

Explorations of the Harmonic Series

Appendix to *A Radical Approach to Real Analysis* 2nd edition
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May 11, 2006

We consider two classic problems that have more in common than might appear at first glance.

1 Stacking Bricks

We stack bricks so that they are held in place simply by gravity. The stack will remain stable even if one brick extends out further than the brick below it, so long as the center of mass of the top brick rests on a solid brick. We can even put a third brick on top of that so that it extends a bit further (see figure 1). How far out can the top brick extend? Is it possible to build a stable stack of bricks, one on each level, so that the top brick is completely to the right of the bottom brick?

To answer this question, we assume that the bricks are identical, each brick has length 1, we ignore the width of the brick, and we also do not care how thick it is. As a practical matter, just to keep the stack from getting very high, we want thin bricks, but they have to be thick enough that they will not bend. We put the bottom brick so that its left-hand edge is at 0. We are interested in the

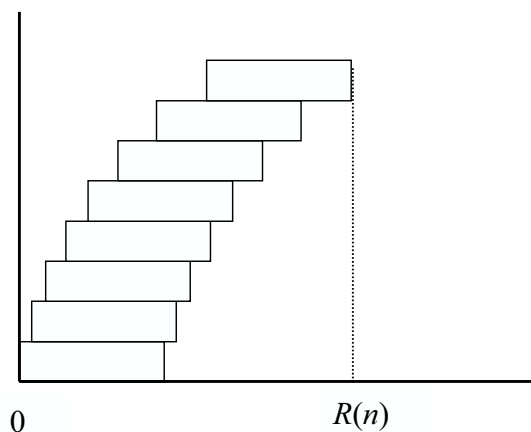


Figure 1: A leaning—but stable—stack of bricks.

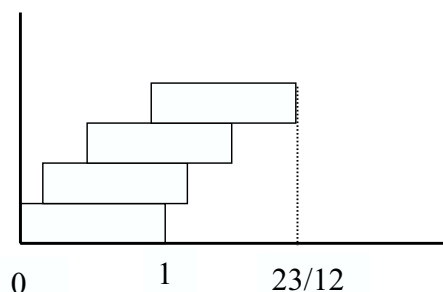


Figure 2: Stacking four bricks.

numerical value of the right-hand edge of the top brick. Let $R(n)$ be the distance from the vertical line at 0 to the right edge of the top brick in a stack of n bricks: $R(1) = 1$.

If we have two bricks, they will just balance if the center of the top brick is directly over the right edge of the bottom brick: $R(2) = 3/2$.

How far to the right can we place the third brick? The top two bricks must balance by themselves, so the top brick extends $1/2$ unit further than the middle brick. The combined center of mass of the top two bricks must lie over the right-hand edge of the bottom brick. This center of mass is at $R(3) - 3/4$. Therefore,

$$R(3) = 1 + \frac{3}{4} = \frac{7}{4}.$$

Can we find an n so that $R(n) \geq 2$? The evidence is still inconclusive. The fourth brick begins to illuminate what is happening (see figure 2). Again, the top three bricks must be stable, so the top brick is $1/2$ a unit to the right of the second brick from the top. The second brick from the top is $1/4$ of a unit to the right of the third brick from the top. How far to the right can we move this third brick? The center of mass of the top three bricks must lie over the right-hand edge of the bottom brick.

The center of mass of these three bricks is at

$$\frac{1}{3} \left[\left(R(4) - \frac{1}{2} \right) + (R(4) - 1) + \left(R(4) - \frac{5}{4} \right) \right] = R(4) - \frac{11}{12}.$$

That means that

$$R(4) = 1 + \frac{11}{12} = \frac{23}{12}.$$

We see that what we really need to know is the distance from the center of mass of a stack of n bricks to the right-hand edge of the stack. If we call this $C(n)$, then

$$R(n+1) = 1 + C(n).$$

As we have seen,

$$C(1) = \frac{1}{2}, \quad C(2) = \frac{3}{4}, \quad C(3) = \frac{11}{12}.$$

When we put our three bricks on top of a fourth, that moves the center of mass to the left by $(1/4) \times (1/2)$ because the fourth brick is $1/4$ of the total mass and its center of mass is $1/2$ unit to the left of the previous center of mass,

$$C(4) = C(3) + \frac{1}{8} = \frac{11}{12} + \frac{1}{8} = \frac{25}{24}.$$

With five bricks, $R(5) = 49/24$, and this fifth brick lies completely to the right of the bottom brick.

