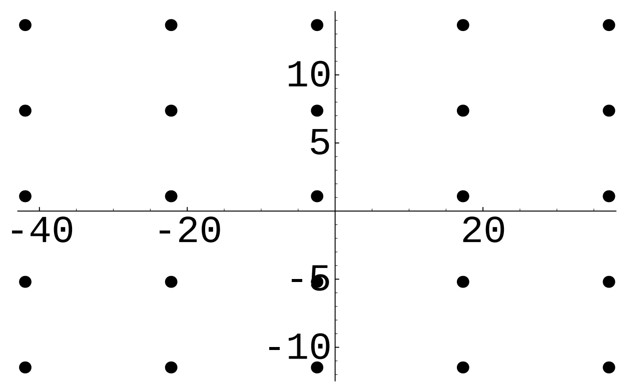


Figure 7.44: The values of $\log\left((1+\imath)^{\imath\pi}\right)$.

Solution 7.17

1.

$$
\begin{aligned}
\mathrm{e}^z &= \imath \\
z &= \log \imath \\
z &= \ln |\imath| + \imath \arg(\imath) \\
z &= \ln(1) + \imath \left(\frac{\pi}{2} + 2\pi n \right), \quad n \in \mathbb{Z} \\
&\boxed{z = \imath \frac{\pi}{2} + \imath 2\pi n, \quad n \in \mathbb{Z}}
\end{aligned}
$$

2. We can solve the equation by writing the cosine and sine in terms of the exponential.

$$
\begin{aligned}
\cos z &= \sin z \\
\frac{\mathrm{e}^{\imath z} + \mathrm{e}^{-\imath z}}{2} &= \frac{\mathrm{e}^{\imath z} - \mathrm{e}^{-\imath z}}{\imath 2} \\
(1 + \imath)\, \mathrm{e}^{\imath z} &= (-1 + \imath)\, \mathrm{e}^{-\imath z} \\
\mathrm{e}^{\imath 2 z} &= \frac{-1 + \imath}{1 + \imath} \\
\mathrm{e}^{\imath 2 z} &= \imath \\
\imath 2 z &= \log(\imath) \\
\imath 2 z &= \imath \frac{\pi}{2} + \imath 2\pi n, \quad n \in \mathbb{Z} \\
&\boxed{z = \frac{\pi}{4} + \pi n, \quad n \in \mathbb{Z}}
\end{aligned}
$$

3.

$$
\begin{gathered}
\tan^2 z = -1 \\
\sin^2 z = -\cos^2 z \\
\cos z = \pm\imath \sin z \\
\frac{\mathrm{e}^{\imath z} + \mathrm{e}^{-\imath z}}{2} = \pm\imath \frac{\mathrm{e}^{\imath z} - \mathrm{e}^{-\imath z}}{\imath 2} \\
\mathrm{e}^{-\imath z} = -\mathrm{e}^{-\imath z} \quad \text{or} \quad \mathrm{e}^{\imath z} = -\mathrm{e}^{\imath z} \\
\mathrm{e}^{-\imath z} = 0 \quad \text{or} \quad \mathrm{e}^{\imath z} = 0 \\
\mathrm{e}^{y - \imath x} = 0 \quad \text{or} \quad \mathrm{e}^{-y + \imath x} = 0 \\
\mathrm{e}^{y} = 0 \quad \text{or} \quad \mathrm{e}^{-y} = 0 \\
\boxed{z = \emptyset}
\end{gathered}
$$

There are no solutions for finite z.

Solution 7.18

1.

$$
\begin{aligned}
w &= \arctan(z) \\
z &= \tan(w) \\
z &= \frac{\sin(w)}{\cos(w)} \\
z &= \frac{\left(e^{\imath w} - e^{-\imath w}\right)/(\imath 2)}{\left(e^{\imath w} + e^{-\imath w}\right)/2} \\
z\, e^{\imath w} + z\, e^{-\imath w} &= -\imath\, e^{\imath w} + \imath\, e^{-\imath w} \\
(\imath + z)\, e^{\imath 2 w} &= (\imath - z) \\
e^{\imath w} &= \left(\frac{\imath - z}{\imath + z}\right)^{1/2} \\
w &= -\imath \log\left(\frac{\imath - z}{\imath + z}\right)^{1/2} \\
&\boxed{\arctan(z) = \frac{\imath}{2} \log\left(\frac{\imath + z}{\imath - z}\right)}
\end{aligned}
$$

We identify the branch points of the arctangent.

$$
\arctan(z) = \frac{\imath}{2}\left(\log(\imath + z) - \log(\imath - z)\right)
$$

There are branch points at $z = \pm\imath$ due to the logarithm terms. We examine the point at infinity with the change of variables $\zeta = 1/z$.

$$
\begin{aligned}
\arctan(1/\zeta) &= \frac{\imath}{2} \log\left(\frac{\imath + 1/\zeta}{\imath - 1/\zeta}\right) \\
\arctan(1/\zeta) &= \frac{\imath}{2} \log\left(\frac{\imath\zeta + 1}{\imath\zeta - 1}\right)
\end{aligned}
$$

As $\zeta \to 0$, the argument of the logarithm term tends to -1 The logarithm does not have a branch point at that point. Since $\arctan(1/\zeta)$ does not have a branch point at $\zeta = 0$, $\arctan(z)$ does not have a branch point at infinity.

2.

$$
\begin{gathered}
w = \operatorname{arctanh}(z) \\
z = \tanh(w) \\
z = \frac{\sinh(w)}{\cosh(w)} \\
z = \frac{\left(\mathrm{e}^{w} - \mathrm{e}^{-w}\right)/2}{\left(\mathrm{e}^{w} + \mathrm{e}^{-w}\right)/2} \\
z\,\mathrm{e}^{w} + z\,\mathrm{e}^{-w} = \mathrm{e}^{w} - \mathrm{e}^{-w} \\
(z-1)\,\mathrm{e}^{2w} = -z - 1 \\
\mathrm{e}^{w} = \left(\frac{-z-1}{z-1}\right)^{1/2} \\
w = \log\left(\frac{z+1}{1-z}\right)^{1/2} \\
\boxed{\operatorname{arctanh}(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)}
\end{gathered}
$$

We identify the branch points of the hyperbolic arctangent.

$$
\operatorname{arctanh}(z) = \frac{1}{2}\left(\log(1+z) - \log(1-z)\right)
$$

There are branch points at $z = \pm 1$ due to the logarithm terms. We examine the point at infinity with the change of variables $\zeta = 1/z$.

$$\operatorname{arctanh}(1/\zeta) = \frac{1}{2}\log\left(\frac{1+1/\zeta}{1-1/\zeta}\right)$$
$$\operatorname{arctanh}(1/\zeta) = \frac{1}{2}\log\left(\frac{\zeta+1}{\zeta-1}\right)$$

As $\zeta \to 0$, the argument of the logarithm term tends to -1 The logarithm does not have a branch point at that point. Since $\operatorname{arctanh}(1/\zeta)$ does not have a branch point at $\zeta = 0$, $\operatorname{arctanh}(z)$ does not have a branch point at infinity.

3.

$$w = \operatorname{arccosh}(z)$$
$$z = \cosh(w)$$
$$z = \frac{\mathrm{e}^{w} + \mathrm{e}^{-w}}{2}$$
$$\mathrm{e}^{2w} - 2z\,\mathrm{e}^{w} + 1 = 0$$
$$\mathrm{e}^{w} = z + \left(z^2 - 1\right)^{1/2}$$
$$w = \log\left(z + \left(z^2 - 1\right)^{1/2}\right)$$
$$\boxed{\operatorname{arccosh}(z) = \log\left(z + \left(z^2 - 1\right)^{1/2}\right)}$$

We identify the branch points of the hyperbolic arc-cosine.

$$\operatorname{arccosh}(z) = \log\left(z + (z-1)^{1/2}(z+1)^{1/2}\right)$$

First we consider branch points due to the square root. There are branch points at $z = \pm 1$ due to the square root terms. If we walk around the singularity at $z = 1$ and no other singularities, the $\left(z^2 - 1\right)^{1/2}$ term changes sign. This will change the value of $\operatorname{arccosh}(z)$. The same is true for the point $z=-1$. The point at infinity is not a branch point for $(z^2-1)^{1/2}$. We factor the expression to verify this.

$$\left(z^2-1\right)^{1/2} = \left(z^2\right)^{1/2}\left(1-z^{-2}\right)^{1/2}$$

$(z^2)^{1/2}$ does not have a branch point at infinity. It is multi-valued, but it has no branch points. $(1-z^{-2})^{1/2}$ does not have a branch point at infinity, The argument of the square root function tends to unity there. In summary, there are branch points at $z=\pm 1$ due to the square root. If we walk around either one of the these branch points. the square root term will change value. If we walk around both of these points, the square root term will not change value.

Now we consider branch points due to logarithm. There may be branch points where the argument of the logarithm vanishes or tends to infinity. We see if the argument of the logarithm vanishes.

$$\begin{gathered} z+\left(z^2-1\right)^{1/2}=0 \\ z^2=z^2-1 \end{gathered}$$

$z+(z^2-1)^{1/2}$ is non-zero and finite everywhere in the complex plane. The only possibility for a branch point in the logarithm term is the point at infinity. We see if the argument of $z+(z^2-1)^{1/2}$ changes when we walk around infinity but no other singularity. We consider a circular path with center at the origin and radius greater than unity. We can either say that this path encloses the two branch points at $z=\pm 1$ and no other singularities or we can say that this path encloses the point at infinity and no other singularities. We examine the value of the argument of the logarithm on this path.

$$z+\left(z^2-1\right)^{1/2} = z+\left(z^2\right)^{1/2}\left(1-z^{-2}\right)^{1/2}$$

Neither $(z^2)^{1/2}$ nor $(1-z^{-2})^{1/2}$ changes value as we walk the path. Thus we can use the principal branch of the square root in the expression.

$$z+\left(z^2-1\right)^{1/2} = z \pm z\sqrt{1-z^{-2}} = z\left(1 \pm \sqrt{1-z^{-2}}\right)$$

First consider the "+" branch.

$$z\left(1+\sqrt{1-z^{-2}}\right)$$

As we walk the path around infinity, the argument of z changes by 2π while the argument of $\left(1+\sqrt{1-z^{-2}}\right)$ does not change. Thus the argument of $z+\left(z^2-1\right)^{1/2}$ changes by 2π when we go around infinity. This makes the value of the logarithm change by $\imath 2\pi$. There is a branch point at infinity.

