THE AMAZING NUMBER Π

PETER BORWEIN

ABSTRACT. This text accompanies an address given at the celebration to replace the lost tombstone of Ludolph van Ceulen at the Pieterskerk (St Peter's Church) in Leiden on the fifth of July, 2000. A version of this paper appears in the Nieuw Archief voor Wiskunde **1** (2000), pp254–258.

It honours the particular achievements of Ludolph as well as the long and important tradition of intellectual inquiry associated with understanding the number π and numbers generally.

The author would like pay tribute to the mathematical community of the Netherlands on the occasion of its honouring one of its founding fathers the mathematician, Ludolph van Ceulen (1540–1610).

The amazing number π:

The history of π parallels virtually the entire history of Mathematics. At times it has been of central interest and at times the interest has been quite peripheral (no pun intended). Certainly Lindemann's proof of the transcendence of π was one of the highlights of nineteenth century mathematics and stands as one of the seminal achievements of the millennium (very loosely this result says that π is not an easy number). One of the low points was the Indiana State legislatures attempts to legislate a value of π in 1897: an attempt as plausible as repealing the law of gravity.

The amount of human ingenuity that has gone into understanding the nature of π and computing its digits is quite phenomenal and begs the question "why π?". After all there are more numbers than one can reasonably contemplate that could get a similar treatment. And π is just one of the very infinite firmament of numbers. Part of the answer is historical. It is the earliest and the most naturally occurring hard number (technically, hard means transcendental which means not the solution of a simple equation). Even the choice of label "transcendental" gives it something of a mystical aura.

What is pi? First and foremost it is a number, between 3 and 4 (3.14159...). It arises in any computations involving circles: the area of a circle of radius 1 or equivalently, though not obviously, the perimeter of a circle of radius $1/2$. The nomenclature π is presumably the Greek letter "p" in periphery. The most basic properties of π were understood in the period of classical Greek mathematics by the time of the death of Archimedes in 212 BC.

The Greek notion of number was quite different from ours, so the Greek numbers were our whole numbers: 1,2, 3... . In Greek geometry the essential idea was not number but continuous magnitude, e.g. line segments. It was based on the notion of multiplicity of units and, in this sense, numbers that existed were numbers that could be drawn with just an unmarked ruler and compass. The rules allowed for starting with a fixed length of 1 and seeing what could be constructed with straight edge and compass alone. (Our current notion is much more based on counting.) The question of whether π is a constructible magnitude had been explicitly raised as a question by the sixth century BC and the time of the Pythagoreans. Unfortunately π is not constructible, though a proof of this would not be available for several thousand years. In this context there isn't a more basic question than "is π a number?" Of course, our more modern notion of number embraces the Greek notion of constructible and doesn't depend on construction. The existence of π as a number given by an infinite (albeit unknown) decimal expansion poses little problem.

Very early on the Greeks had hypothesized that π wasn't constructible, Aristophanes already makes fun of "circle squarers" in his fifth century BC play "The Birds."

Lindemman's proof of the transcendence of π in 1882 settles the issue that π is not constructible by the Greek rules and a truly marvelous proof was given a few years later by Hilbert. Not that this has stopped cranks from still trying to construct π.

Does this tell us everything we wish to know about π. No, our ignorance is still much more profound than our knowledge! For example, the second most natural hard number is e which is provably transcendental. But what about $\pi + e$? This embarrassingly easy question is currently totally intractable (we don't even know how to show that $\pi + e$ is irrational). The number π is a mathematical apple and e is a mathematical orange and we have no idea how to mix them.

Why compute the digits of π? Sometimes it is necessary to do so, though hardly ever more than the 6 or so digits that Archimedes computed several thousand years ago are needed for physical applications. Even far fetched computations like the volume of a spherical universe only require a few dozen digits. There is also the "Everest Hypothesis" ("because its there"). Probably the number of people involved and the effort in time has been similar in the two quests. A few thousand people have reached the computational level that requires the carrying of oxygen – though so far I know of no π related fatalities. There has been significant knowledge accumulated in this slightly quixotic pursuit. But this knowledge could have been derived from computing a host of other numbers in a variety of different bases. Once again the answer to "why π" is largely historical and cultural. These are good but not particularly scientific reasons. Pi was first, pi is hard and pi has captured the educated imagination. (Have you ever seen a cartoon about $\log 2$ - a number very similar to π?)

