Here is a collection of practice problems suitable for the final exam.

1. Consider the real valued function:

$$f(x) = \frac{1}{\sqrt{1+x^4}} - \cos(x^2)\,.$$

(a) Find the *behavior* of $f(x)$ as $x \to 0$.

(b) Determine the value of $f(x)$ at $x = 0.01$ to four significant figures.

2. Evaluate the quantities,

$$\text{(a)}\quad \cos\left[2i\ln\left(\frac{1-i}{1+i}\right)\right]\,, \qquad\qquad \text{(b)}\quad i^{2/3}\,.$$

If either of these quantities is multivalued, you should provide all possible values.

3. Evaluate the conditionally convergent sum,

$$S \equiv \sum_{n=1}^{\infty} \frac{i^n}{n}\,.$$

*HINT:* If you replace $i$ with a variable $z$ in $S$, you should recognize the resulting power series.

4. Compute the inverse $A^{-1}$ of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\,.$$

5. In special relativity, the space-time coordinates of two inertial frames moving at relative constant velocity $v$ are related by

$$x' = \gamma(x - vt)\,,$$
$$t' = \gamma(t - vx/c^2)\,,$$

where $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ and $c$ is the velocity of light.

(a) Rewrite this system of equations in matrix form.

(b) Using Cramer's rule, solve for $x$ and $t$ in terms of $x'$ and $t'$.

6. If one of the eigenvalues of the matrix $A$ is $\lambda = 0$, prove that $A^{-1}$ does not exist.

7. If $A^{\mathsf{T}} = -A$, then we say that $A$ is a skew-symmetric (or antisymmetric) matrix. Prove that if $A$ is antisymmetric and $B$ is symmetric, then $\mathrm{Tr}(AB) = 0$.

*HINT:* Note that for any matrix $M$, $\mathrm{Tr}\, M^{\mathsf{T}} = \mathrm{Tr}\, M$.

8. Let $A$ be a complex $n \times n$ matrix. Prove that the eigenvalues of $AA^\dagger$ are real and non-negative.

*HINT:* Let $\vec{\boldsymbol{w}} = A^\dagger \vec{\boldsymbol{v}}$, where $\vec{\boldsymbol{v}}$ is an eigenvector of $AA^\dagger$. Investigate the consequence of the fact that the inner product $\langle \vec{\boldsymbol{w}}, \vec{\boldsymbol{w}} \rangle$ is non-negative in a complex Euclidean space.

9. Suppose that the $n \times n$ matrix $M$ is diagonalizable. Then, one can find an invertible matrix $S$ such that $S^{-1}MS = D$ where $D$ is diagonal. Show that $M^n = SD^nS^{-1}$, where $n$ is a positive integer. This provides a simple way to compute $M^n$ since raising a diagonal matrix to a power is easy. Using these considerations, compute $M^{10}$ where

$$M = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.$$

10. Determine whether the following matrices are diagonalizable. If diagonalizable, indicate whether it is possible to diagonalize the matrix with a unitary similarity transformation.

$$\text{(a) } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \text{(b) } A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}.$$

11. A linear transformation $T$ that is represented by a matrix $M$ with respect to the standard basis $\mathcal{B}$ is also represented by the matrix $P^{-1}MP$ with respect to a different basis $\mathcal{B}'$, where $P$ is the invertible matrix that transforms the basis $\mathcal{B}$ into the basis $\mathcal{B}'$.

(a) Show that the characteristic equation for determining the eigenvalues of $T$ does not depend on the choice of basis.

*HINT:* Note that $P^{-1}MP - \lambda\mathbf{I} = P^{-1}(M - \lambda\mathbf{I})P$.

(b) Show that the eigenvalues of $T$ do not depend on the choice of basis.

12. Suppose that $U$ is a $n \times n$ matrix whose columns comprise an orthonormal set of column vectors. Prove that $U$ is a unitary matrix. Moreover, show that the rows of $U$ must also comprise an orthonormal set of row vectors.