First consider the "−" branch.

$$\begin{aligned}
z\left(1-\sqrt{1-z^{-2}}\right) &= z\left(1-\left(1-\frac{1}{2}z^{-2}+\mathcal{O}\left(z^{-4}\right)\right)\right)\\
&= z\left(\frac{1}{2}z^{-2}+\mathcal{O}\left(z^{-4}\right)\right)\\
&= \frac{1}{2}z^{-1}\left(1+\mathcal{O}\left(z^{-2}\right)\right)
\end{aligned}$$

As we walk the path around infinity, the argument of z^{-1} changes by -2π while the argument of $\left(1+\mathcal{O}\left(z^{-2}\right)\right)$ does not change. Thus the argument of $z+\left(z^2-1\right)^{1/2}$ changes by -2π when we go around infinity. This makes the value of the logarithm change by $-\imath 2\pi$. Again we conclude that there is a branch point at infinity.

For the sole purpose of overkill, let's repeat the above analysis from a geometric viewpoint. Again we consider the possibility of a branch point at infinity due to the logarithm. We walk along the circle shown in the first plot of Figure 7.45. Traversing this path, we go around infinity, but no other singularities. We consider the mapping $w=z+\left(z^2-1\right)^{1/2}$. Depending on the branch of the square root, the circle is mapped to one one of the contours shown in the second plot. For each branch, the argument of w changes by $\pm 2\pi$ as we traverse the circle in the z-plane. Therefore the value of $\operatorname{arccosh}(z)=\log\left(z+\left(z^2-1\right)^{1/2}\right)$ changes by $\pm\imath 2\pi$ as we traverse the circle. We again conclude that there is a branch point at infinity due to the logarithm.

To summarize: There are branch points at $z=\pm 1$ due to the square root and a branch point at infinity due to the logarithm.

Branch Points and Branch Cuts

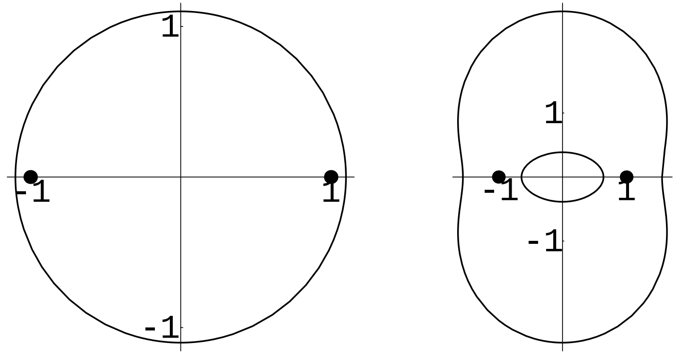


Figure 7.45: The mapping of a circle under $w = z + (z^2 - 1)^{1/2}$.

Solution 7.19

We expand the function to diagnose the branch points in the finite complex plane.

$$f(z) = \log\left(\frac{z(z+1)}{z-1}\right) = \log(z) + \log(z+1) - \log(z-1)$$

The are branch points at $z = -1, 0, 1$. Now we examine the point at infinity. We make the change of variables $z = 1/\zeta$.

$$\begin{aligned} f\left(\frac{1}{\zeta}\right) &= \log\left(\frac{(1/\zeta)(1/\zeta+1)}{(1/\zeta-1)}\right) \\ &= \log\left(\frac{1}{\zeta}\frac{(1+\zeta}{1-\zeta}\right) \\ &= \log(1+\zeta) - \log(1-\zeta) - \log(\zeta) \end{aligned}$$

$\log(\zeta)$ has a branch point at $\zeta = 0$. The other terms do not have branch points there. Since $f(1/\zeta)$ has a branch point at $\zeta = 0$ $f(z)$ has a branch point at infinity.

Note that in walking around either $z = -1$ or $z = 0$ once in the positive direction, the argument of $z(z+1)/(z-1)$ changes by 2π. In walking around $z = 1$, the argument of $z(z+1)/(z-1)$ changes by -2π. This argument does not change if we walk around both $z = 0$ and $z = 1$. Thus we put a branch cut between $z = 0$ and $z = 1$. Next be put a branch cut between $z = -1$ and the point at infinity. This prevents us from walking around either of these branch points. These two branch cuts separate the branches of the function. See Figure 7.46

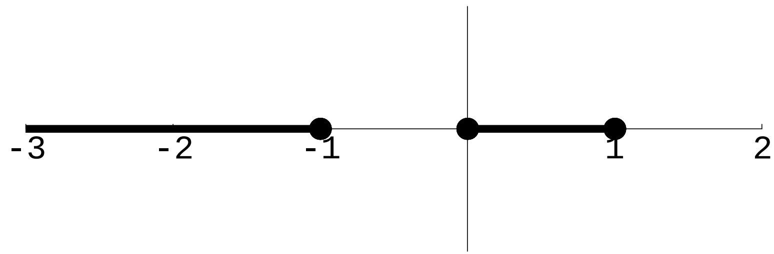


Figure 7.46: Branch cuts for $\log\left(\frac{z(z+1)}{z-1}\right)$.

Solution 7.20
First we factor the function.

$$f(z) = (z(z+3)(z-2))^{1/2} = z^{1/2}(z+3)^{1/2}(z-2)^{1/2}$$

There are branch points at $z = -3, 0, 2$. Now we examine the point at infinity.

$$f\left(\frac{1}{\zeta}\right) = \left(\frac{1}{\zeta}\left(\frac{1}{\zeta}+3\right)\left(\frac{1}{\zeta}-2\right)\right)^{1/2} = \zeta^{-3/2}((1+3\zeta)(1-2\zeta))^{1/2}$$

Since $\zeta^{-3/2}$ has a branch point at $\zeta = 0$ and the rest of the terms are analytic there, $f(z)$ has a branch point at infinity.

Consider the set of branch cuts in Figure 7.47. These cuts do not permit us to walk around any single branch point. We can only walk around none or all of the branch points, (which is the same thing). The cuts can be used to define a single-valued branch of the function.

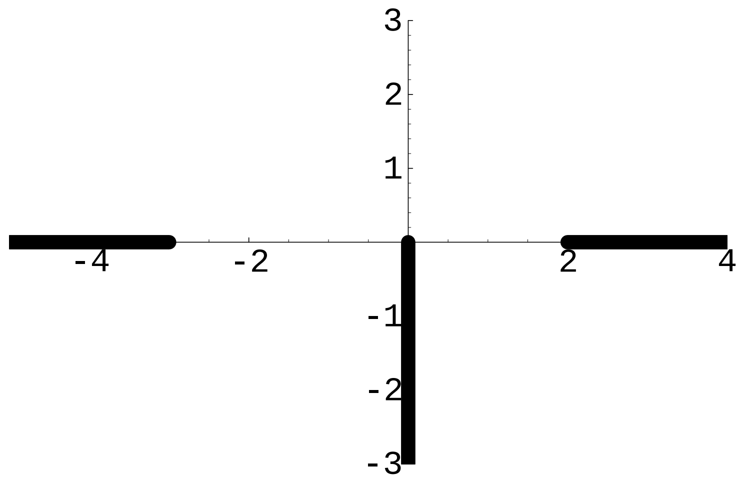


Figure 7.47: Branch cuts for $(z^3 + z^2 - 6z)^{1/2}$.

Now to define the branch. We make a choice of angles.

$$z + 3 = r_1 \mathrm{e}^{\imath\theta_1}, \quad -\pi < \theta_1 < \pi$$
$$z = r_2 \mathrm{e}^{\imath\theta_2}, \quad -\frac{\pi}{2} < \theta_2 < \frac{3\pi}{2}$$
$$z - 2 = r_3 \mathrm{e}^{\imath\theta_3}, \quad 0 < \theta_3 < 2\pi$$

The function is

$$f(z) = \left(r_1 \mathrm{e}^{\imath\theta_1} r_2 \mathrm{e}^{\imath\theta_2} r_3 \mathrm{e}^{\imath\theta_3}\right)^{1/2} = \sqrt{r_1 r_2 r_3}\, \mathrm{e}^{\imath(\theta_1+\theta_2+\theta_3)/2}.$$

We evaluate the function at $z = -1$.

$$f(-1) = \sqrt{(2)(1)(3)}\, \mathrm{e}^{\imath(0+\pi+\pi)/2} = -\sqrt{6}$$

We see that our choice of angles gives us the desired branch.

The stereographic projection is the projection from the complex plane onto a unit sphere with south pole at the origin. The point $z = x + \imath y$ is mapped to the point (X, Y, Z) on the sphere with

$$X = \frac{4x}{|z|^2 + 4}, \quad Y = \frac{4y}{|z|^2 + 4}, \quad Z = \frac{2|z|^2}{|z|^2 + 4}.$$

Figure 7.48 first shows the branch cuts and their stereographic projections and then shows the stereographic projections alone.

Solution 7.21

1. For each value of z, $f(z) = z^{1/3}$ has three values.

$$f(z) = z^{1/3} = \sqrt[3]{z}\, \mathrm{e}^{\imath k 2\pi/3}, \quad k = 0, 1, 2$$

2.

$$g(w) = w^3 = |w|^3 \mathrm{e}^{\imath 3 \arg(w)}$$

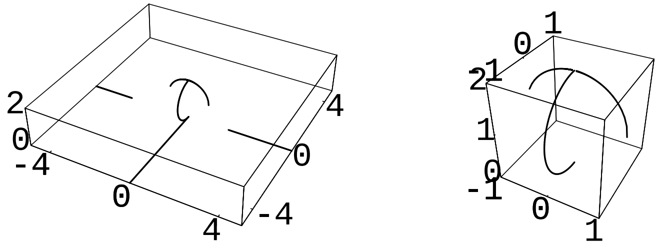


Figure 7.48: Branch cuts for $(z^3 + z^2 - 6z)^{1/2}$ and their stereographic projections.

