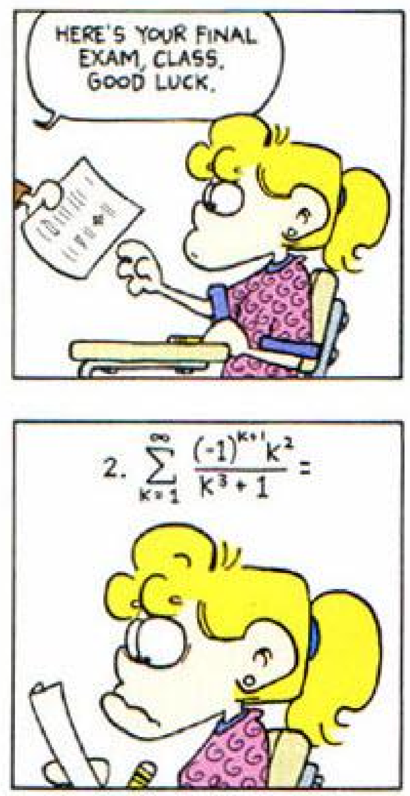
INSTRUCTIONS: During the exam, you may refer to the textbook (Boas), the class handouts (including solution sets to the homeworks and the practice problems and exam) or your own personal notes. Collaboration with your neighbor is strictly forbidden.

In answering the questions, it is not sufficient to simply give a final result. You must provide the intermediate steps needed to arrive at the final solution in order to get full credit. *However, if you are employing any result already obtained in class or in the textbook, you do not need to re-derive it.* The point value of each problem is indicated in brackets. The total number of points available is 100.

1. [30]



(a) In *The Best of Foxtrot*, Volume 1, cartoonist Bill Amend (a physics major from Amherst College), admitted that he made up this problem but it turned out to be "crazy hard." So I will be more forgiving than Paige's professor and simply ask you whether the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^2}{k^3+1}$$

converges absolutely, converges conditionally or diverges. Explain the reasoning behind your conclusion.

(b) Find the interval of convergence of the following power series:

$$g(x) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} ,$$

where x is a real variable.

(c) Using part (b), evaluate $\dfrac{dg}{dx}$ and sum the resulting series to obtain a known elementary function.

2. [20] Consider the real-valued function,

$$f(x) = \frac{1}{x^3}\left[1 + ax - 2\ln(1+ax) - \frac{1}{1+ax}\right],$$

where a is a non-zero finite constant. Find the *behavior* of $f(x)$ as $x \to 0$. (Obtaining the limit of $f(x)$ as $x \to 0$ is not sufficient.)

3. [20] Consider the complex number $w = e^{2\pi i/n}$, where n is an integer larger than 2.

(a) Find the sum of the following finite series,

$$S \equiv \sum_{k=1}^{n} w^{2k} = w^2 + w^4 + w^6 + \cdots + w^{2n}.$$

(b) What is the value of the sum S in part (a) for the special cases of $n = 1$ and $n = 2$?

4. [10] Evaluate the quantity,

$$\mathrm{Im}\big[\ln(i-1)\big].$$

If this quantity is multivalued, you should provide all possible values.

5. [20] In the following problem, c is some (as yet unspecified) number.

(a) For what values of c (if any) will the following equations,

$$x + y + z = 6,$$

$$x + cy + cz = 2,$$

have either a unique solution, an infinite number of solutions, or no solutions.

(b) Determine the values of x, y and z that solve the equations of part (a) as a function of c, in cases where solutions exist. If an infinite number of solutions exist, write the solution set for (x, y, z) that indicates all the possible solutions.

NOTE: Any solution that you write down must explicitly depend on c. Do not choose any specific value for c, but express any solution that you find in terms of this variable.