# A Power Series Paradox Resolved

## 1. A simple example of the paradox and its resolution

Consider the following real-valued function defined via a series,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\,(x^n - x^{2n})\,, \qquad |x| < 1\,. \tag{1}$$

Using the ratio test, it follows that the radius of convergence of this series is $R = 1$. We can also determine the convergence properties of the endpoints of the interval of convergence. At $x = 1$, we trivially find that $f(1) = 0$ by direct substitution. At $x = -1$, we note that

$$(-1)^n - (-1)^{2n} = (-1)^n - 1 = \begin{cases} -2\,, & \text{if } n \text{ is odd}\,, \\ 0\,, & \text{if } n \text{ is even}\,. \end{cases}$$

Thus, $f(-1)$ is proportional to the sum of the odd reciprocals, which diverges.

Let us now evaluate the sum in eq. (1) explicitly, assuming that $|x| < 1$. In this case, the sum is absolutely convergent. Hence, it is permissible to write:

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} - \sum_{n=1}^{\infty} \frac{x^{2n}}{n}\,, \qquad |x| < 1\,.$$

Using the well-known result,

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}\,, \qquad -1 \leq x < 1\,, \tag{2}$$

it immediately follows that

$$f(x) = -\ln(1 - x) + \ln(1 - x^2)\,, \qquad |x| < 1\,. \tag{3}$$

If we write $\ln(1 - x^2) = \ln\left[(1 - x)(1 + x)\right] = \ln(1 - x) + \ln(1 + x)$, then eq. (3) simplifies to:

$$f(x) = \ln(1 + x)\,, \qquad |x| < 1\,.$$

Combining this result with the observation above that $f(1) = 0$, we conclude that:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\,(x^n - x^{2n}) = \begin{cases} \ln(1 + x)\,, & \text{for } |x| < 1\,, \\ 0\,, & \text{for } x = 1\,. \end{cases}$$

Note that $f(x)$ is *not* continuous at $x = 1$. In particular,

$$f(1) = 0 \neq \lim_{x\to 1} f(x) = \ln 2\,, \tag{4}$$

in apparent violation of Abel's Theorem (see the class handout entitled *Theorems About Power Series*).

In fact, this seemingly paradoxical result obtained in eq. (4) is correct, although this is not a violation of Abel's theorem for a subtle reason. If we write out the series in eq. (1) explicitly, we have

$$f(x) = 1 - x^2 + \tfrac{1}{2}(x^2 - x^4) + \tfrac{1}{3}(x^3 - x^6) + \tfrac{1}{4}(x^4 - x^8) + \cdots\,, \qquad |x| < 1\,, \tag{5}$$

which is technically not a power series, since the powers of $x$ are not monotonically increasing. One is allowed to reorder the terms into a proper power series when $|x| < 1$, since the series is absolutely convergent in this regime. However, one is not permitted to reorder the terms when $x = 1$, since the series is only conditionally convergent at this point.

Abel's theorem states that if a power series $f(x) = \sum_n a_n x^n$ is convergent at an endpoint of the interval of convergence (which in the present example is the point $x = 1$), then

$$f(1) = \lim_{x\to 1} \sum_n a_n x^n = \sum_n a_n\,.$$

But if the convergence at $x = 1$ is only conditional, then the order of the terms in $\sum a_n$ must correspond to an index $n$ that is monotonically increasing. Applying the correct version of Abel's theorem to our example, we first rewrite eq. (5) such that the powers of $x$ are monotonically increasing,

$$f(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 + \cdots\,, \qquad |x| < 1\,.$$

Using Abel's theorem, we correctly conclude that

$$\lim_{x\to 1} f(x) = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots = \ln 2\,, \tag{6}$$

in agreement with eq. (4). In contrast, eq. (5) yields

$$f(1) = (1 - 1) + \left(\tfrac{1}{2} - \tfrac{1}{2}\right) + \left(\tfrac{1}{3} - \tfrac{1}{3}\right) + \left(\tfrac{1}{4} - \tfrac{1}{4}\right) + \left(\tfrac{1}{5} - \tfrac{1}{5}\right) + \cdots = 0\,. \tag{7}$$

Removing the parentheses, this series is conditionally convergent. In fact, the series in eqs. (6) and (7) are related by a rearrangement of the terms. Starting with eq. (6), the following result is an identity:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots = 1 + \left(\tfrac{1}{2} - 1\right) + \tfrac{1}{3} + \left(\tfrac{1}{4} - \tfrac{1}{2}\right) + \tfrac{1}{5} + \left(\tfrac{1}{6} - \tfrac{1}{3}\right) + \cdots\,.$$

Removing the parentheses, the latter can be rearranged to yield the series in eq. (7).

The lesson to be learned by this example is that one must be careful in applying Abel's theorem when a power series is conditionally convergent at an endpoint of the interval of convergence.