Any sector of the w plane of angle $2\pi/3$ maps one-to-one to the whole z-plane.

$$g : \left\{ r\,\mathrm{e}^{\imath\theta} \mid r \geq 0, \theta_0 \leq \theta < \theta_0 + 2\pi/3 \right\} \mapsto \left\{ r^3\,\mathrm{e}^{\imath 3\theta} \mid r \geq 0, \theta_0 \leq \theta < \theta_0 + 2\pi/3 \right\}$$
$$g : \left\{ r\,\mathrm{e}^{\imath\theta} \mid r \geq 0, \theta_0 \leq \theta < \theta_0 + 2\pi/3 \right\} \mapsto \left\{ r\,\mathrm{e}^{\imath\theta} \mid r \geq 0, 3\theta_0 \leq \theta < 3\theta_0 + 2\pi \right\}$$
$$g : \left\{ r\,\mathrm{e}^{\imath\theta} \mid r \geq 0, \theta_0 \leq \theta < \theta_0 + 2\pi/3 \right\} \mapsto \mathbb{C}$$

See Figure 7.49 to see how $g(w)$ maps the sector $0 \leq \theta < 2\pi/3$.

3. See Figure 7.50 for a depiction of the Riemann surface for $f(z) = z^{1/3}$. We show two views of the surface and a curve that traces the edge of the shown portion of the surface. The depiction is misleading because the surface is not self-intersecting. We would need four dimensions to properly visualize the this Riemann surface.

4. $f(z) = z^{1/3}$ has branch points at $z = 0$ and $z = \infty$. Any branch cut which connects these two points would prevent us from walking around the points singly and would thus separate the branches of the function. For example, we could put a branch cut on the negative real axis. Defining the angle $-\pi < \theta < \pi$ for the mapping

$$f\left(r\,\mathrm{e}^{\imath\theta}\right) = \sqrt[3]{r}\,\mathrm{e}^{\imath\theta/3}$$

defines a single-valued branch of the function.

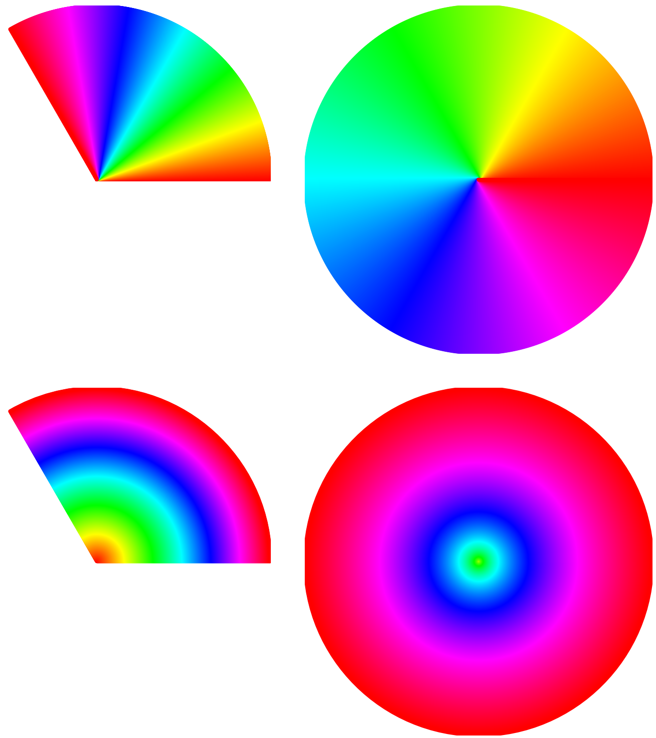


Figure 7.49: The function $g(w) = w^3$ maps the sector $0 \leq \theta < 2\pi/3$ one-to-one to the whole z-plane.

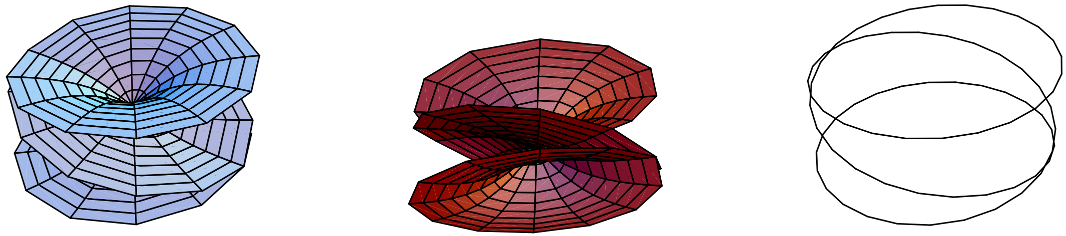


Figure 7.50: Riemann surface for $f(z) = z^{1/3}$.

Solution 7.22
The cube roots of 1 are

$$\left\{1, \mathrm{e}^{\imath 2\pi/3}, \mathrm{e}^{\imath 4\pi/3}\right\} = \left\{1, \frac{-1+\imath\sqrt{3}}{2}, \frac{-1-\imath\sqrt{3}}{2}\right\}.$$

We factor the polynomial.

$$\left(z^3-1\right)^{1/2} = (z-1)^{1/2}\left(z+\frac{1-\imath\sqrt{3}}{2}\right)^{1/2}\left(z+\frac{1+\imath\sqrt{3}}{2}\right)^{1/2}$$

There are branch points at each of the cube roots of unity.

$$z = \left\{1, \frac{-1+\imath\sqrt{3}}{2}, \frac{-1-\imath\sqrt{3}}{2}\right\}$$

Now we examine the point at infinity. We make the change of variables $z = 1/\zeta$.

$$f(1/\zeta) = \left(1/\zeta^3 - 1\right)^{1/2} = \zeta^{-3/2}\left(1-\zeta^3\right)^{1/2}$$

$\zeta^{-3/2}$ has a branch point at $\zeta = 0$, while $(1-\zeta^3)^{1/2}$ is not singular there. Since $f(1/\zeta)$ has a branch point at $\zeta = 0$, $f(z)$ has a branch point at infinity.

There are several ways of introducing branch cuts to separate the branches of the function. The easiest approach is to put a branch cut from each of the three branch points in the finite complex plane out to the branch point at infinity. See Figure 7.51a. Clearly this makes the function single valued as it is impossible to walk around any of the branch points. Another approach is to have a branch cut from one of the branch points in the finite plane to the branch point at infinity and a branch cut connecting the remaining two branch points. See Figure 7.51bcd. Note that in walking around any one of the finite branch points, (in the positive direction), the argument of the function changes by π. This means that the value of the function changes by $e^{\imath\pi}$, which is to say the value of the function changes sign. In walking around any two of the finite branch points, (again in the positive direction), the argument of the function changes by 2π. This means that the value of the function changes by $e^{\imath 2\pi}$, which is to say that the value of the function does not change. This demonstrates that the latter branch cut approach makes the function single-valued.

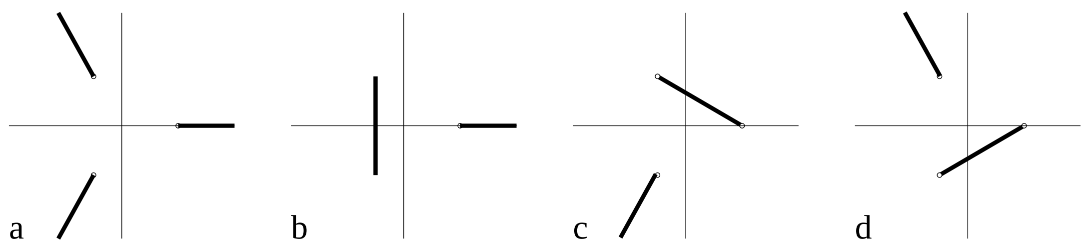


Figure 7.51: Suitable branch cuts for $(z^3-1)^{1/2}$.

Now we construct a branch. We will use the branch cuts in Figure 7.51a. We introduce variables to measure radii and angles from the three finite branch points.

$$
\begin{aligned}
z - 1 &= r_1 \mathrm{e}^{\imath\theta_1}, \quad 0 < \theta_1 < 2\pi \\
z + \frac{1 - \imath\sqrt{3}}{2} &= r_2 \mathrm{e}^{\imath\theta_2}, \quad -\frac{2\pi}{3} < \theta_2 < \frac{\pi}{3} \\
z + \frac{1 + \imath\sqrt{3}}{2} &= r_3 \mathrm{e}^{\imath\theta_3}, \quad -\frac{\pi}{3} < \theta_3 < \frac{2\pi}{3}
\end{aligned}
$$

We compute $f(0)$ to see if it has the desired value.

$$
\begin{gathered}
f(z) = \sqrt{r_1 r_2 r_3}\, \mathrm{e}^{\imath(\theta_1+\theta_2+\theta_3)/2} \\
f(0) = \mathrm{e}^{\imath(\pi - \pi/3 + \pi/3)/2} = \imath
\end{gathered}
$$

Since it does not have the desired value, we change the range of θ_1.

$$
z - 1 = r_1 \mathrm{e}^{\imath\theta_1}, \quad 2\pi < \theta_1 < 4\pi
$$

$f(0)$ now has the desired value.

$$
f(0) = \mathrm{e}^{\imath(3\pi - \pi/3 + \pi/3)/2} = -\imath
$$

We compute $f(-1)$.

$$
f(-1) = \sqrt{2}\, \mathrm{e}^{\imath(3\pi - 2\pi/3 + 2\pi/3)/2} = -\imath\sqrt{2}
$$

Solution 7.23

First we factor the function.

$$
w(z) = ((z+2)(z-1)(z-6))^{1/2} = (z+2)^{1/2}(z-1)^{1/2}(z-6)^{1/2}
$$

There are branch points at $z = -2, 1, 6$. Now we examine the point at infinity.

$$
w\left(\frac{1}{\zeta}\right) = \left(\left(\frac{1}{\zeta} + 2\right)\left(\frac{1}{\zeta} - 1\right)\left(\frac{1}{\zeta} - 6\right)\right)^{1/2} = \zeta^{-3/2}\left(\left(1 + \frac{2}{\zeta}\right)\left(1 - \frac{1}{\zeta}\right)\left(1 - \frac{6}{\zeta}\right)\right)^{1/2}
$$

Since $\zeta^{-3/2}$ has a branch point at $\zeta = 0$ and the rest of the terms are analytic there, $w(z)$ has a branch point at infinity.

Consider the set of branch cuts in Figure 7.52. These cuts let us walk around the branch points at $z = -2$ and $z = 1$ together or if we change our perspective, we would be walking around the branch points at $z = 6$ and $z = \infty$ together. Consider a contour in this cut plane that encircles the branch points at $z = -2$ and $z = 1$. Since the argument of $(z - z_0)^{1/2}$ changes by π when we walk around z_0, the argument of $w(z)$ changes by 2π when we traverse the contour. Thus the value of the function does not change and it is a valid set of branch cuts.

