Theorems About Power Series

Consider a power series,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \,, \tag{1}$$

where the a_n are real coefficients and x is a real variable. There exists a real non-negative number R, called the *radius of convergence* such that

1. If $R = 0$, then the series in eq. (1) converges for $x = 0$ and diverges for any non-zero real value of x.

2. If $R = \infty$, then the series in eq. (1) converges absolutely for any (finite) real number x.

3. If $0 < R < \infty$, then the series in eq. (1) converges absolutely for every real number x such that $|x| < R$, and diverges for every real number x such that $|x| > R$.

In many cases, R can be determined by the ratio test, which yields*

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \,. \tag{2}$$

Examples of the three possible cases exhibited above are:

$$\text{(i)} \quad \sum_{n=0}^{\infty} n!\, x^n \,, \qquad \text{(ii)} \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} \,, \qquad \text{(iii)} \quad \sum_{n=0}^{\infty} x^n \,.$$

In particular, using eq. (2), it follows that for the three series listed above, $R = 0$ for series (i), $R = \infty$ for series (ii) and $R = 1$ for series (iii).

The *interval of convergence* for the series in eq. (1) is defined to be the set of all possible values of x for which the series converges. Note that if if $0 < R < \infty$,

*If the limit in eq. (2) does not exist, then a different test, called the root test, can typically be used to determine the radius of convergence. The root test yields $1/R = \lim_{n\to\infty} |a_n|^{1/n}$. If the latter fails to exist, one can modify the root test slightly by denoting by μ the upper limit of the positive infinite sequence of numbers, $\{|a_1|,\, |a_2|^{1/2},\, |a_3|^{1/3}, \ldots,\, |a_n|^{1/n}, \ldots\}$, in which case the radius of convergence can be identified as $R \equiv \mu^{-1}$. This last result, called the Cauchy-Hadamard formula, is taken in advanced textbooks as the definition of R. If both the ratio test and the root test apply, one can show that they both yield the same value for the radius of convergence R.

then the convergence properties of eq. (1) for $x = R$ and $x = -R$ are not specified, and must be determined by other means. Thus, the interval of convergence may or may not include one or both of the endpoints of the interval $-R \leq x \leq R$. The possible convergence properties at an endpoint are: absolute convergence, conditional convergence or divergence.

Theorem 1: The power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is *absolutely convergent* for $|x| < R$, where R is the (nonzero) radius of convergence. Moreover, $\sum_{n=0}^{\infty} a_n x^n$ is continuous and infinitely differentiable within the interval of convergence, $|x| < R$.

Proof: The convergence properties of the power series are a consequence of the ratio test. The proof of continuity and differentiability can be found in the references at the end of this note.

As noted above, at the endpoints of the interval of convergence, $x = R$ and $x = -R$, the corresponding power series may be absolute convergent, conditionally convergent or divergent. It is straightforward to show that if the power series is absolutely convergent at $x = R$ then it is absolutely convergent at $x = -R$ and vice versa (this result follows from the definition of absolute convergence). Hence there are are five possible scenarios. For simplicity, in the following we shall examine examples of power series where the radius of convergence is $R = 1$.

First, consider the case of the geometric series,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \,. \tag{3}$$

Note that eq. (3) is divergent at both $x = 1$ and $x = -1$. In contrast, consider the power series,

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \,. \tag{4}$$

Eq. (4) is divergent at $x = 1$ and conditionally convergent at $x = -1$. Likewise, the power series for $\ln(1+x)$ [cf. eq. (7)] is divergent at $x = -1$ and conditionally convergent at $x = 1$. A related example is,

$$\ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n} \,, \tag{5}$$

which is conditionally convergent at both endpoints of the interval of convergence, $x = \pm 1$.