*HINT:* In this problem, orthonormality must be defined with respect to a *complex* vector space.

13. A linear transformation $A$ is represented by the matrix:

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix},$$

with respect to the standard basis $\mathcal{B} = \{(1,0,0),(0,1,0),(0,0,1)\}$. Consider a new basis, $\mathcal{B}' = \{(2,-2,1),(1,1,0),(-1,1,4)\}$, where the components of the basis vectors of $\mathcal{B}'$ are given with respect to the standard basis $\mathcal{B}$.

(a) What are the components of the basis vectors of $\mathcal{B}$ when expressed relative to the basis $\mathcal{B}'$?

(b) Determine the matrix representation of $A$ relative to the basis $\mathcal{B}'$.

14. Eq. (11.27) on p. 154 of Boas is imprecisely worded. The correct statement is that a matrix has real eigenvalues and can be diagonalized by an orthogonal similarity transformation transformation if and only if it is a *real* symmetric matrix. (Boas omitted the qualification that the symmetric matrix must be real.) To see that it is important to be precise, consider the following *complex* symmetric matrix,

$$A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

(a) Determine the eigenvalues and eigenvectors of $A$.

(b) How many linearly independent eigenvectors of $A$ exist?

(c) Is $A$ diagonalizable?

15. Consider the matrix

$$M = \begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix},$$

where $a$ and $b$ are arbitrary complex numbers.

(a) Compute the eigenvalues of $M$.

(b) Find a matrix $C$ such that $C^{-1}MC$ is diagonal.

(c) Compute $e^M$.

*HINT:* Denote $D = C^{-1}MC$ where $D$ is the diagonal matrix obtained in part (b). Show that

$$e^M = e^{CDC^{-1}} = Ce^D C^{-1}. \qquad (1)$$

Using the results of parts (a) and (b), first evaluate $e^D$ and then use eq. (1) to compute $e^M$.

(d) Verify that $\det(e^M) = e^{\mathrm{Tr}\, M}$.

(e) In light of the hint given in part (c) and result of part (d), can you see how to prove that $\det e^A = e^{\mathrm{Tr} A}$ is true for any diagonalizable matrix $A$?

16. A population of black bears consists of $A_k$ adults and $J_k$ juveniles, where $k$ is the year under consideration. Each year, 40% of the juveniles die and 10% of the juveniles mature into adults. Meanwhile one new cub is born on average for every 2.5 adults in the population thereby adding to the population of the juveniles. In equations,

$$J_{k+1} = 0.5J_k + 0.4A_k\,. \tag{2}$$

Meanwhile, some of the adults die off each year as well, and after one year 80% of the adults remain. That is,

$$A_{k+1} = 0.1J_k + 0.8A_k\,. \tag{3}$$

(a) The equations governing the black bear population can be written in the form

$$\begin{pmatrix} J_{k+1} \\ A_{k+1} \end{pmatrix} = M \begin{pmatrix} J_k \\ A_k \end{pmatrix}\,, \tag{4}$$

where $M$ is a $2 \times 2$ matrix. Determine this matrix $M$.

(b) Assume that at $k = 0$, the population of juvenile and adult black bears is given by $J_0$ and $A_0$, respectively. Write a matrix equation that determines $J_n$ and $A_n$ in terms of $J_0$ and $A_0$, under the assumption that the dynamics of the black bear population as described by eq. (4) does not change over time.

(c) By computing $\lim_{n\to\infty} M^n$, determine whether or not black bears will eventually go extinct.

(d) Based on the result of part (c), can you state a general criterion on the property of $M$ which determines whether or not extinction is inevitable?