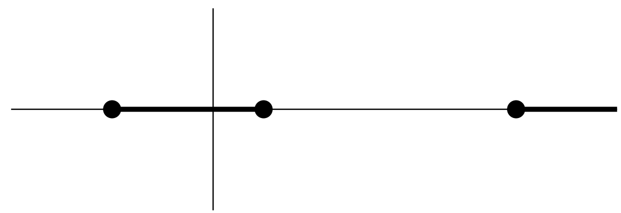


Figure 7.52: Branch cuts for $((z+2)(z-1)(z-6))^{1/2}$.

Now to define the branch. We make a choice of angles.

$$
\begin{aligned}
z + 2 &= r_1 \mathrm{e}^{\imath\theta_1}, \quad \theta_1 = \theta_2 \text{ for } z \in (1 \ldots 6), \\
z - 1 &= r_2 \mathrm{e}^{\imath\theta_2}, \quad \theta_2 = \theta_1 \text{ for } z \in (1 \ldots 6), \\
z - 6 &= r_3 \mathrm{e}^{\imath\theta_3}, \quad 0 < \theta_3 < 2\pi
\end{aligned}
$$

The function is

$$
w(z) = \left(r_1 \mathrm{e}^{\imath\theta_1} r_2 \mathrm{e}^{\imath\theta_2} r_3 \mathrm{e}^{\imath\theta_3}\right)^{1/2} = \sqrt{r_1 r_2 r_3}\, \mathrm{e}^{\imath(\theta_1+\theta_2+\theta_3)/2}.
$$

We evaluate the function at $z = 4$.

$$
w(4) = \sqrt{(6)(3)(2)}\, \mathrm{e}^{\imath(2\pi n + 2\pi n + \pi)/2} = \imath 6
$$

We see that our choice of angles gives us the desired branch.

Solution 7.24

1.

$$\cos\left(z^{1/2}\right) = \cos\left(\pm\sqrt{z}\right) = \cos\left(\sqrt{z}\right)$$

This is a single-valued function. There are no branch points.

2.

$$\begin{aligned}(z+\imath)^{-z} &= \mathrm{e}^{-z\log(z+\imath)} \\ &= \mathrm{e}^{-z(\ln|z+\imath|+\imath\operatorname{Arg}(z+\imath)+\imath 2\pi n)}, \quad n \in \mathbb{Z}\end{aligned}$$

There is a branch point at $z = -\imath$. There are an infinite number of branches.

Solution 7.25

1.

$$f(z) = \left(z^2+1\right)^{1/2} = (z+\imath)^{1/2}(z-\imath)^{1/2}$$

We see that there are branch points at $z = \pm\imath$. To examine the point at infinity, we substitute $z = 1/\zeta$ and examine the point $\zeta = 0$.

$$\left(\left(\frac{1}{\zeta}\right)^2+1\right)^{1/2} = \frac{1}{\left(\zeta^2\right)^{1/2}}\left(1+\zeta^2\right)^{1/2}$$

Since there is no branch point at $\zeta = 0$, $f(z)$ has no branch point at infinity.

A branch cut connecting $z = \pm\imath$ would make the function single-valued. We could also accomplish this with two branch cuts starting $z = \pm\imath$ and going to infinity.

2.

$$f(z) = \left(z^3-z\right)^{1/2} = z^{1/2}(z-1)^{1/2}(z+1)^{1/2}$$

There are branch points at $z = -1, 0, 1$. Now we consider the point at infinity.

$$f\left(\frac{1}{\zeta}\right) = \left(\left(\frac{1}{\zeta}\right)^3 - \frac{1}{\zeta}\right)^{1/2} = \zeta^{-3/2}\left(1-\zeta^2\right)^{1/2}$$

There is a branch point at infinity.

One can make the function single-valued with three branch cuts that start at $z = -1, 0, 1$ and each go to infinity. We can also make the function single-valued with a branch cut that connects two of the points $z = -1, 0, 1$ and another branch cut that starts at the remaining point and goes to infinity.

3.
$$f(z) = \log\left(z^2 - 1\right) = \log(z - 1) + \log(z + 1)$$

There are branch points at $z = \pm 1$.

$$f\left(\frac{1}{\zeta}\right) = \log\left(\frac{1}{\zeta^2} - 1\right) = \log\left(\zeta^{-2}\right) + \log\left(1 - \zeta^2\right)$$

$\log\left(\zeta^{-2}\right)$ has a branch point at $\zeta = 0$.

$$\log\left(\zeta^{-2}\right) = \ln\left|\zeta^{-2}\right| + \imath \arg\left(\zeta^{-2}\right) = \ln\left|\zeta^{-2}\right| - \imath 2 \arg(\zeta)$$

Every time we walk around the point $\zeta = 0$ in the positive direction, the value of the function changes by $-\imath 4\pi$. $f(z)$ has a branch point at infinity.

We can make the function single-valued by introducing two branch cuts that start at $z = \pm 1$ and each go to infinity.

4.
$$f(z) = \log\left(\frac{z + 1}{z - 1}\right) = \log(z + 1) - \log(z - 1)$$

There are branch points at $z = \pm 1$.

$$f\left(\frac{1}{\zeta}\right) = \log\left(\frac{1/\zeta + 1}{1/\zeta - 1}\right) = \log\left(\frac{1 + \zeta}{1 - \zeta}\right)$$

There is no branch point at $\zeta = 0$. $f(z)$ has no branch point at infinity.

We can make the function single-valued by introducing two branch cuts that start at $z = \pm 1$ and each go to infinity. We can also make the function single-valued with a branch cut that connects the points $z = \pm 1$. This is because $\log(z+1)$ and $-\log(z-1)$ change by $\imath 2\pi$ and $-\imath 2\pi$, respectively, when you walk around their branch points once in the positive direction.

Solution 7.26

1. The cube roots of -8 are

$$\left\{-2, -2\,\mathrm{e}^{\imath 2\pi/3}, -2\,\mathrm{e}^{\imath 4\pi/3}\right\} = \left\{-2, 1 + \imath\sqrt{3}, 1 - \imath\sqrt{3}\right\}.$$

Thus we can write

$$\left(z^3 + 8\right)^{1/2} = (z+2)^{1/2}\left(z - 1 - \imath\sqrt{3}\right)^{1/2}\left(z - 1 + \imath\sqrt{3}\right)^{1/2}.$$

There are three branch points on the circle of radius 2.

$$z = \left\{-2, 1 + \imath\sqrt{3}, 1 - \imath\sqrt{3}\right\}.$$

We examine the point at infinity.

$$f(1/\zeta) = \left(1/\zeta^3 + 8\right)^{1/2} = \zeta^{-3/2}\left(1 + 8\zeta^3\right)^{1/2}$$

Since $f(1/\zeta)$ has a branch point at $\zeta = 0$, $f(z)$ has a branch point at infinity.

There are several ways of introducing branch cuts outside of the disk $|z| < 2$ to separate the branches of the function. The easiest approach is to put a branch cut from each of the three branch points in the finite complex plane out to the branch point at infinity. See Figure 7.53a. Clearly this makes the function single valued as it is impossible to walk around any of the branch points. Another approach is to have a branch cut from one of the branch points in the finite plane to the branch point at infinity and a branch cut connecting the remaining two branch points. See Figure 7.53bcd. Note that in walking around any one of the finite branch points, (in the positive direction), the argument of the function changes by π. This means that the value of the function changes by $\mathrm{e}^{\imath\pi}$, which is to say the value of the function changes sign. In walking around any two of the finite

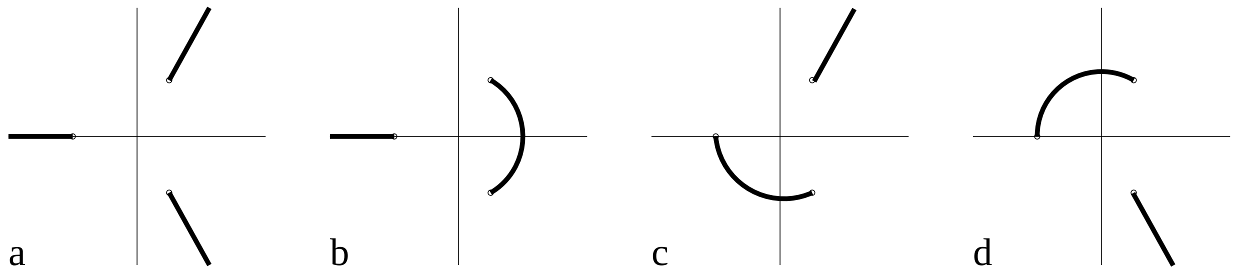


Figure 7.53: Suitable branch cuts for $(z^3+8)^{1/2}$.

branch points, (again in the positive direction), the argument of the function changes by 2π. This means that the value of the function changes by $e^{\imath 2\pi}$, which is to say that the value of the function does not change. This demonstrates that the latter branch cut approach makes the function single-valued.

2.
$$f(z) = \log\left(5 + \left(\frac{z+1}{z-1}\right)^{1/2}\right)$$

First we deal with the function
$$g(z) = \left(\frac{z+1}{z-1}\right)^{1/2}$$

Note that it has branch points at $z = \pm 1$. Consider the point at infinity.

$$g(1/\zeta) = \left(\frac{1/\zeta+1}{1/\zeta-1}\right)^{1/2} = \left(\frac{1+\zeta}{1-\zeta}\right)^{1/2}$$

Since $g(1/\zeta)$ has no branch point at $\zeta = 0$, $g(z)$ has no branch point at infinity. This means that if we walk around both of the branch points at $z = \pm 1$, the function does not change value. We can verify this with another method: When we walk around the point $z = -1$ once in the positive direction, the argument of $z+1$ changes by 2π, the argument of $(z+1)^{1/2}$ changes by π and thus the value of $(z+1)^{1/2}$ changes by $e^{\imath\pi} = -1$. When we walk around the point $z = 1$ once in the positive direction, the argument of $z - 1$ changes by 2π, the argument of $(z-1)^{-1/2}$ changes by $-\pi$ and thus the value of $(z-1)^{-1/2}$ changes by $\mathrm{e}^{-\imath\pi} = -1$. $f(z)$ has branch points at $z = \pm 1$. When we walk around both points $z = \pm 1$ once in the positive direction, the value of $\left(\frac{z+1}{z-1}\right)^{1/2}$ does not change. Thus we can make the function single-valued with a branch cut which enables us to walk around either none or both of these branch points. We put a branch cut from -1 to 1 on the real axis.

