INSTRUCTIONS: This is a three-hour exam. During the exam, you may refer to the textbook, the class handouts, or your own personal notes. Collaboration with your neighbor is strictly forbidden. In answering the questions, it is not sufficient to simply write the final result. You must provide the intermediate steps needed to arrive at the solution in order to get full credit. You may quote results that have been derived in the textbook, homework solutions and class handouts (but if you do so, please cite explicitly the source of any such quotations).

The exam consists of six problems with a total of 18 parts. Each part is worth ten points, for a total of 180 points.

1. The inverse hyperbolic tangent can be defined via the following integral:

$$\frac{\tanh^{-1} x}{x} = \int_0^1 \frac{dt}{1 - x^2 t^2}\,.$$

(a) By expanding the denominator of the integral, derive the power series expansion for $\tanh^{-1} x$ about $x = 0$. Determine the general form for the nth term of this power series.

(b) What is the radius of convergence, R, of the power series obtained in part (a)? Does this series converge at $x = R$? Does this series converge at $x = -R$?

(c) Evaluate the integral:

$$\int_0^1 \frac{\tanh^{-1} t}{t}\,dt\,, \tag{1}$$

and express the result as an infinite series.

HINT: Using the power series expansion of part (a), you can integrate term by term. [*EXTRA CREDIT:* Sum the resulting series.]

2. Consider the matrix

$$M = \begin{pmatrix} x & 1 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{pmatrix}.$$

(a) Find all values of x for which the inverse of M does *not* exist. [*HINT:* You do not have to compute M^{-1} to answer this question.]

(b) Determine the rank of M as a function of x. [*HINT:* Consider the explicit form of M for the values of x found in part (a).]

3. Consider the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$

(a) Compute the eigenvalues and eigenvectors of A. Find the diagonalizing matrix P such that $D \equiv P^{-1}AP$ is a diagonal matrix.

(b) Compute e^D. Then, using the result of part (a), compute e^A.

HINT: e^A is related in a simple way to e^D, which is easy to evaluate.

(c) Using the Cayley-Hamilton theorem, find a general expression for A^n in terms of A. Then compute e^A directly from its series expansion. Verify that the result agrees with the one obtained in part (b).

(d) Compute e^B and e^{A+B} using one of the two methods above (your choice!) for evaluating the matrix exponential. Verify that $e^{A+B} \neq e^A e^B$ in this problem.

4. We have learned two methods in this class for computing the inverse of a matrix. One method involves row reduction and the second method involves the transpose of the cofactor matrix. Consider the matrix

$$M = \begin{pmatrix} 4 & 0 & -1 \\ -2 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}. \tag{2}$$

(a) Using one of the two methods mentioned above, compute M^{-1}. Check your result by computing MM^{-1}.

(b) Here is a third method for computing M^{-1}. Diagonalize M and take the inverse of the diagonalizing equation. Then solve for M^{-1} (your formula should involve the inverse of a diagonal matrix, which can be obtained by inspection). Apply this technique to the matrix M given by eq. (2). Verify that the result obtained for M^{-1} by this method is correct.

(c) Here is a fourth method for computing M^{-1}. By the Cayley-Hamilton theorem, M solves its own characteristic equation. Compute the characteristic equation for the matrix M given by eq. (2). Multiply this equation by M^{-1}, and show that M^{-1} can be expressed in terms of M^2, M and the identity matrix. Use this result to evaluate M^{-1}, and compare with the results of parts (a) and (b).

5. An epidemic spreads through a population at a rate proportional to the product of the number of people already infected and the number of people susceptible, but not yet infected. Let S denote the total population of susceptible people and $I(t)$ denote the number of infected people at time t. However, there is a medication that cures the infected population at a rate proportional to the number of infected individuals.

A simple differential equation that describes how the epidemic spreads is,

$$\frac{dI}{dt} = rI(S - I) - qI \,, \tag{3}$$

where r and q are positive real constants.

(a) At time $t = 0$, the number of infected people is given by I_0 (assumed to be less than the number of susceptible people S). Solve the differential equation given by eq. (3) subject to the stated initial condition, assuming that $q \neq rS$.

(b) How many susceptible people will still be infected in the $t \to \infty$ limit? Consider separately the cases of $q > rS$ and $q < rS$. (Note: to simplify matters, we have ignored the fact that people do not live forever.)

(c) In the limiting case of $q = rS$, answer questions (a) and (b).

HINT: In the case of $q = rS$, the best strategy is to go back to eq. (3) and solve the problem from the beginning.

6. Consider the differential equation,

$$y'' - 7y' + 12y = 5e^{4x} \,, \tag{4}$$

subject to the initial conditions, $y_0 = 0$ and $y'_0 = 1$, where $y_0 \equiv y(0)$ and y'_0 is the value of dy/dx at $x = 0$.

(a) Show that no particular solution to eq. (4) of the form $y(x) = Ae^{4x}$ exists for any choice of the constant A.

(b) Find the solution to eq. (4) subject to the initial conditions stated above by employing the method of undetermined coefficients.

(c) Find the solution to eq. (4) subject to the initial conditions stated above by employing the Laplace transform method.

Did you get the same answer in parts (b) and (c)?