17. Four equal mass balls lying along the $x$-axis are attached by three springs. The two outermost balls are fixed, while the two innermost balls are free to oscillate in the $x$ direction. Denote the displacement from equilibrium of the two balls by $x_1$ and $x_2$. The total potential energy of the system is:

$$V = \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_2 - x_1)^2\,.$$

(a) Show that the equations of motion for the two displacements can be cast into the matrix equation:

$$\frac{d^2\vec{\boldsymbol{x}}}{dt^2} = A\vec{\boldsymbol{x}}\,,$$

where $\vec{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A$ is a $2 \times 2$ matrix.

(b) Diagonalize the matrix $A$ and determine the two possible frequencies of vibrations. These are the *normal modes* of the system.

18. In Example 4 on pp. 441–442 of Boas, the following system of differential equations is solved by the Laplace transform method,

$$y' - 2y + z = 0\,, \qquad y_0 \equiv y(t=0) = 1\,, \tag{5}$$
$$z' - y - 2z = 0\,, \qquad z_0 \equiv z(t=0) = 0\,, \tag{6}$$

where $y' \equiv dy/dt$ and $z' = dz/dt$. In this problem, you will solve this system of equations by another technique.

(a) Show that eqs. (5) and (6) can be rewritten as a matrix differential equation of the form

$$\frac{d\vec{\boldsymbol{x}}}{dt} = A\vec{\boldsymbol{x}}\,, \qquad \vec{\boldsymbol{x}}_0 \equiv \vec{\boldsymbol{x}}(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{7}$$

where

$$\vec{\boldsymbol{x}} = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}.$$

Identify the matrix $A$ such that eq. (7) is equivalent to eqs. (5) and (6).

(b) Show that if $A$ is a matrix of constant coefficients, then the solution to eq. (7) is given by

$$\vec{\boldsymbol{x}}(t) = e^{At}\,\vec{\boldsymbol{x}}_0\,. \tag{8}$$

(c) Using the matrix $A$ obtained in part (a), compute $e^{At}$. Use the method outlined in parts (a)–(c) of problem 15.

(d) Insert the result of part (c) into eq. (8), and show that the solution you have obtained for eqs. (5) and (6) matches the result obtain by Boas on p. 442.

19. Radioactive decay is governed by an exponential decay law,

$$\frac{dN}{dt} = -\lambda N\,. \tag{9}$$

(a) Suppose that the number of atoms of an unstable nucleus at time $t = 0$ is equal to $N(t=0) = N_0$. The half-life, denoted by $t_{1/2}$, is the time it takes for half of the atoms to decay. That is, $N(t_{1/2}) = \frac{1}{2}N_0$. Solve eq. (9) subject to the stated initial condition and find a formula for $t_{1/2}$ in terms of the parameter $\lambda$ that appears in eq. (9).

(b) An unstable nucleus, $A$, with a half-life of $t^A_{1/2}$, decays into a nucleus, $B$. The nucleus $B$ is also unstable, with a half-life of $t^B_{1/2}$, which you may assume is different from $t^A_{1/2}$. The unstable nucleus $B$ decays into a stable nucleus, $C$.

At time $t = 0$, there are $N_0$ nuclei of type $A$. Assume that no nuclei of type $B$ and $C$ are present at time $t = 0$. The exponential decay equations that govern the decays of $A$ and $B$ have been given in Example 2 on pp. 402–403 of Boas. Suppose that $t^A_{1/2} = 30$ minutes and $t^B_{1/2} = 60$ minutes. If $N_0 = 1024$, how many nuclei of types $A$, $B$ and $C$ remain after four hours?

20. Solve the equation,

$$y'' + y = 8x \sin x\,, \tag{10}$$

subject to the initial conditions, $y_0 \equiv y(0) = 0$ and $y'_0 \equiv (dy/dx)\big|_{x=0} = 0$. In this problem, you will solve eq. (10) by four different methods.