Whatever the personal motivations π has been much computed and a surprising amount has been learnt along the way.

In constructing the all star hockey team of great mathematicians, there seems to be pretty wide agreement that the front line consists of Archimedes, Newton, and Gauss. Both Archimedes and Newton invented methods for computing pi. In Newton's case this was an application of his newly invented calculus. I know of no such calculation from Gauss though his exploration of the Arithmetic-Geometric mean iteration laid the foundation of the most successful methods for doing such calculations. There is less consensus about who comes next. I might add Hilbert and Euler next (on defense). Both of these mathematicians also contribute to the story of π. Perhaps von Neumann is in goal – certainly he is a candidate for the most versatile and smartest mathematician of the twentieth century. One of the first calculations done on ENIAC (one of the first real computers) was the computations of roughly a thousand digits of π and von Neumann was part of the team that did the calculation.

One doesn't often think of a problem like this having economic benefits. But as is often the case with pure mathematics and curiosity driven research the rewards can be surprising. Large recent records depend on three things: better algorithms for pi; larger and faster computers; and an understanding of how to do arithmetic with numbers that are billions of digits long

The better algorithms are due to a variety of people including Ramanujan, Brent, Salamin, the Chudnovsky brothers and ourselves. Some of the mathematics is both beautiful and subtle. (The Ramanujan type series listed in the appendix are, for me, of this nature.)

The better computers are, of course, the most salient technological advance of the second half ot the twentieth century.

Understanding arithmetic is an interesting and illuminating story in its own right. A hundred years ago we knew how to add and multiply – do it the way we all learned in school. Now we are not so certain except that we now know that the "high school method" is a disaster for multiplying really big numbers. The mathematical technology that allows for multiplying very large numbers together is essentially the same as the mathematical technology that allows image processing devices like

CAT scanners to work (FFTs). In making the record setting algorithms work, David Bailey tuned the FFT algorithms in several of the standard implementations and saved the US economy millions of dollars annually. Most recent records are set when new computers are being installed and tested. (Recent records are more or less how many digits can be computed in a day – a reasonable amount of test time on a costly machine.) The computation of pi seems to stretch the machine and there is a history of uncovering subtle and sometimes not so subtle bugs at this stage.

What do the calculations of π reveal and what does one expect? One expects that the digits of pi should look random – that roughly one out each ten digits should be a 7 etc. This appears to be true at least for the first few hundred billion. But this is far from a proof – an actual proof of this is way out of the reach of current mathematics. As is so often the case in mathematics some of the most basic questions are some of the most intractable. What mathematicians believe is that every pattern possible eventually occurs in the digits of pi – with a suitable encoding the Bible is written in entirety in the digits, as is the New York phone book and everything else imaginable.

The question of whether there are subtle patterns in the digits is an interesting one. (Perhaps every billionth digit is a seven after a while. While unlikely this is not provably impossible. Or perhaps pi is buried within pi in some predictable way.) Looking for subtle patterns in long numbers is exactly the kind of problem one needs to tackle in handling the human genome (a chromosome is just a large number base 4, at least to a mathematician).

I have included two appendices. One is from David H. Bailey, Jonathan M. Borwein, Peter B. Borwein, and Simon Plouffe, "The Quest for Pi," (June, 1996) *The Mathematical Intelligencer.* It is a chronology of the computation of digits of π. The second is taken from: Lennert Berggren, Jonathan M. Borwein and Peter B. Borwein, "Pi: A Source Book" *Springer-Verlag 1988.* It is a list of significant mathematical formulae related to π. These are reproduced with permission from Springer–Verlag New York.

The previously mentioned chronology is of the problem of computing all of the initial digits of pi. There is also a shorter chronology of computing just a few very distant bits of pi. The record here is 40 trillion and is due to Colin Percival using the methods described in the last reference above. It is surprising that this is possible at all.