Finally, here is an example of a power series that is absolutely convergent at both endpoints of the interval of convergence, $x = \pm 1$,

$$\sum_{n=1}^{\infty} \frac{x^n}{n^p} \,, \quad \text{for any real } p > 1 \,. \tag{6}$$

Theorem 2: If the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is convergent at $x = R$, then it is a continuous function within the interval of convergence *including the endpoint* at $x = R$. In this case, we have

$$f(R) = \lim_{x \to R^-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \lim_{x \to R^-} a_n x^n = \sum_{n=0}^{\infty} a_n R^n \,,$$

where $\lim_{x \to R^-}$ means that x approaches R from the left, i.e. from inside the interval of convergence, $|x| < R$. That is, in this case it is permissible to interchange the order of the limit and the infinite sum. Likewise, if the power series is convergent at $x = -R$, then it is a continuous function within the interval of convergence *including the endpoint* at $x = -R$. In this case, we have

$$f(-R) = \lim_{x \to -R^+} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \lim_{x \to -R^+} a_n x^n = \sum_{n=0}^{\infty} (-1)^n a_n R^n \,,$$

where $\lim_{x \to -R^+}$ means that x approaches R from the right, i.e. from inside the interval of convergence, $|x| < R$.

Theorem 2 is known as *Abel's theorem.* As an example of its application, consider the power series,

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \,, \qquad |x| < 1 \,. \tag{7}$$

In this case, the radius of convergence is $R = 1$. Moreover, if we set $x = 1$ above, the resulting series is conditionally convergent (as a consequence of the alternating series test). Thus, the power series for $\ln(1 + x)$ is continuous at $x = 1$, which allows us to conclude that:

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \,.$$

The converse of Abel's theorem is sometimes false. As an example, we consider the infinite geometric series,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \,. \tag{8}$$

Setting $x = 1$ above yields the divergent series $1-1+1-1+1-1+\cdots$. Hence, the conditions of Abel's theorem are *not* satisfied, in which case we *cannot* conclude that $\sum_{n=0}^{\infty} (-1)^n x^n$ is continuous at $x = 1$. In particular, for $x = 1$, the left hand side of eq. (8) yields $\frac{1}{2}$. Although one can make a case for assigning $\frac{1}{2}$ to the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$, the latter series is clearly *not* convergent according to the standard mathematical definition of convergence.

It is instructive to examine one more application of Abel's theorem. Consider the power series,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{\lfloor n/2 \rfloor}}{n+1}\, x^n\,, \quad \text{for } |x| < 1\,, \tag{9}$$

where the exponent in eq. (9) is the floor function, $\lfloor n/2 \rfloor$, which is defined as the greatest integer less than or equal to $n/2$. In particular, for any integer n,

$$\lfloor n/2 \rfloor = \begin{cases} \tfrac{1}{2}n\,, & \text{for } n \text{ even}\,, \\ \tfrac{1}{2}(n-1)\,, & \text{for } n \text{ odd}\,. \end{cases}$$

By considering the terms of eq. (9) with even n and odd n separately, one can rewrite eq. (9) [after an appropriate relabeling of the summation index], as follows,

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x^{2k}}{2k+1} + \frac{x^{2k+1}}{2k+2} \right)\,, \quad \text{for } |x| < 1\,. \tag{10}$$

Note that the terms in the sum given by eq. (9) have been reordered in writing eq. (10). This step is justified since these sums are absolutely convergent for $|x| < 1$. We can now evaluate these sums by recalling that

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}\,, \tag{11}$$

and employing the result of eq. (5). The end result is,

$$f(x) = \frac{1}{x}\left[\arctan x + \tfrac{1}{2}\ln(1+x^2)\right]\,. \tag{12}$$

We can now use eq. (12) to determine the values of $f(x)$ at the boundary of the interval of convergence, $x = \pm 1$. Note that the series for $f(1)$ and $f(-1)$ are convergent, since we can apply the alternating series test to eq. (10) when $x = \pm 1$. Consequently, $f(x)$ must be continuous at $x = \pm 1$ due to Abel's theorem. It then follows that

$$f(1) = 1 + \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} - \ldots = \tfrac{1}{4}\pi + \tfrac{1}{2}\ln 2\,, \tag{13}$$

$$f(-1) = 1 - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} - \ldots = \tfrac{1}{4}\pi - \tfrac{1}{2}\ln 2\,. \tag{14}$$

Indeed the two series above are both conditionally convergent (since the corresponding series with all plus signs is the harmonic series which is divergent).