(a) Solve eq. (10) by using the method of undetermined coefficients [cf. eq. (6.24) on p. 421 of Boas] by first finding the most general solution to

$$y'' + y = 8xe^{ix}\,. \tag{11}$$

Then by taking the imaginary part of the solution to eq. (11), one obtains the general solution to eq. (10) before imposing the two initial conditions stated above, Finally, impose those initial conditions to obtain the final result.

(b) Use the Laplace transform technique to solve eq. (10).

(c) In the class handout, *Applications of the Wronskian to linear differential equations*, it is shown that for the differential equation $y' + a(x)y' + b(x)y = f(x)$, a particular solution is given by,

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)}\, dx\,, \tag{12}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions to the homogeneous equation, $y' + a(x)y' + b(x)y = 0$, and the Wronskian is given by $W(x) \equiv y_1(x)y'_2(x) - y'_1(x)y_2(x)$. Apply the result of eq. (12) to obtain the most general solution of eq. (11). Then, use this solution to solve eq. (10) subject to the initial conditions stated above.

(d) This method corresponds to the reduction of order technique described in eqs. (7.21) and (7.22) on p. 434 of Boas. Consider the differential equation.

$$y'' + a(x)y' + b(x)y = f(x)\,, \tag{13}$$

If one solution, $y_1(x)$ of the corresponding homogeneous equation, $y'' + a(x)y' + b(x)y = 0$ is known, then the solution to eq. (13) can be obtained with the change of variables,

$$y(x) = y_1(x)v(x)\,.$$

Then, eq. (13) is converted into a differential equation where the variable $y$ is replaced by $v$. Since $y_1(x)$ solves the homogeneous equation, you will find that the resulting differential equation in terms of $v$ is of the form,

$$y_1(x)v'' + g(x)v' = f(x)\,, \tag{14}$$

where $g(x)$ is a function that depends on $y_1(x)$ and $a(x)$. Note that there is no term on the left hand side of eq. (14) that is proportional to $v$. Thus, one can now introduce $w = v'$ and obtain a first order linear differential equation in $w$, which can be exactly solved.

Apply this technique to solve eq. (11). Show that if $y_1(x) = e^{ix}$, then the resulting equation for $w$ is given by,

$$w' + 2iw = 8ix\,.$$

Solve this equation for $w$, and then use this result to find the most general solution of eq. (11). Finally, use this solution to solve eq. (10) subject to the initial conditions stated above.

21. Solve the following differential equations. Your solution should correspond to the most general form (depending on some number of arbitrary constants), unless initial conditions are specified.

(a) $(2x - y\sin 2x)dx + (2y - \sin^2 x)dy = 0$.

(b) $x^2y' - xy = 1/x$.

(c) $y' = \dfrac{y}{x} - \tan\left(\dfrac{y}{x}\right)$.

(d) $2xy'' + y' = 2x^{5/2}$, assuming $x > 0$.

(e) $yy'' + (y')^2 + 4 = 0$, subject to the initial conditions, $y(1) = 3$ and $(dy/dx)_{x=1} = 0$.

[*HINT*: Introduce a new variable $v = y'$ and use the chain rule to obtain $v' = v\dfrac{dv}{dy}$. In the resulting first order differential equation, $v$ becomes the dependent variable and $y$ becomes the independent variable.]

(f) $x^2y'' - 2xy' + 2y = x\ln x$, assuming that $x > 0$.

(g) $(x^2 + 1)y'' - 2xy' + 2y = 0$, subject to the initial conditions $y_0 = y_0' = 1$.

[*HINT*: Note that $y(x) = x$ is a solution of $(x^2 + 1)y'' - 2xy' + 2y = 0$ prior to imposing the initial conditions.]

22. Using the Laplace transform technique, solve the following system of differential equations,

$$\begin{aligned} y' + z &= 2\cos t\,, \\ z' - y &= 1\,, \end{aligned}$$

subject to the initial conditions $y_0 = -1$ and $z_0 = 1$, where $y' \equiv dy/dt$ and $z' \equiv dz/dt$.