Babylonians	2000? BCE	1	3.125 ($3\frac{1}{8}$)
Egyptians	2000? BCE	1	3.16045 ($4(\frac{8}{9})^2$)
China	1200? BCE	1	3
Bible (1 Kings 7:23)	550? BCE	1	3
Archimedes	250? BCE	3	3.1418 (ave.)
Hon Han Shu	130 AD	1	3.1622 ($= \sqrt{10}$?)
Ptolemy	150	3	3.14166
Chung Hing	250?	1	3.16227 ($\sqrt{10}$)
Wang Fau	250?	1	3.15555 ($\frac{142}{45}$)
Liu Hui	263	5	3.14159
Siddhanta	380	3	3.1416
Tsu Ch'ung Chi	480?	7	3.1415926
Aryabhata	499	4	3.14156
Brahmagupta	640?	1	3.162277 ($= \sqrt{10}$)
Al-Khowarizmi	800	4	3.1416
Fibonacci	1220	3	3.141818
Al-Kashi	1429	14	
Otho	1573	6	3.1415929
Viete	1593	9	3.1415926536 (ave.)
Romanus	1593	15	
Van Ceulen	1596	20	
Van Ceulen	1610	35	
Newton	1665	16	
Sharp	1699	71	
Seki	1700?	10	
Kamata	1730?	25	
Machin	1706	100	
De Lagny	1719	127	(112 correct)
Takebe	1723	41	
Matsunaga	1739	50	
Vega	1794	140	
Rutherford	1824	208	(152 correct)
Strassnitzky and Dase	1844	200	
Clausen	1847	248	
Lehmann	1853	261	
Rutherford	1853	440	
Shanks	1874	707	(527 correct)

TABLE 1. History of π Calculations (Pre 20th Century)

Ferguson	1946	620
Ferguson	Jan. 1947	710
Ferguson and Wrench	Sep. 1947	808
Smith and Wrench	1949	1,120
Reitwiesner et al. (ENIAC)	1949	2,037
Nicholson and Jeenel	1954	3,092
Felton	1957	7,480
Genuys	Jan. 1958	10,000
Felton	May 1958	10,021
Guilloud	1959	16,167
Shanks and Wrench	1961	100,265
Guilloud and Filliatre	1966	250,000
Guilloud and Dichampt	1967	500,000
Guilloud and Bouyer	1973	1,001,250
Miyoshi and Kanada	1981	2,000,036
Guilloud	1982	2,000,050
Tamura	1982	2,097,144
Tamura and Kanada	1982	8,388,576
Kanada, Yoshino and Tamura	1982	16,777,206
Ushiro and Kanada	Oct. 1983	10,013,395
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada, Tamura, Kubo, et. al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Chudnovskys	Jun. 1989	525,229,270
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1989	1,011,196,691
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Takahashi and Kanada	Jun. 1995	3,221,225,466
Kanada	Aug. 1995	4,294,967,286
Kanada	Oct. 1995	6,442,450,938
Kanada	Jun. 1997	51,539,600,000
Kanada	Sep. 1999	206,158,430,000

TABLE 2. History of π Calculations (20th Century)

Selected Formulae for PI

Archimedes

Let $a_0 := 2\sqrt{3}$, $b_0 := 3$ and

$$a_{n+1} := \frac{2a_n b_n}{a_n + b_n} \quad \text{and} \quad b_{n+1} := \sqrt{a_{n+1} b_n}.$$

Then a_n and b_n converge linearly to π (with an error $O(4^{-n})$.)

(ca 250 BC)

Francois Viète

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

(ca 1579)

John Wallis

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdots}$$

(ca 1650)

William Brouncker

$$\pi = \frac{4}{1+\frac{1}{2+\frac{9}{2+\frac{25}{2+\cdots}}}}$$

(ca 1650)

Mādhava, James Gregory, Gottfried Wilhelm Leibnitz

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(1450–1671)

Isaac Newton

$$\pi \quad = \quad \frac{3\sqrt{3}}{4} + 24\left(\frac{2}{3\cdot 2^3} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \cdots\right)$$

(ca 1666)