$f(z)$ has branch points where

$$5 + \left(\frac{z+1}{z-1}\right)^{1/2}$$

is either zero or infinite. The only place in the extended complex plane where the expression becomes infinite is at $z = 1$. Now we look for the zeros.

$$\begin{aligned}
5 + \left(\frac{z+1}{z-1}\right)^{1/2} &= 0 \\
\left(\frac{z+1}{z-1}\right)^{1/2} &= -5 \\
\frac{z+1}{z-1} &= 25 \\
z + 1 &= 25z - 25 \\
z &= \frac{13}{12}
\end{aligned}$$

Note that

$$\left(\frac{13/12+1}{13/12-1}\right)^{1/2} = 25^{1/2} = \pm 5.$$

On one branch, (which we call the positive branch), of the function $g(z)$ the quantity

$$5 + \left(\frac{z+1}{z-1}\right)^{1/2}$$

is always nonzero. On the other (negative) branch of the function, this quantity has a zero at $z = 13/12$.

The logarithm introduces branch points at $z = 1$ on both the positive and negative branch of $g(z)$. It introduces a branch point at $z = 13/12$ on the negative branch of $g(z)$. To determine if additional branch cuts are needed to separate the branches, we consider

$$w = 5 + \left(\frac{z+1}{z-1}\right)^{1/2}$$

and see where the branch cut between ± 1 gets mapped to in the w plane. We rewrite the mapping.

$$w = 5 + \left(1 + \frac{2}{z-1}\right)^{1/2}$$

The mapping is the following sequence of simple transformations:

(a) $z \mapsto z - 1$

(b) $z \mapsto \dfrac{1}{z}$

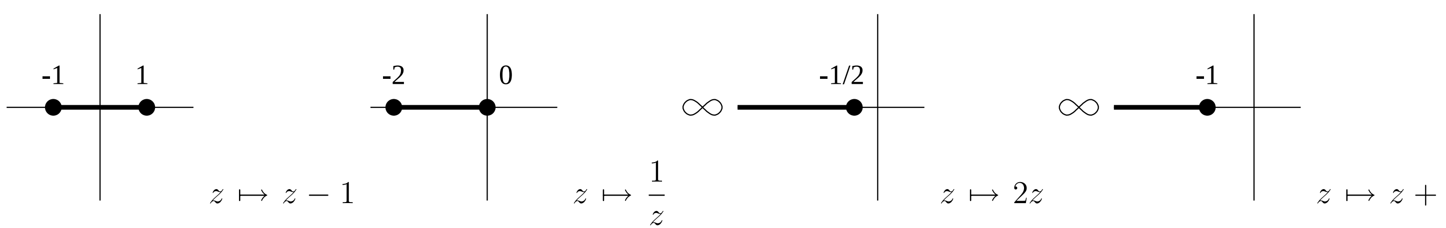
(c) $z \mapsto 2z$

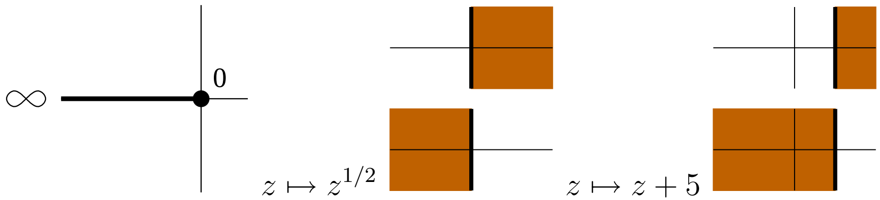
(d) $z \mapsto z + 1$

(e) $z \mapsto z^{1/2}$

(f) $z \mapsto z + 5$

We show these transformations graphically below.





For the positive branch of $g(z)$, the branch cut is mapped to the line $x = 5$ and the z plane is mapped to the half-plane $x > 5$. $\log(w)$ has branch points at $w = 0$ and $w = \infty$. It is possible to walk around only one of these points in the half-plane $x > 5$. Thus no additional branch cuts are needed in the positive sheet of $g(z)$.

For the negative branch of $g(z)$, the branch cut is mapped to the line $x = 5$ and the z plane is mapped to the half-plane $x < 5$. It is possible to walk around either $w = 0$ or $w = \infty$ alone in this half-plane. Thus we need an additional branch cut. On the negative sheet of $g(z)$, we put a branch cut beteen $z = 1$ and $z = 13/12$. This puts a branch cut between $w = \infty$ and $w = 0$ and thus separates the branches of the logarithm.

Figure 7.54 shows the branch cuts in the positive and negative sheets of $g(z)$.

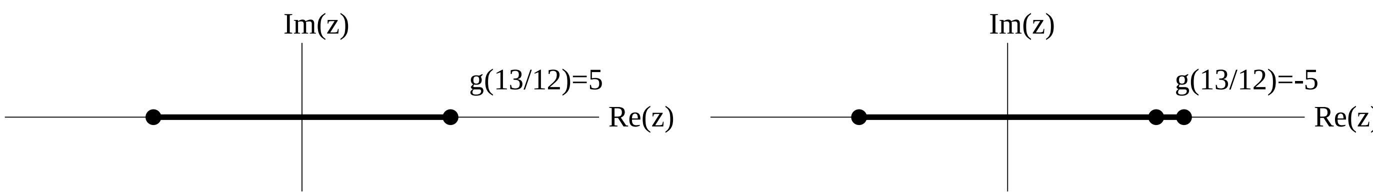


Figure 7.54: The branch cuts for $f(z) = \log\left(5 + \left(\frac{z+1}{z-1}\right)^{1/2}\right)$.

3. The function $f(z) = (z + \imath 3)^{1/2}$ has a branch point at $z = -\imath 3$. The function is made single-valued by connecting this point and the point at infinity with a branch cut.

Solution 7.27

Note that the curve with opposite orientation goes around infinity in the positive direction and does not enclose any branch points. Thus the value of the function does not change when traversing the curve, (with either orientation, of course). This means that the argument of the function must change my an integer multiple of 2π. Since the branch cut only allows us to encircle all three or none of the branch points, it makes the function single valued.

Solution 7.28

We suppose that $f(z)$ has only one branch point in the finite complex plane. Consider any contour that encircles this branch point in the positive direction. $f(z)$ changes value if we traverse the contour. If we reverse the orientation of the contour, then it encircles infinity in the positive direction, but contains no branch points in the finite complex plane. Since the function changes value when we traverse the contour, we conclude that the point at infinity must be a branch point. If $f(z)$ has only a single branch point in the finite complex plane then it must have a branch point at infinity.

If $f(z)$ has two or more branch points in the finite complex plane then it may or may not have a branch point at infinity. This is because the value of the function may or may not change on a contour that encircles all the branch points in the finite complex plane.

Solution 7.29

First we factor the function,

$$f(z) = \left(z^4+1\right)^{1/4} = \left(z - \frac{1+\imath}{\sqrt{2}}\right)^{1/4} \left(z - \frac{-1+\imath}{\sqrt{2}}\right)^{1/4} \left(z - \frac{-1-\imath}{\sqrt{2}}\right)^{1/4} \left(z - \frac{1-\imath}{\sqrt{2}}\right)^{1/4}.$$

There are branch points at $z = \frac{\pm 1 \pm \imath}{\sqrt{2}}$. We make the substitution $z = 1/\zeta$ to examine the point at infinity.

$$\begin{aligned} f\left(\frac{1}{\zeta}\right) &= \left(\frac{1}{\zeta^4} + 1\right)^{1/4} \\ &= \frac{1}{\left(\zeta^4\right)^{1/4}} \left(1+\zeta^4\right)^{1/4} \end{aligned}$$

$\left(\zeta^{1/4}\right)^4$ has a removable singularity at the point $\zeta = 0$, but no branch point there. Thus $\left(z^4+1\right)^{1/4}$ has no branch point at infinity.

Note that the argument of $\left(z^4 - z_0\right)^{1/4}$ changes by $\pi/2$ on a contour that goes around the point z_0 once in the positive direction. The argument of $\left(z^4+1\right)^{1/4}$ changes by $n\pi/2$ on a contour that goes around n of its branch points.

Thus any set of branch cuts that permit you to walk around only one, two or three of the branch points will not make the function single valued. A set of branch cuts that permit us to walk around only zero or all four of the branch points will make the function single-valued. Thus we see that the first two sets of branch cuts in Figure 7.32 will make the function single-valued, while the remaining two will not.

Consider the contour in Figure 7.32. There are two ways to see that the function does not change value while traversing the contour. The first is to note that each of the branch points makes the argument of the function increase by $\pi/2$. Thus the argument of $(z^4+1)^{1/4}$ changes by $4(\pi/2) = 2\pi$ on the contour. This means that the value of the function changes by the factor $\mathrm{e}^{\imath 2\pi} = 1$. If we change the orientation of the contour, then it is a contour that encircles infinity once in the positive direction. There are no branch points inside the this contour with opposite orientation. (Recall that the inside of a contour lies to your left as you walk around it.) Since there are no branch points inside this contour, the function cannot change value as we traverse it.

Solution 7.30

$$f(z) = \left(\frac{z}{z^2+1}\right)^{1/3} = z^{1/3}(z-\imath)^{-1/3}(z+\imath)^{-1/3}$$

There are branch points at $z = 0, \pm\imath$.

$$f\left(\frac{1}{\zeta}\right) = \left(\frac{1/\zeta}{(1/\zeta)^2+1}\right)^{1/3} = \frac{\zeta^{1/3}}{(1+\zeta^2)^{1/3}}$$

There is a branch point at $\zeta = 0$. $f(z)$ has a branch point at infinity.

We introduce branch cuts from $z = 0$ to infinity on the negative real axis, from $z = \imath$ to infinity on the positive imaginary axis and from $z = -\imath$ to infinity on the negative imaginary axis. As we cannot walk around any of the branch points, this makes the function single-valued.

