Here is a collection of practice problems suitable for the midterm exam.

1. Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0}\left(\frac{1+x}{x}-\frac{1}{\sin x}\right)$,

(b) $\displaystyle\lim_{n\to\infty}\sqrt{n^2+3n}-n$,

(c) $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$.

2. Find the radius of convergence of the following three series:

$$\text{(a)}\ \sum_{n=1}^{\infty}\frac{x^n}{\ln(n+1)}\,,\qquad \text{(b)}\ \sum_{n=0}^{\infty}\frac{(n!)^2x^n}{(2n)!}\,,\qquad \text{(c)}\ \sum_{n=0}^{\infty}\frac{n^2(x-5)^n}{5^n(n^2+1)}\,.$$

3. Determine whether the following series are absolutely convergent, conditionally convergent or divergent.

$$\text{(a)}\ \sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}\,,\qquad \text{(b)}\ \sum_{n=1}^{\infty}\frac{(-1)^n}{2^{\ln n}}\,,\qquad \text{(c)}\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}\,.$$

4. Without using your calculator, compute the cube root of 1.09, with an accuracy of four decimal places.

5. What is the *behavior* of the function:

$$f(x)=-1+\frac{1}{x^2}\left[\frac{1}{(1+x^2)^{3/2}}-\frac{1}{(1+x^2)^{5/2}}\right]\,,$$

as $x\to 0$? (Obtaining the limit as $x\to 0$ is not sufficient.)

6. Evaluate $f(x)=\ln\sqrt{(1+x)/(1-x)}-\tan x$ at $x=0.0015$ without a calculator. Determine the numerical accuracy of your result. Is your calculator a useful tool for this problem? (Try it!)

7. For each expression find all possible values and express your result both in the form $x + iy$ and in polar form $re^{i\theta}$, where θ is the principal value of the argument.

(a) $i^{77} + i^{202}$ (b) $\dfrac{3+i}{2+i}$ (c) $\sqrt{-2 + 2i\sqrt{3}}$

(d) $\left(\dfrac{1+i}{1-i}\right)^4$ (e) $\sqrt[4]{16}$

8. Let $z = 1 - i$. Express each of the following in the form of $x + iy$. For any multi-valued function, you should indicate all possible values of the result.

(a) $\cos(1/z)$ (b) z^z (c) $\tan(z-1)$

(d) $\operatorname{Ln} z$ (e) $\arg z$

9. Solve for all possible values of the real numbers x and y in the following equations:

(a) $x + iy = y + ix$.

(b) $\dfrac{x+iy}{x-iy} = -i$.

10. Find the disk of convergence of the following complex power series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{z^{2n}}{(2n+1)!}$. (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(z-i)^n}{n}$.

11. Evaluate the integral

$$\int_0^{\pi} \sin 3x \cos 4x \, dx \, .$$

HINT: Rewrite the trigonometric functions in exponential form.

12. Evaluate the following quantities:

(a) $(1)^{\pi}$

(b) $\arg(e^{x+iy})$, where x and y are real numbers

Be sure to indicate all possible values if the quantity in question is multi-valued. Simplify your expressions as much as possible.

13. Find all complex number solutions z to the equation, $e^z = 1 - i$.

14. Consider the real-valued function:

$$f(x) = \tfrac{1}{2} \ln \left(\frac{1+x}{1-x} \right) .$$

(a) Determine the Taylor series expansion of $f(x)$ about the point $x = 0$. Write the series using summation notation (that is, you will need to determine the general term in the series).

(b) Determine all possible values of x for which the series obtained in part (a) converges.

(c) Evaluate explicitly the sum

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{1}{2n+1} .$$

Use your calculator to compute the sum of the first four terms of the series, and compare this numerical approximation with the exact result.

HINT: You may find the results of part (a) helpful in this regard.

15. Assume that p is a real parameter such that $-1 < p < 1$.

(a) Compute the following sum:

$$\sum_{n=0}^{\infty} p^n e^{in\theta} .$$

(b) Using the results of part (a), compute the sum

$$\sum_{n=0}^{\infty} p^n \cos(n\theta) .$$

Verify that your result for the sum in part (b) has the correct form in the $\theta \to 0$ limit.

16. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 0 \\ 0 & 5 & 1 \end{pmatrix} , \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{pmatrix} ,$$

Compute AB, BA, det A, det B, det AB and det BA. Verify that $AB \neq BA$ and det $AB = (\det A)(\det B)$.

17. Let A be a 3×3 matrix. Assume that $A \neq 0$. The determinant of A is denoted by $\det(A)$.

(a) Is the equation $\det(3A) = 3\det(A)$ true or false? Explain.

(b) Suppose that $\det(A) = 1$. Let B be a matrix obtained from A by permuting the order of the rows so that the first row of A is the second row of B, the second row of A is the third row of B and the third row of A is the first row of B. (This is called a *cyclic permutation*.) What is the value of $\det(B)$?

(c) Suppose that the 3×3 matrix $A \neq 0$ but $\det(A) = 0$. What can you say about the rank of A?

18. Consider the system of equations:

$$\begin{aligned} 5x + 2y + z &= 2 \\ x + y + 2z &= 1 \\ 3x - 3z &= 0\,. \end{aligned}$$

(a) What is the augmented matrix for this system of equations?

(b) Solve this system of equations using Gaussian elimination.

(c) What is the rank of the augmented matrix of part (b)?

(d) Remove the third equation above, and solve the new system of two equations and three unknowns using Gaussian elimination. What is the rank of the corresponding augmented matrix.

19. Evaluate the following determinant by hand:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}\,.$$

HINT: Simplify the matrix using appropriate row and/or column operations.

20. Find z by Cramer's rule (*NOTE*: you are not being asked to find x and y),

$$\begin{cases} (a-b)x - (a-b)y + 3b^2 z = 3ab\,, \\ (a+2b)x - (a-2b)y - (3ab^2 + 3b^2)z = 3b^2\,, \\ bx + ay - (2b^2 + a^2)z = 0\,, \end{cases} \tag{1}$$

where a and b are arbitrary real numbers.

21. A complex number $x+iy$ can be represented by the 2×2 matrix

$$Z=\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\,, \tag{2}$$

where x and y are real numbers. Verify that this is a sensible representation by answering the following questions.

(a) Show that the matrix representation of $(x+iy)(a+ib)$ is equal to

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\,.$$

To show this, you should express the product $(x+iy)(a+ib)$ in the form of $X+iY$ and show that the matrix product above, when evaluated, is consistent with the form given by eq. (2).

(b) Show that the matrix representation of the complex number $(x+iy)^{-1}$ is correctly given by the inverse of eq. (2). Here, the inverse of Z (denoted by Z^{-1}) satisfies the matrix equation, $ZZ^{-1}=\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(c) How is the determinant of the matrix given in eq. (2) related to the corresponding complex number, $x+iy$?