We can now pick up the pattern. If we have a stack of $n - 1$ bricks and place them on top of an n th brick, this moves the center of mass $1/2n$ units to the left, so

$$\begin{aligned} C(n) &= C(n-1) + \frac{1}{2n} \\ &= C(n-2) + \frac{1}{2(n-1)} + \frac{1}{2n} \\ &= \dots \\ &= C(1) + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n} \\ &= \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right), \\ R(n) &= 1 + \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} \right). \end{aligned}$$

Questions

1. What is the smallest n so that $R(n) \geq 3$? ≥ 10 ? ≥ 100 ? (Ignore all practical considerations.)
2. We have stacked these bricks so that each is as far to the right as possible. The slightest breeze would cause this stack to topple. Redo this problem so that instead of placing the center of mass of the $n - 1$ bricks above the right-hand edge of the bottom brick, at each iteration we put it $1/4$ of a unit to the left of the right-hand edge of the bottom brick, to gain greater stability. How many bricks are now needed so that the top brick is completely to the right of the bottom brick?

2 Traversing the Desert

We are faced with the problem of crossing 1000 miles of desert using a truck that gets 5 miles to the gallon and that can only carry 80 gallons, enough gasoline to go 400 miles before it needs to refill. We assume that it can carry its gasoline in containers that can be deposited at any point

along the route for later use. Thus, for example, we could travel 100 miles into the desert, drop off 40 gallons, and have just enough to get back to the starting point and refill. We can now make a second trip. When we get to the drop-off point, we have 60 gallons left. We refill from our deposit and go another 100 miles into the desert. There we drop off 40 gallons, get back to the first depository, and refill, getting just enough gas to get back to the starting point. We now have no gasoline at the 100 mile mark, but we have 40 gallons sitting 200 miles into the desert. If we now started out from home, we could refill when we got to the 200 mile mark and go another 400 miles into the desert. We do not want to do that because we would run out of gas 400 miles short of our destination. But can we get across the desert, and, if so, how many trips would it take?

There are many solutions to this problem. We can extend the method used in the first paragraph to make 40 gallon deposits at various 100 mile marks. Show by induction that using the procedure described above, it takes 2^{n-1} trips to get 40 gallons out to the $100n$ -mile mark. If we deposited 40 gallons at the 400-mile mark and 80 gallons at the 600-mile mark, then we could make a trip completely across the desert. It would take 72 trips to set up these deposits. We would cross the desert on our 73rd trip.

We can do better than that. If we deposit 40 gallons at the 200-, 400-, and 600-mile marks, then on the last trip we top up every 200 miles and can get across the desert. This only requires 42 trips to set up the deposits, and we get across the desert with 43 trips.

What is the fewest number of trips that will get us across the desert?

To understand the solution to this problem, it is useful to change how we visualize it. Instead of making n trips, we begin with n trucks. Each truck can share its gasoline with the subsequent trucks down the line. All but one of them will have to return to the starting point (so that they can refill to make the next trip). One truck, which represents our last trip across the desert, does not return. It needs to have filled up from the last truck to turn back at the 600-mile mark.

The n trucks all start out together. The first truck to turn back will share its gasoline with the other trucks. It needs to be able to share all of its gasoline, leaving just enough gas for it and all but the last truck to get back from its turn-around point. Let x_1 be the mile at which it turns around. Each of the n trucks has consumed $x_1/400$ of its gasoline. All but the last truck will need to return from this drop-off point, which means that they will need $x_1/400$ of their capacity for the return trip. We use the gasoline in the first truck most efficiently if

$$n \frac{x_1}{400} + (n-1) \frac{x_1}{400} = 1, \quad x_1 = \frac{400}{2n-1}.$$

The remaining $n-1$ trucks are filled when they leave mile $x_1 = 400/(2n-1)$, and there is enough gasoline deposited at this position for $n-2$ of them to get back to the starting point. The second truck should go a further $400/(2n-3)$ miles, turning around at mile

$$x_2 = 400 \left(\frac{1}{2n-1} + \frac{1}{2n-3} \right).$$

Continuing in this way, we see that the $n-1$ st truck turns around at mile

$$x_{n-1} = 400 \left(\frac{1}{2n-1} + \frac{1}{2n-3} + \cdots + \frac{1}{3} \right).$$

When it leaves mile x_{n-1} , the last truck is full. It will make it across the desert provided that $x_{n-1} \geq 600$. We now have an equation we can solve:

$$\begin{aligned}400 \left(\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} \right) &\geq 600, \\ \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} &\geq \frac{3}{2}, \\ 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} &\geq \frac{5}{2}, \\ \sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^n \frac{1}{k} &\geq 2.5.\end{aligned}\tag{1}$$

Questions

1. Find the smallest value of n that satisfies equation (1).
2. If the truck can carry enough gasoline to travel r miles and if the desert is d miles wide, find the fewest trips needed to cross the desert, expressed as a function of r and d .