## 2. How to sum a series the wrong way and the right way

Determine the sum of the following series:

$$S \equiv \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} \,.$$

This is an absolutely convergent series. Using the method of partial fractions, this sum can be rewritten as:

$$S = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{2}{2n+1} \right) \,. \tag{8}$$

Consider the function,

$$g(x) = \sum_{n=1}^{\infty} \left( \frac{x^n}{n} - \frac{2x^{2n+1}}{2n+1} \right) \,, \qquad |x| < 1 \,.$$

where the radius of convergence of this series is given by $R = 1$. Suppose we could evaluate this sum explicitly (see the Appendix to this note). Then, it would be tempting to employ Abel's theorem to conclude that

$$S \stackrel{?}{=} \lim_{x \to 1} g(x) \,.$$

However, this result is *not correct.* Note that the series representation for $g(x)$ is explicitly given by:

$$g(x) = x - \tfrac{2}{3}x^3 + \tfrac{1}{2}x^2 - \tfrac{2}{5}x^5 + \tfrac{1}{3}x^3 - \tfrac{2}{7}x^7 + \cdots \,, \qquad |x| < 1 \tag{9}$$

which is not a proper power series since the powers in eq. (14) are not monotonically increasing. Hence, we *cannot* conclude that $g(x)$ is continuous at $x = 1$.

A better strategy is to introduce a slightly different function,

$$h(x) = 2\sum_{n=1}^{\infty} \left( \frac{x^{2n}}{2n} - \frac{x^{2n+1}}{2n+1} \right) \,, \qquad |x| < 1 \,. \tag{10}$$

Explicitly,

$$h(x) = 2\left( \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 + \tfrac{1}{6}x^6 - \tfrac{1}{7}x^7 + \cdots \right) \,, \qquad |x| < 1 \,, \tag{11}$$

which is a proper power series with monotonically increasing powers. Using eq. (2), we recognize this series immediately:

$$h(x) = 2\sum_{k=2}^{\infty} (-1)^k \, \frac{x^k}{k} = 2\left( x + \sum_{k=1}^{\infty} (-1)^k \, \frac{x^k}{k} \right) = 2\left[ x - \ln(1+x) \right] \,, \qquad |x| < 1 \,.$$

Since the series representation of $h(1)$ is convergent, Abel's theorem implies that $h(x)$ is continuous at $x = 1$. Hence, it follows that:

$$S = \lim_{x\to 1} h(x) = 2 - 2\ln 2\,. \tag{12}$$

Thus, we conclude that

$$\boxed{S \equiv \sum_{n=1}^{\infty} \frac{1}{n(2n+1)} = 2 - 2\ln 2} \tag{13}$$

**Appendix: Evaluation of the function $g(x)$**

Consider the function,

$$g(x) = \sum_{n=1}^{\infty} \left( \frac{x^n}{n} - \frac{2x^{2n+1}}{2n+1} \right)\,, \qquad |x| < 1\,. \tag{14}$$

To evaluate $g(x)$ explicitly, we first evaluate $g(1)$ directly from the series. Since the series representation of $g(1)$ is conditionally convergent, we must keep the order of the terms as given in eq. (9). Comparing eqs. (9) and (11) then yields

$$g(1) = h(1) = 2 - 2\ln 2\,, \tag{15}$$

as a consequence of eq. (12). Next, we compute the derivative of $g(x)$ using eq. (14),

$$\frac{dg}{dx} = \sum_{n=1}^{\infty} x^{n-1} - 2\sum_{n=1}^{\infty} x^{2n}\,, \qquad |x| < 1\,. \tag{16}$$

One can now evaluate eq. (16) by noting that:

$$\sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,, \qquad |x| < 1\,,$$

$$\sum_{n=1}^{\infty} x^{2n} = -1 + \sum_{n=0}^{\infty} x^{2n} = -1 + \frac{1}{1-x^2} = \frac{x^2}{1-x^2}\,, \qquad |x| < 1\,.$$

Hence,

$$\frac{dg}{dx} = \frac{1}{1-x} - \frac{2x^2}{1-x^2} = \frac{1+x-2x^2}{1-x^2} = \frac{1+2x}{1+x} = 2 - \frac{1}{1+x}\,, \qquad |x| < 1\,.$$

Integrating this expression and noting that $g(0) = 0$ [which fixes the integration constant] determines $g(x)$ in the range $|x| < 1$. We conclude that:

$$g(x) = \begin{cases} 2x - \ln(1+x)\,, & |x| < 1\,, \\ 2 - 2\ln 2\,, & x = 1\,. \end{cases}$$

Note that $g(x)$ is not continuous at $x = 1$, as expected. In particular,

$$g(1) = 2 - 2\ln 2 \neq \lim_{x\to 1} g(x) = 2 - \ln 2\,.$$