Here is one pitfall to avoid. If one inserts $x = -1$ into eq. (10), one obtains,

$$f(-1) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{2k+1} - \frac{1}{2k+2} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(2k+2)}\,. \tag{15}$$

Since the last sum on the right hand side of eq. (15) is absolutely convergent, one might erroneously conclude that $f(-1)$ is absolutely convergent. Of course, this conclusion is false. This is just an illustration of the fact that a conditionally convergent series written in some specified order can be numerically equal to another series that is absolutely convergent.

Theorem 3: Consider a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence R. Then, term-by-term differentiation and integration of the power series is permitted, and does not change the radius of convergence. That is,

$$\frac{df}{dx} = \frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1}\,, \qquad |x| < R\,, \quad (16)$$

$$\int f(x)dx = \int dx \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \int dx a_n x^n = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}\,. \qquad |x| < R\,. \quad (17)$$

In particular, for values of x within the interval of convergence, $|x| < R$, it is permissible to interchange the order of the infinite summation and the differentiation or integration. This feature is one of the reasons that power series are so nice—they behave for the most part like ordinary polynomials.[†]

Proof: This theorem is a simple consequence of the ratio test. Note that the ratio test is inconclusive at the endpoints of the interval of convergence, so that the convergence properties at $x = R$ and $x = -R$ must be separately investigated.

Although a power series, its derivative and its integral possess the same radius of convergence, this does not mean that they have the same interval of convergence. In particular, the intervals of convergence of the power series representations of $f(x)$, df/dx and $\int f(x)dx$ can differ at the endpoints of the interval of convergence. In general, by differentiating a function defined by a power series with radius of convergence R, one may *lose* convergence at an endpoint of the interval of convergence of $f(x)$. In contrast, by integrating a function defined by a power series with radius of convergence R, one may *gain* convergence at an endpoint of the interval of convergence of $f(x)$. On the other hand, the series comparison test implies if $f(x)$ diverges at an endpoint, then df/dx must also diverge at that endpoint, whereas if $f(x)$ converges at an endpoint, then $\int f(x)dx$ must also converge at that endpoint.

The following two examples are instructive. First, we define the *dilogarithm* $\text{Li}_2(x)$ via the power series,

$$\text{Li}_2(x) \equiv \sum_{n=1}^{\infty} \frac{x^n}{n^2}\,, \qquad |x| \leq 1\,. \quad (18)$$

The ratio test implies that the radius of convergence is $R = 1$, and the p-series test implies that the power series converges absolutely at both endpoints of the interval of convergence. Taking a derivative of eq. (18) yields

$$\frac{d}{dx}\text{Li}_2(x) = \frac{d}{dx}\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}\frac{d}{dx}x^n = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}\,, \qquad |x| < 1\,. \quad (19)$$

[†]In general, if $f(x) = \sum_n f_n(x)$ is a pointwise convergent sum, it may happen that the integral of the infinite sum is not equal to the infinite sum of the integrals, and/or the derivative of the infinite sum is not equal to the infinite sum of the derivatives. However, this cannot happen for a power series when x lies within the interval of convergence.

At $x = -1$ the resulting series is the alternating harmonic series which converges, whereas at $x = 1$ the resulting series is the harmonic series which diverges. Using Abel's theorem, we can extend the domain of validity of eq. (19) to include the endpoint $x = -1$ (but not the endpoint $x = 1$). That is, even though the series given by eq. (18) is convergent at $x = 1$, the series representation of the derivative of $\mathrm{Li}_2(x)$ is divergent at $x = 1$. Using eqs. (7) and (19), it follows that:

$$\frac{d}{dx}\mathrm{Li}_2(x) = -\frac{\ln(1-x)}{x}\,. \tag{20}$$

Strictly speaking, this result is only valid in the range $-1 \leq x < 1$.