We define a branch by defining angles from the branch points. Let

$$\begin{aligned} z &= r\,\mathrm{e}^{\imath\theta} & -\pi < \theta < \pi, \\ (z-\imath) &= s\,\mathrm{e}^{\imath\phi} & -3\pi/2 < \phi < \pi/2, \\ (z+\imath) &= t\,\mathrm{e}^{\imath\psi} & -\pi/2 < \psi < 3\pi/2. \end{aligned}$$

With

$$
\begin{aligned}
f(z) &= z^{1/3}(z-\imath)^{-1/3}(z+\imath)^{-1/3} \\
&= \sqrt[3]{r}\,\mathrm{e}^{\imath\theta/3}\,\frac{1}{\sqrt[3]{s}}\,\mathrm{e}^{-\imath\phi/3}\,\frac{1}{\sqrt[3]{t}}\,\mathrm{e}^{-\imath\psi/3} \\
&= \sqrt[3]{\frac{r}{st}}\,\mathrm{e}^{\imath(\theta-\phi-\psi)/3}
\end{aligned}
$$

we have an explicit formula for computing the value of the function for this branch. Now we compute $f(1)$ to see if we chose the correct ranges for the angles. (If not, we'll just change one of them.)

$$
f(1) = \sqrt[3]{\frac{1}{\sqrt{2}\sqrt{2}}}\,\mathrm{e}^{\imath(0-\pi/4-(-\pi/4))/3} = \frac{1}{\sqrt[3]{2}}
$$

We made the right choice for the angles. Now to compute $f(1+\imath)$.

$$
f(1+\imath) = \sqrt[3]{\frac{\sqrt{2}}{1\sqrt{5}}}\,\mathrm{e}^{\imath(\pi/4-0-\mathrm{Arctan}(2))/3} = \sqrt[6]{\frac{2}{5}}\,\mathrm{e}^{\imath(\pi/4-\mathrm{Arctan}(2))/3}
$$

Consider the value of the function above and below the branch cut on the negative real axis. Above the branch cut the function is

$$
f(-x+\imath 0) = \sqrt[3]{\frac{x}{\sqrt{x^2+1}\sqrt{x^2+1}}}\,\mathrm{e}^{\imath(\pi-\phi-\psi)/3}
$$

Note that $\phi = -\psi$ so that

$$
f(-x+\imath 0) = \sqrt[3]{\frac{x}{x^2+1}}\,\mathrm{e}^{\imath\pi/3} = \sqrt[3]{\frac{x}{x^2+1}}\frac{1+\imath\sqrt{3}}{2}.
$$

Below the branch cut $\theta = -\pi$ and

$$
f(-x-\imath 0) = \sqrt[3]{\frac{x}{x^2+1}}\,\mathrm{e}^{\imath(-\pi)/3} = \sqrt[3]{\frac{x}{x^2+1}}\frac{1-\imath\sqrt{3}}{2}.
$$

For the branch cut along the positive imaginary axis,

$$
\begin{aligned}
f(\imath y+0) &= \sqrt[3]{\frac{y}{(y-1)(y+1)}}\,\mathrm{e}^{\imath(\pi/2-\pi/2-\pi/2)/3}\\
&= \sqrt[3]{\frac{y}{(y-1)(y+1)}}\,\mathrm{e}^{-\imath\pi/6}\\
&= \sqrt[3]{\frac{y}{(y-1)(y+1)}}\,\frac{\sqrt{3}-\imath}{2},
\end{aligned}
$$

$$
\begin{aligned}
f(\imath y-0) &= \sqrt[3]{\frac{y}{(y-1)(y+1)}}\,\mathrm{e}^{\imath(\pi/2-(-3\pi/2)-\pi/2)/3}\\
&= \sqrt[3]{\frac{y}{(y-1)(y+1)}}\,\mathrm{e}^{\imath\pi/2}\\
&= \imath\sqrt[3]{\frac{y}{(y-1)(y+1)}}.
\end{aligned}
$$

For the branch cut along the negative imaginary axis,

$$
\begin{aligned}
f(-\imath y+0) &= \sqrt[3]{\frac{y}{(y+1)(y-1)}}\,\mathrm{e}^{\imath(-\pi/2-(-\pi/2)-(-\pi/2))/3}\\
&= \sqrt[3]{\frac{y}{(y+1)(y-1)}}\,\mathrm{e}^{\imath\pi/6}\\
&= \sqrt[3]{\frac{y}{(y+1)(y-1)}}\,\frac{\sqrt{3}+\imath}{2},
\end{aligned}
$$

$$
\begin{aligned}
f(-\imath y-0) &= \sqrt[3]{\frac{y}{(y+1)(y-1)}}\,\mathrm{e}^{\imath(-\pi/2-(-\pi/2)-(3\pi/2))/3} \\
&= \sqrt[3]{\frac{y}{(y+1)(y-1)}}\,\mathrm{e}^{-\imath\pi/2} \\
&= -\imath\sqrt[3]{\frac{y}{(y+1)(y-1)}}.
\end{aligned}
$$

Solution 7.31

First we factor the function.

$$f(z) = ((z-1)(z-2)(z-3))^{1/2} = (z-1)^{1/2}(z-2)^{1/2}(z-3)^{1/2}$$

There are branch points at $z = 1, 2, 3$. Now we examine the point at infinity.

$$f\left(\frac{1}{\zeta}\right) = \left(\left(\frac{1}{\zeta}-1\right)\left(\frac{1}{\zeta}-2\right)\left(\frac{1}{\zeta}-3\right)\right)^{1/2} = \zeta^{-3/2}\left(\left(1-\frac{1}{\zeta}\right)\left(1-\frac{2}{\zeta}\right)\left(1-\frac{3}{\zeta}\right)\right)^{1/2}$$

Since $\zeta^{-3/2}$ has a branch point at $\zeta = 0$ and the rest of the terms are analytic there, $f(z)$ has a branch point at infinity.

The first two sets of branch cuts in Figure 7.33 do not permit us to walk around any of the branch points, including the point at infinity, and thus make the function single-valued. The third set of branch cuts lets us walk around the branch points at $z = 1$ and $z = 2$ together or if we change our perspective, we would be walking around the branch points at $z = 3$ and $z = \infty$ together. Consider a contour in this cut plane that encircles the branch points at $z = 1$ and $z = 2$. Since the argument of $(z - z_0)^{1/2}$ changes by π when we walk around z_0, the argument of $f(z)$ changes by 2π when we traverse the contour. Thus the value of the function does not change and it is a valid set of branch cuts. Clearly the fourth set of branch cuts does not make the function single-valued as there are contours that encircle the branch point at infinity and no other branch points. The other way to see this is to note that the argument of $f(z)$ changes by 3π as we traverse a contour that goes around the branch points at $z = 1, 2, 3$ once in the positive direction.

Now to define the branch. We make the preliminary choice of angles,

$$
\begin{aligned}
z-1 &= r_1\,\mathrm{e}^{\imath\theta_1}, \quad 0 < \theta_1 < 2\pi, \\
z-2 &= r_2\,\mathrm{e}^{\imath\theta_2}, \quad 0 < \theta_2 < 2\pi, \\
z-3 &= r_3\,\mathrm{e}^{\imath\theta_3}, \quad 0 < \theta_3 < 2\pi.
\end{aligned}
$$

The function is

$$f(z) = \left(r_1 \, e^{\imath\theta_1} \, r_2 \, e^{\imath\theta_2} \, r_3 \, e^{\imath\theta_3}\right)^{1/2} = \sqrt{r_1 r_2 r_3} \, e^{\imath(\theta_1+\theta_2+\theta_3)/2}.$$

The value of the function at the origin is

$$f(0) = \sqrt{6} \, e^{\imath(3\pi)/2} = -\imath\sqrt{6},$$

which is not what we wanted. We will change range of one of the angles to get the desired result.

$$\begin{aligned}
z-1 &= r_1 \, e^{\imath\theta_1}, \quad 0 < \theta_1 < 2\pi, \\
z-2 &= r_2 \, e^{\imath\theta_2}, \quad 0 < \theta_2 < 2\pi, \\
z-3 &= r_3 \, e^{\imath\theta_3}, \quad 2\pi < \theta_3 < 4\pi.
\end{aligned}$$

$$f(0) = \sqrt{6} \, e^{\imath(5\pi)/2} = \imath\sqrt{6},$$

Solution 7.32

$$w = \left(\left(z^2-2\right)(z+2)\right)^{1/3} \left(z+\sqrt{2}\right)^{1/3} \left(z-\sqrt{2}\right)^{1/3} (z+2)^{1/3}$$

There are branch points at $z = \pm\sqrt{2}$ and $z = -2$. If we walk around any one of the branch points once in the positive direction, the argument of w changes by $2\pi/3$ and thus the value of the function changes by $e^{\imath 2\pi/3}$. If we walk around all three branch points then the argument of w changes by $3 \times 2\pi/3 = 2\pi$. The value of the function is unchanged as $e^{\imath 2\pi} = 1$. Thus the branch cut on the real axis from -2 to $\sqrt{2}$ makes the function single-valued.

Now we define a branch. Let

$$z-\sqrt{2} = a \, e^{\imath\alpha}, \quad z+\sqrt{2} = b \, e^{\imath\beta}, \quad z+2 = c \, e^{\imath\gamma}.$$

We constrain the angles as follows: On the positive real axis, $\alpha = \beta = \gamma$. See Figure 7.55.

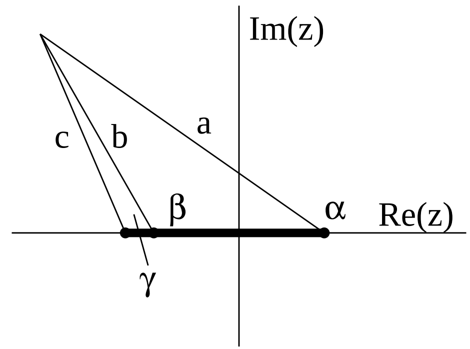


Figure 7.55: A branch of $\left((z^2-2)(z+2)\right)^{1/3}$.