Machin Type Formulae

$$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$$

$$\frac{\pi}{4} = \arctan(\frac{1}{2}) + \arctan(\frac{1}{3})$$

$$\frac{\pi}{4} = 2\arctan(\frac{1}{2}) - \arctan(\frac{1}{7})$$

$$\frac{\pi}{4} = 2\arctan(\frac{1}{3}) + \arctan(\frac{1}{7})$$

(1706–1776)

Leonard Euler

$$\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots$$
$$\frac{\pi^4}{90} = 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\cdots$$

$$\frac{\pi^2}{6} = 3\sum_{m=1}^{\infty}\frac{1}{m^2\binom{2m}{m}}$$

(ca 1748)

Srinivasa Ramanujan

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{42n+5}{2^{12n+4}}.$$

$$\frac{1}{\pi}=\frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\frac{[1103+26390n]}{396^{4n}}.$$

Each additional term of the latter series adds roughly 8 digits.

(1914)

Louis Comtet

$$\frac{\pi^4}{90} = \frac{36}{17}\sum_{m=1}^{\infty}\frac{1}{m^4\binom{2m}{m}}$$

(1974)

Eugene Salamin , Richard Brent

Set $a_0 = 1, b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. For $k = 1, 2, 3, \cdots$ compute

$$\begin{aligned} a_k &= \frac{a_{k-1} + b_{k-1}}{2} \\ b_k &= \sqrt{a_{k-1} b_{k-1}} \\ c_k &= a_k^2 - b_k^2 \\ s_k &= s_{k-1} - 2^k c_k \\ p_k &= \frac{2a_k^2}{s_k} \end{aligned}$$

Then p_k converges *quadratically* to π.

---(1976)

Jonathan Borwein and Peter Borwein

Set $a_0 = 1/3$ and $s_0 = (\sqrt{3} - 1)/2$. Iterate

$$\begin{aligned} r_{k+1} &= \frac{3}{1 + 2(1 - s_k^3)^{1/3}} \\ s_{k+1} &= \frac{r_{k+1} - 1}{2} \\ a_{k+1} &= r_{k+1}^2 a_k - 3^k (r_{k+1}^2 - 1) \end{aligned}$$

Then $1/a_k$ converges *cubically* to π.

---(1991)

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$\begin{aligned} y_{k+1} &= \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}} \\ a_{k+1} &= a_k (1 + y_{k+1})^4 - 2^{2k+3} y_{k+1} (1 + y_{k+1} + y_{k+1}^2) \end{aligned}$$

Then a_k converges *quartically* to $1/\pi$.

---(1985)

David Chudnovsky and Gregory Chudnovsky

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3 (3n)!} \frac{13591409 + n545140134}{(640320^3)^{n+1/2}}.$$

Each additional term of the series adds roughly 15 digits.

---(1989)

Jonathan Borwein and Peter Borwein

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(n!)^3 (3n)!} \frac{(A + nB)}{C^{n+1/2}}$$

where

$$\begin{aligned} A &:= 212175710912\sqrt{61} + 1657145277365 \\ B &:= 13773980892672\sqrt{61} + 107578229802750 \\ C &:= [5280(236674 + 30303\sqrt{61})]^3. \end{aligned}$$

Each additional term of the series adds roughly 31 digits.

(1989)

The following is not an identity but is correct to over 42 billion digits

$$\left(\frac{1}{10^5}\sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{10^{10}}}\right)^2 \doteq \pi.$$

(1985)

Roy North

Gregory's series for π, truncated at 500,000 terms gives to forty places

$$4\sum_{k=1}^{500,000}\frac{(-1)^{k-1}}{2k-1}$$

$$= 3.14159\underline{0}65358979324\underline{40}46264338326\underline{6}9502884197.$$

Only the underlined digits are incorrect.

(1989)

David Bailey, Peter Borwein and Simon Plouffe

$$\pi = \sum_{i=0}^{\infty}\frac{1}{16^i}\left(\frac{4}{8i+1}-\frac{2}{8i+4}-\frac{1}{8i+5}-\frac{1}{8i+6}\right)$$

(1996)

Department of Mathematics and Statistics, Simon Fraser University, Burnaby, B.C., Canada V5A 1S6