For our second example, we start with the infinite geometric series given in eq. (8), which diverges at both endpoints of the interval of convergence. Computing the integral of eq. (8) yields:

$$\begin{aligned}\int \frac{dx}{1+x} = \ln(1+x) = \int dx \sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n \int x^n\, dx \\ = \sum_{n=0}^{\infty}(-1)^n\, \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\, \frac{x^n}{n}\,, \qquad |x| < 1\,. \end{aligned} \tag{21}$$

At $x = +1$ the resulting series is the alternating harmonic series which converges, whereas at $x = -1$ the resulting series is the negative of the harmonic series which diverges. Using Abel's theorem, we can extend the domain of validity of eq. (21) to include the endpoint $x = 1$ (but not the endpoint $x = -1$). That is, even though the infinite geometric series given in eq. (8) is divergent at $x = 1$, the series representation of the integral of $1/(1+x)$ is convergent at $x = 1$.

Theorem 4: Given two power series with radii of convergence R_1 and R_2, respectively, i.e.

$$f_1(x) = \sum_{n=0}^{\infty} a_n x^n\,, \qquad |x| < R_1\,, \tag{22}$$

$$f_2(x) = \sum_{n=0}^{\infty} b_n x^n\,, \qquad |x| < R_2\,, \tag{23}$$

then the sum and product of the two power series are given respectively by:

$$f_1(x) + f_2(x) = \sum_{n=0}^{\infty}(a_n + b_n)x^n\,, \qquad |x| < R\,, \tag{24}$$

$$f_1(x)f_2(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} x^n\,, \qquad |x| < R\,, \tag{25}$$

where the radius of convergence of the sum and of the product is at least as large as the minimum of R_1 and R_2, i.e. $R \geq \min\{R_1, R_2\}$. The subtraction of two series is then defined simply by changing the signs of all the b_n above before adding the two series. The division of the two series, $f_1(x)/f_2(x)$, can be performed if and only if $b_0 \neq 0$. Assuming that this condition holds,

$$\frac{f_1(x)}{f_2(x)} = \sum_{n=0}^{\infty} c_n x^n \,, \qquad |x| < R' \,, \tag{26}$$

where the radius of convergence satisfies $R' \geq \min\{R_1, R_2, x_0\}$, with x_0 identified as the zero of $f_2(x)$ nearest to $x = 0$. The coefficients c_n in eq. (26) are determined recursively using:

$$c_0 = \frac{a_0}{b_0} \,, \qquad c_n = \frac{1}{b_0}\left[a_n - \sum_{k=1}^{n} b_k c_{n-k}\right] \quad \text{for} \quad n = 1, 2, 3, \ldots \,.$$

In the generic case, $R = \min\{R_1, R_2\}$ and $R' = \min\{R_1, R_2, x_0\}$. However, in special cases the radius of convergence may be larger. Here is one such example:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \,, \qquad |z| < 1 \,, \tag{27}$$

$$\frac{-z}{(2-z)(1-z)} = \frac{1}{1-\frac{1}{2}z} - \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \left(\frac{1}{2^n} - 1\right) \,, \qquad |z| < 1 \,, \tag{28}$$

have radii of convergence $R_1 = R_2 = 1$. Nevertheless, the sum of the two series defined in eqs. (27) and (28) has a radius of convergence $R = 2 > \min\{R_1, R_2\}$,

$$\frac{1}{1-\frac{1}{2}z} = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \,, \qquad |z| < 2 \,.$$

Theorem 5: The power series representation of a function, $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with a non-zero radius of convergence $|x| < R$, is unique.

Proof: This is a consequence of Taylor's theorem in calculus, which provides an explicit formula for the coefficients of a power series,

$$a_n = \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=0} \,.$$

References

All of the results obtained in these notes can be found in standard mathematical references. In particular, the following two references are elementary and highly readable:

1. Earl D. Rainville, *Infinite Series* (The Macmillan Company, New York, 1967).

2. O.E. Stanaitis, *An Introduction to Sequences, Series, and Improper Integrals* (Holden-Day, Inc., San Francisco, 1967).

At a slightly higher level, but still accessible, I also recommend:

3. T.J.I'a. Bromwich, *An Introduction to the Theory of Infinite Series* (Macmillan & Co. Ltd., London, 1959).

4. Konrad Knapp, *Theory and Application of Infinite Series* (Dover Publications, Inc., Mineola, NY, 1990).

5. Brian S. Thomson, Judith B. Bruckner and Andrew M. Bruckner, *Elementary Real Analysis* (Prentice-Hall, Inc., Englewood Cliffs, NJ, 2001).