INSTRUCTIONS: During the exam, you may refer to the textbook, the class handouts, or your own personal notes. Collaboration with your neighbor is strictly forbidden. In answering the questions, it is not sufficient to simply write the final result. You must provide the intermediate steps needed to arrive at the solution in order to get full credit. You may quote results that have been derived in the textbook, homework solutions and class handouts (but if you do so, please cite explicitly the source of any such quotations).

The exam consists of six problems with a total of 18 parts. Each part is worth ten points, for a total of 180 points.

1. Consider the function $F(x) = \dfrac{x}{1-x-2x^2}$.

(a) Show that $F(x)$ can be written as a sum or difference of two simpler terms. In particular, find the constants A and B such that

$$\frac{x}{1-x-2x^2} = \frac{A}{1-2x} + \frac{B}{1+x}\,.$$

(b) Express $F(x)$ as a power series about $x = 0$,

$$F(x) = \sum_{n=0}^{\infty} a_n x^n\,. \tag{1}$$

This is most easily done by separately expanding the two terms obtained in part (a) and then combining the two sums. Determine a closed-form expression for a_n as a function of n. Write out the first seven values of a_n (for $n = 0, 1, 2, \ldots, 6$).

(c) What is the radius of convergence of the series obtained in eq. (1)?

2. Consider the matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{2}$$

(a) Evaluate A^2. Then show that $A^n = 0$ for all integer $n \geq 3$.

(b) Evaluate $(\mathbf{I} - A)^{-1}$, where $\mathbf{I}$ is the 3×3 identity matrix. Choose any technique for computing the inverse that you prefer.

(c) Check your answer to part (b) by the following sophisticated method. Use the geometric series to define:

$$(\mathbf{I} - A)^{-1} = \sum_{k=0}^{\infty} A^k\,.$$

where $A^0 = \mathbf{I}$ (assuming that the sum converges). Using the result of part (a), sum the series when A is given by eq. (2), and verify that you reproduce the result of part (b).

3. Consider the following system of equations:

$$\begin{aligned} 2x + 2y + 3z &= 0\,, \\ 4x + 8y + 12z &= -4\,, \\ 6x + 2y + cz &= 4\,. \end{aligned}$$

(a) Determine all values of c for which there is a unique solution, and compute the solution for these cases.

(b) Determine all values of c for which there are an infinite number of solutions, and give the general solution for these cases.

(c) Determine all values of c, if any, for which this system of equations is inconsistent.

4. Consider the following interesting series of numbers:

$$0, 1, 1, 3, 5, 11, 21, \ldots \tag{3}$$

This series has been generated by the following rules. First, we define

$$x_0 = 0 \quad \text{and} \quad x_1 = 1\,. \tag{4}$$

Then for all positive integers $n = 1, 2, 3, \ldots,$

$$x_{n+1} = x_n + 2x_{n-1}\,. \tag{5}$$

Starting with $x_0 = 0$ and $x_1 = 1$, we can derive the values for x_2, x_3, $x_4, \ldots$ sequentially. For example, setting $n = 1$ in eq. (5) yields $x_2 = x_1 + 2x_0 = 1$. Next we can determine $x_3 = x_2 + 2x_1 = 3$ followed by $x_4 = x_3 + 2x_2 = 5$, etc. However, this is a very inefficient way of computing x_n for some large value of n (as it would take n separate computations).

Matrix methods can help us derive a simple rule for directly determining an arbitrary term x_n in the series. Consider the matrix equation:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}\,. \tag{6}$$

(a) Show that this matrix equation is equivalent to the rule given in eq. (5).

(b) Defining the matrix:

$$M \equiv \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\,,$$

which appears in eq. (6), prove that for any non-negative integer n:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,. \tag{7}$$

HINT: Verify eq. (7) for $n = 0$ and $n = 1$. Then iterate the process using eq. (6).

(c) Compute M^n (for arbitrary n) by first diagonalizing the matrix M and raising the resulting diagonal matrix to the nth power. Once you have obtained an expression for M^n, use eq. (7) to write an explicit formula for x_n as a function of n. Check that your formula reproduces the series given in eq. (3).

5. Consider the differential equation,

$$y'' + 2y' = 4x\,,$$

subject to the initial conditions $y_0 = 0$ and $y'_0 = 1$, where $y_0 \equiv y(0)$ and y'_0 is the value of dy/dx evaluated at $x = 0$.

(a) Solve this problem by converting it into a first order differential equation by making an appropriate substitution.

(b) Solve this problem by employing the method of undetermined coefficients as described in Section 6 in Chapter 8 of Boas.

Did you get the same answer by using the methods suggested in parts (a) and (b)?

6. Consider the following system of linear differential equations that has been used to model the dynamics of love affairs,

$$\frac{dr}{dt} = -ar + bj\,, \tag{8}$$

$$\frac{dj}{dt} = br - aj\,, \tag{9}$$

where the constants a and b are positive. With apologies to Shakespeare, in this model $r(t)$ is a measure of Romeo's love/hate for Juliet at time t, and $j(t)$ is a measure of Juliet's love/hate for Romeo at time t. Positive values of $r(t)$ and $j(t)$ indicate love whereas negative values of $r(t)$ and $j(t)$ indicate hate. Note that $r(t) = j(t) = 0$ would indicate complete indifference at time t.

Due to the minus signs in eqs. (8) and (9), a is a measure of cautiousness (Romeo and Juliet each try to avoid being too eager), whereas b is a measure of responsiveness (where both get excited by the other's advances). The question now is: what is the fate of Romeo and Juliet's relationship as t gets very large? The answer depends on the initial condition and the relative size of the constants a and b.

(a) Solve eqs. (8) and (9) by the Laplace transform method for $r(t)$ and $j(t)$ assuming an initial condition of $r_0 = 1$ and $j_0 = 0$, where $r_0 \equiv r(0)$ and $j_0 \equiv j(0)$.

(b) If $a > b$, what is the fate of the relationship (i.e., what is the limiting behavior of $r(t)$ and $j(t)$ as t becomes very large)?

(c) If $b > a$, what is the fate of the relationship (as t becomes very large)?

(d) If the initial condition is $r_0 = 0$ and $j_0 = -1$, how would your conclusions in parts (b) and (c) change?