Now we determine $w(2)$.

$$
\begin{aligned}
w(2) &= \left(2-\sqrt{2}\right)^{1/3}\left(2+\sqrt{2}\right)^{1/3}(2+2)^{1/3} \\
&= \sqrt[3]{2-\sqrt{2}}\,\mathrm{e}^{\imath 0}\,\sqrt[3]{2+\sqrt{2}}\,\mathrm{e}^{\imath 0}\,\sqrt[3]{4}\,\mathrm{e}^{\imath 0} \\
&= \sqrt[3]{2}\sqrt[3]{4} \\
&= 2.
\end{aligned}
$$

Note that we didn't have to choose the angle from each of the branch points as zero. Choosing any integer multiple of 2π would give us the same result.

$$
\begin{aligned}
w(-3) &= \left(-3-\sqrt{2}\right)^{1/3}\left(-3+\sqrt{2}\right)^{1/3}(-3+2)^{1/3} \\
&= \sqrt[3]{3+\sqrt{2}}\,\mathrm{e}^{\imath\pi/3}\,\sqrt[3]{3-\sqrt{2}}\,\mathrm{e}^{\imath\pi/3}\,\sqrt[3]{1}\,\mathrm{e}^{\imath\pi/3} \\
&= \sqrt[3]{7}\,\mathrm{e}^{\imath\pi} \\
&= -\sqrt[3]{7}
\end{aligned}
$$

The value of the function is

$$w = \sqrt[3]{abc}\,\mathrm{e}^{\imath(\alpha+\beta+\gamma)/3}.$$

Consider the interval $\left(-\sqrt{2}\ldots\sqrt{2}\right)$. As we approach the branch cut from above, the function has the value,

$$w = \sqrt[3]{abc}\,\mathrm{e}^{\imath\pi/3} = \sqrt[3]{\left(\sqrt{2}-x\right)\left(x+\sqrt{2}\right)(x+2)}\,\mathrm{e}^{\imath\pi/3}.$$

As we approach the branch cut from below, the function has the value,

$$w = \sqrt[3]{abc}\,\mathrm{e}^{-\imath\pi/3} = \sqrt[3]{\left(\sqrt{2}-x\right)\left(x+\sqrt{2}\right)(x+2)}\,\mathrm{e}^{-\imath\pi/3}.$$

Consider the interval $\left(-2\ldots-\sqrt{2}\right)$. As we approach the branch cut from above, the function has the value,

$$w = \sqrt[3]{abc}\,\mathrm{e}^{\imath 2\pi/3} = \sqrt[3]{\left(\sqrt{2}-x\right)\left(-x-\sqrt{2}\right)(x+2)}\,\mathrm{e}^{\imath 2\pi/3}.$$

As we approach the branch cut from below, the function has the value,

$$w = \sqrt[3]{abc}\,\mathrm{e}^{-\imath 2\pi/3} = \sqrt[3]{\left(\sqrt{2}-x\right)\left(-x-\sqrt{2}\right)(x+2)}\,\mathrm{e}^{-\imath 2\pi/3}.$$

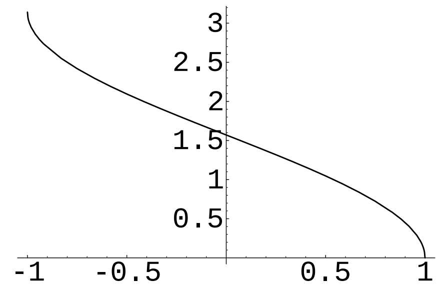


Figure 7.56: The principal branch of the arc cosine, $\operatorname{Arccos}(x)$.

Solution 7.33

$\operatorname{Arccos}(x)$ is shown in Figure 7.56 for real variables in the range $[-1 \ldots 1]$.

First we write $\arccos(z)$ in terms of $\log(z)$. If $\cos(w) = z$, then $w = \arccos(z)$.

$$\begin{gathered}
\cos(w) = z \\
\frac{\mathrm{e}^{\imath w} + \mathrm{e}^{-\imath w}}{2} = z \\
\left(\mathrm{e}^{\imath w}\right)^2 - 2z\,\mathrm{e}^{\imath w} + 1 = 0 \\
\mathrm{e}^{\imath w} = z + \left(z^2 - 1\right)^{1/2} \\
w = -\imath \log\left(z + \left(z^2 - 1\right)^{1/2}\right)
\end{gathered}$$

Thus we have

$$\boxed{\arccos(z) = -\imath \log\left(z + \left(z^2 - 1\right)^{1/2}\right).}$$

Since $\operatorname{Arccos}(0) = \frac{\pi}{2}$, we must find the branch such that

$$\begin{gathered}
-\imath \log\left(0 + \left(0^2 - 1\right)^{1/2}\right) = 0 \\
-\imath \log\left((-1)^{1/2}\right) = 0.
\end{gathered}$$

Since

$$-\imath \log(\imath) = -\imath \left(\imath \frac{\pi}{2} + \imath 2\pi n \right) = \frac{\pi}{2} + 2\pi n$$

and

$$-\imath \log(-\imath) = -\imath \left(-\imath \frac{\pi}{2} + \imath 2\pi n \right) = -\frac{\pi}{2} + 2\pi n$$

we must choose the branch of the square root such that $(-1)^{1/2} = \imath$ and the branch of the logarithm such that $\log(\imath) = \imath \frac{\pi}{2}$.

First we construct the branch of the square root.

$$\left(z^2 - 1\right)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}$$

We see that there are branch points at $z = -1$ and $z = 1$. In particular we want the Arccos to be defined for $z = x$, $x \in [-1 \ldots 1]$. Hence we introduce branch cuts on the lines $-\infty < x \leq -1$ and $1 \leq x < \infty$. Define the local coordinates

$$z + 1 = r \, \mathrm{e}^{\imath\theta}, \qquad z - 1 = \rho \, \mathrm{e}^{\imath\phi} .$$

With the given branch cuts, the angles have the possible ranges

$$\{\theta\} = \{\ldots, (-\pi \ldots \pi), (\pi \ldots 3\pi), \ldots\}, \qquad \{\phi\} = \{\ldots, (0 \ldots 2\pi), (2\pi \ldots 4\pi), \ldots\}.$$

Now we choose ranges for θ and ϕ and see if we get the desired branch. If not, we choose a different range for one of the angles. First we choose the ranges

$$\theta \in (-\pi \ldots \pi), \qquad \phi \in (0 \ldots 2\pi).$$

If we substitute in $z = 0$ we get

$$\left(0^2 - 1\right)^{1/2} = \left(1 \, \mathrm{e}^{\imath 0}\right)^{1/2} \left(1 \, \mathrm{e}^{\imath\pi}\right)^{1/2} = \mathrm{e}^{\imath 0} \, \mathrm{e}^{\imath\pi/2} = \imath$$

Thus we see that this choice of angles gives us the desired branch.

Now we go back to the expression

$$\arccos(z) = -\imath \log \left(z + \left(z^2 - 1\right)^{1/2} \right).$$

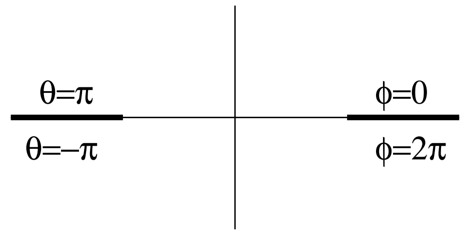


Figure 7.57: Branch cuts and angles for $(z^2-1)^{1/2}$.

We have already seen that there are branch points at $z=-1$ and $z=1$ because of $(z^2-1)^{1/2}$. Now we must determine if the logarithm introduces additional branch points. The only possibilities for branch points are where the argument of the logarithm is zero.

$$z+\left(z^2-1\right)^{1/2}=0$$
$$z^2=z^2-1$$
$$0=-1$$

We see that the argument of the logarithm is nonzero and thus there are no additional branch points. Introduce the variable, $w=z+(z^2-1)^{1/2}$. What is the image of the branch cuts in the w plane? We parameterize the branch cut connecting $z=1$ and $z=+\infty$ with $z=r+1$, $r\in[0\dots\infty)$.

$$\begin{aligned} w &= r+1+\left((r+1)^2-1\right)^{1/2} \\ &= r+1\pm\sqrt{r(r+2)} \\ &= r\left(1\pm r\sqrt{1+2/r}\right)+1 \end{aligned}$$

$r\left(1+\sqrt{1+2/r}\right)+1$ is the interval $[1\dots\infty)$; $r\left(1-\sqrt{1+2/r}\right)+1$ is the interval $(0\dots1]$. Thus we see that this branch cut is mapped to the interval $(0\dots\infty)$ in the w plane. Similarly, we could show that the branch cut $(-\infty\dots-1]$

in the z plane is mapped to $(-\infty \dots 0)$ in the w plane. In the w plane there is a branch cut along the real w axis from $-\infty$ to ∞. Thus cut makes the logarithm single-valued. For the branch of the square root that we chose, all the points in the z plane get mapped to the upper half of the w plane.

With the branch cuts we have introduced so far and the chosen branch of the square root we have

$$
\begin{aligned}
\arccos(0) &= -\imath \log\left(0 + \left(0^2 - 1\right)^{1/2}\right) \\
&= -\imath \log \imath \\
&= -\imath\left(\imath \frac{\pi}{2} + \imath 2\pi n\right) \\
&= \frac{\pi}{2} + 2\pi n
\end{aligned}
$$

Choosing the $n = 0$ branch of the logarithm will give us $\operatorname{Arccos}(z)$. We see that we can write

$$
\operatorname{Arccos}(z) = -\imath \operatorname{Log}\left(z + \left(z^2 - 1\right)^{1/2}\right).
$$

Solution 7.34

We consider the function $f(z) = \left(z^{1/2} - 1\right)^{1/2}$. First note that $z^{1/2}$ has a branch point at $z = 0$. We place a branch cut on the negative real axis to make it single valued. $f(z)$ will have a branch point where $z^{1/2} - 1 = 0$. This occurs at $z = 1$ on the branch of $z^{1/2}$ on which $1^{1/2} = 1$. ($1^{1/2}$ has the value 1 on one branch of $z^{1/2}$ and -1 on the other branch.) For this branch we introduce a branch cut connecting $z = 1$ with the point at infinity. (See Figure 7.58.)

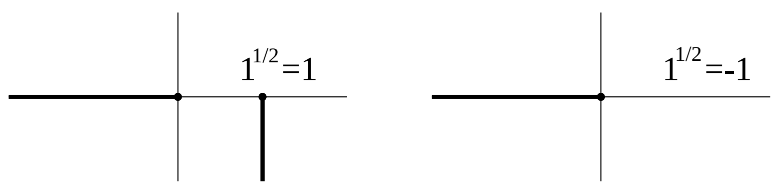


Figure 7.58: Branch cuts for $\left(z^{1/2} - 1\right)^{1/2}$.

Solution 7.35

The distance between the end of rod a and the end of rod c is b. In the complex plane, these points are $a\,\mathrm{e}^{\imath\theta}$ and $l + c\,\mathrm{e}^{\imath\phi}$, respectively. We write this out mathematically.

$$\left|l + c\,\mathrm{e}^{\imath\phi} - a\,\mathrm{e}^{\imath\theta}\right| = b$$
$$\left(l + c\,\mathrm{e}^{\imath\phi} - a\,\mathrm{e}^{\imath\theta}\right)\left(l + c\,\mathrm{e}^{-\imath\phi} - a\,\mathrm{e}^{-\imath\theta}\right) = b^2$$
$$l^2 + cl\,\mathrm{e}^{-\imath\phi} - al\,\mathrm{e}^{-\imath\theta} + cl\,\mathrm{e}^{\imath\phi} + c^2 - ac\,\mathrm{e}^{\imath(\phi-\theta)} - al\,\mathrm{e}^{\imath\theta} - ac\,\mathrm{e}^{\imath(\theta-\phi)} + a^2 = b^2$$
$$\boxed{cl\cos\phi - ac\cos(\phi-\theta) - al\cos\theta = \frac{1}{2}\left(b^2 - a^2 - c^2 - l^2\right)}$$

This equation relates the two angular positions. One could differentiate the equation to relate the velocities and accelerations.

Solution 7.36

1. Let $w = u + \imath v$. First we do the strip: $|\Re(z)| < 1$. Consider the vertical line: $z = c + \imath y$, $y \in \mathbb{R}$. This line is mapped to

$$w = 2(c + \imath y)^2$$
$$w = 2c^2 - 2y^2 + \imath 4cy$$
$$u = 2c^2 - 2y^2, \quad v = 4cy$$

This is a parabola that opens to the left. For the case $c = 0$ it is the negative u axis. We can parametrize the curve in terms of v.

$$u = 2c^2 - \frac{1}{8c^2}v^2, \quad v \in \mathbb{R}$$

The boundaries of the region are both mapped to the parabolas:

$$u = 2 - \frac{1}{8}v^2, \quad v \in \mathbb{R}.$$

The image of the mapping is

$$\boxed{\left\{w = u + \imath v : v \in \mathbb{R} \text{ and } u < 2 - \frac{1}{8}v^2\right\}.}$$

Note that the mapping is two-to-one.

Now we do the strip $1 < \Im(z) < 2$. Consider the horizontal line: $z = x + \imath c$, $x \in \mathbb{R}$. This line is mapped to

$$w = 2(x + \imath c)^2$$
$$w = 2x^2 - 2c^2 + \imath 4cx$$
$$u = 2x^2 - 2c^2, \quad v = 4cx$$

This is a parabola that opens upward. We can parametrize the curve in terms of v.

$$u = \frac{1}{8c^2}v^2 - 2c^2, \quad v \in \mathbb{R}$$

The boundary $\Im(z) = 1$ is mapped to

$$u = \frac{1}{8}v^2 - 2, \quad v \in \mathbb{R}.$$

The boundary $\Im(z) = 2$ is mapped to

$$u = \frac{1}{32}v^2 - 8, \quad v \in \mathbb{R}$$

The image of the mapping is

$$\boxed{\left\{w = u + \imath v : v \in \mathbb{R} \text{ and } \frac{1}{32}v^2 - 8 < u < \frac{1}{8}v^2 - 2\right\}.}$$

2. We write the transformation as

$$\frac{z+1}{z-1} = 1 + \frac{2}{z-1}.$$

Thus we see that the transformation is the sequence:

(a) translation by -1

(b) inversion

(c) magnification by 2

(d) translation by 1

Consider the strip $|\Re(z)| < 1$. The translation by -1 maps this to $-2 < \Re(z) < 0$. Now we do the inversion. The left edge, $\Re(z) = 0$, is mapped to itself. The right edge, $\Re(z) = -2$, is mapped to the circle $|z+1/4| = 1/4$. Thus the current image is the left half plane minus a circle:

$$\Re(z) < 0 \quad \text{and} \quad \left|z + \frac{1}{4}\right| > \frac{1}{4}.$$

The magnification by 2 yields

$$\Re(z) < 0 \quad \text{and} \quad \left|z + \frac{1}{2}\right| > \frac{1}{2}.$$

The final step is a translation by 1.

$$\boxed{\Re(z) < 1 \quad \text{and} \quad \left|z - \frac{1}{2}\right| > \frac{1}{2}.}$$

Now consider the strip $1 < \Im(z) < 2$. The translation by -1 does not change the domain. Now we do the inversion. The bottom edge, $\Im(z) = 1$, is mapped to the circle $|z + \imath/2| = 1/2$. The top edge, $\Im(z) = 2$, is mapped to the circle $|z + \imath/4| = 1/4$. Thus the current image is the region between two circles:

$$\left|z + \frac{\imath}{2}\right| < \frac{1}{2} \quad \text{and} \quad \left|z + \frac{\imath}{4}\right| > \frac{1}{4}.$$

The magnification by 2 yields

$$|z + \imath| < 1 \quad \text{and} \quad \left|z + \frac{\imath}{2}\right| > \frac{1}{2}.$$

The final step is a translation by 1.

$$\boxed{|z - 1 + \imath| < 1 \quad \text{and} \quad \left|z - 1 + \frac{\imath}{2}\right| > \frac{1}{2}.}$$

Solution 7.37

1. There is a simple pole at $z = -2$. The function has a branch point at $z = -1$. Since this is the only branch point in the finite complex plane there is also a branch point at infinity. We can verify this with the substitution $z = 1/\zeta$.

$$
\begin{aligned}
f\left(\frac{1}{\zeta}\right) &= \frac{(1/\zeta + 1)^{1/2}}{1/\zeta + 2} \\
&= \frac{\zeta^{1/2}(1+\zeta)^{1/2}}{1 + 2\zeta}
\end{aligned}
$$

   Since $f(1/\zeta)$ has a branch point at $\zeta = 0$, $f(z)$ has a branch point at infinity.

2. $\cos z$ is an entire function with an essential singularity at infinity. Thus $f(z)$ has singularities only where $1/(1+z)$ has singularities. $1/(1+z)$ has a first order pole at $z = -1$. It is analytic everywhere else, including the point at infinity. Thus we conclude that $f(z)$ has an essential singularity at $z = -1$ and is analytic elsewhere. To explicitly show that $z = -1$ is an essential singularity, we can find the Laurent series expansion of $f(z)$ about $z = -1$.

$$
\cos\left(\frac{1}{1+z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z+1)^{-2n}
$$

3. $1 - \mathrm{e}^z$ has simple zeros at $z = \imath 2n\pi$, $n \in \mathbb{Z}$. Thus $f(z)$ has second order poles at those points.

   The point at infinity is a non-isolated singularity. To justify this: Note that

$$
f(z) = \frac{1}{(1 - \mathrm{e}^z)^2}
$$

   has second order poles at $z = \imath 2n\pi$, $n \in \mathbb{Z}$. This means that $f(1/\zeta)$ has second order poles at $\zeta = \frac{1}{\imath 2n\pi}$, $n \in \mathbb{Z}$. These second order poles get arbitrarily close to $\zeta = 0$. There is no deleted neighborhood around $\zeta = 0$ in which $f(1/\zeta)$ is analytic. Thus the point $\zeta = 0$, ($z = \infty$), is a non-isolated singularity. There is no Laurent series expansion about the point $\zeta = 0$, ($z = \infty$).

The point at infinity is neither a branch point nor a removable singularity. It is not a pole either. If it were, there would be an n such that $\lim_{z \to \infty} z^{-n} f(z) = \text{const} \neq 0$. Since $z^{-n} f(z)$ has second order poles in every deleted neighborhood of infinity, the above limit does not exist. Thus we conclude that the point at infinity is an essential singularity.

Solution 7.38
We write $\sinh z$ in Cartesian form.

$$w = \sinh z = \sinh x \cos y + \imath \cosh x \sin y = u + \imath v$$

Consider the line segment $x = c$, $y \in (0 \ldots \pi)$. Its image is

$$\{\sinh c \cos y + \imath \cosh c \sin y \mid y \in (0 \ldots \pi)\}.$$

This is the parametric equation for the upper half of an ellipse. Also note that u and v satisfy the equation for an ellipse.

$$\frac{u^2}{\sinh^2 c} + \frac{v^2}{\cosh^2 c} = 1$$

The ellipse starts at the point $(\sinh(c), 0)$, passes through the point $(0, \cosh(c))$ and ends at $(-sinh(c), 0)$. As c varies from zero to ∞ or from zero to $-\infty$, the semi-ellipses cover the upper half w plane. Thus the mapping is 2-to-1.

Consider the infinite line $y = c$, $x \in (-\infty \ldots \infty)$.Its image is

$$\{\sinh x \cos c + \imath \cosh x \sin c \mid x \in (-\infty \ldots \infty)\}.$$

This is the parametric equation for the upper half of a hyperbola. Also note that u and v satisfy the equation for a hyperbola.

$$-\frac{u^2}{\cos^2 c} + \frac{v^2}{\sin^2 c} = 1$$

As c varies from 0 to $\pi/2$ or from $\pi/2$ to π, the semi-hyperbola cover the upper half w plane. Thus the mapping is 2-to-1.

We look for branch points of $\sinh^{-1} w$.

$$
\begin{gathered}
w = \sinh z \\
w = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2} \\
\mathrm{e}^{2z} - 2w \,\mathrm{e}^z - 1 = 0 \\
\mathrm{e}^z = w + \left(w^2 + 1\right)^{1/2} \\
z = \log\left(w + (w - \imath)^{1/2}(w + \imath)^{1/2}\right)
\end{gathered}
$$

There are branch points at $w = \pm\imath$. Since $w + \left(w^2 + 1\right)^{1/2}$ is nonzero and finite in the finite complex plane, the logarithm does not introduce any branch points in the finite plane. Thus the only branch point in the upper half w plane is at $w = \imath$. Any branch cut that connects $w = \imath$ with the boundary of $\Im(w) > 0$ will separate the branches under the inverse mapping.

Consider the line $y = \pi/4$. The image under the mapping is the upper half of the hyperbola

$$
2u^2 + 2v^2 = 1.
$$

Consider the segment $x = 1$.The image under the mapping is the upper half of the ellipse

$$
\frac{u^2}{\sinh^2 1} + \frac{v^2}{\cosh^2 1} = 1.
$$