Applications of the Wronskian to ordinary linear differential equations

Consider a of n continuous functions $y_i(x)$ $[i = 1, 2, 3, \ldots, n]$, each of which is differentiable at least n times. Then if there exist a set of constants λ_i that are not all zero such that

$$\lambda_i y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) = 0\,, \tag{1}$$

then we say that the set of functions $\{y_i(x)\}$ are *linearly dependent*. If the only solution to eq. (1) is $\lambda_i = 0$ for all i, then the set of functions $\{y_i(x)\}$ are *linearly independent*.

The Wronskian matrix is defined as:

$$\Phi[y_i(x)] = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix},$$

where

$$y_i' \equiv \frac{dy_i}{dx}\,, \quad y_i'' \equiv \frac{d^2 y_i}{dx^2}\,, \quad \cdots\,, \quad y_i^{(n-1)} \equiv \frac{d^{(n-1)} y_i}{dx^{n-1}}\,.$$

The *Wronskian* is defined to be the determinant of the Wronskian matrix,

$$W(x) \equiv \det\, \Phi[y_i(x)]\,. \tag{2}$$

According to the contrapositive of eq. (8.5) on p. 133 of Boas, if $\{y_i(x)\}$ is a linearly dependent set of functions then the Wronskian must vanish. However, the converse is not necessarily true, as one can find cases in which the Wronskian vanishes without the functions being linearly dependent. (For further details, see problem 3.8–16 on p. 136 of Boas.)

Nevertheless, if the $y_i(x)$ are solutions to an nth order ordinary linear differential equation, then the converse does hold. That is, if the $y_i(x)$ are solutions to an nth order ordinary linear differential equation and the Wronskian of the $y_i(x)$ vanishes, then $\{y_i(x)\}$ is a linearly dependent set of functions. Moreover, if the Wronskian does not vanish for some value of x, then it is does not vanish for all values of x, in which case an arbitrary linear combination of the $y_i(x)$ constitutes the most general solution to the nth order ordinary linear differential equation.

To demonstrate that the Wronskian either vanishes for all values of x or it is never equal to zero, if the $y_i(x)$ are solutions to an nth order ordinary linear differential equation, we shall derive a formula for the Wronskian. Consider the differential equation,

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0\,. \tag{3}$$

We are interested in solving this equation over an interval of the real axis $a < x < b$ in which $a_0(x) \neq 0$. We can rewrite eq. (3) as a first order matrix differential equation. Defining the vector

$$\vec{\boldsymbol{Y}} = \begin{pmatrix} y \\ y' \\ y'' \\ \cdots \\ y^{(n-1)} \end{pmatrix} ,$$

It is straightforward to verify that eq. (3) is equivalent to

$$\frac{d\vec{\boldsymbol{Y}}}{dx} = A(x)\vec{\boldsymbol{Y}} ,$$

where the matrix $A(x)$ is given by

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{a_n(x)}{a_0(x)} & -\dfrac{a_{n-1}(x)}{a_0(x)} & -\dfrac{a_{n-2}(x)}{a_0(x)} & -\dfrac{a_{n-3}(x)}{a_0(x)} & \cdots & -\dfrac{a_1(x)}{a_0(x)} \end{pmatrix} . \tag{4}$$

It immediately follows that if the $y_i(x)$ are linearly independent solutions to eq. (3), then the Wronskian matrix satisfies the first order matrix differential equation,

$$\frac{d\Phi}{dx} = A(x)\Phi . \tag{5}$$

Using eq. (25) of Appendix A, it follows that

$$\frac{d}{dx}\det\Phi = \det\Phi\,\mathrm{Tr}\left(\Phi^{-1}\frac{d\Phi}{dx}\right) = \det\Phi\,\mathrm{Tr}\;A(x) ,$$

after using eq. (5) and the cyclicity property of the trace (i.e. the trace is unchanged by cyclically permuting the matrices inside the trace). In terms of the Wronskian W defined in eq. (2),

$$\frac{dW}{dx} = W\,\mathrm{Tr}\;A(x) . \tag{6}$$

This is a first order differential equation for W that is easily integrated,

$$W(x) = W(x_0)\exp\left\{\int_{x_0}^{x}\mathrm{Tr}\,A(t)dt\right\} .$$

Using eq. (4), it follows that $\mathrm{Tr}\,A(t) = -a_1(t)/a_0(t)$. Hence, we arrive at *Liouville's formula* (also called Abel's formula),

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x}\frac{a_1(t)}{a_0(t)}dt\right\} . \tag{7}$$

Note that if $W(x_0) \neq 0$, then the result for $W(x)$ is strictly positive or strictly negative depending on the sign of $W(x_0)$. This confirms our assertion that the Wronskian either vanishes for all values of x or it is never equal to zero.

Let us apply these results to an ordinary second order linear differential equation,

$$y'' + a(x)y' + b(x)y = 0\,, \tag{8}$$

where for convenience, we have divided out by the function that originally appeared multiplied by y''. Then, eq. (7) yields the Wronskian, which we shall write in the form:

$$W(x) = c \exp\left\{-\int^x a(x)dx\right\}\,, \tag{9}$$

where c is an arbitrary nonzero constant and $\int^x a(x)dx$ is the indefinite integral of $a(x)$.

For the case of a second order linear differential equation, there is a simpler and more direct derivation of eq. (9). Suppose that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of eq. (8). Then the Wronskian is non-vanishing,

$$W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2 \neq 0\,. \tag{10}$$

Taking the derivative of the above equation,

$$\frac{dW}{dx} = \frac{d}{dx}(y_1 y_2' - y_1' y_2) = y_1 y_2'' - y_1'' y_2\,,$$

since the terms proportional to $y_1' y_2'$ exactly cancel. Using the fact that y_1 and y_2 are solutions to eq. (8), we have

$$y_1'' + a(x)y_1' + b(x)y_1 = 0\,, \tag{11}$$

$$y_2'' + a(x)y_2' + b(x)y_2 = 0\,. \tag{12}$$

Next, we multiply eq. (12) by y_1 and multiply eq. (11) by y_2, and subtract the resulting equations. The end result is:

$$y_1 y_2'' - y_1'' y_2 + a(x)\left[y_1 y_2' - y_1' y_2\right] = 0\,.$$

or equivalently [cf. eq. (6)],

$$\frac{dW}{dx} + a(x)W = 0\,, \tag{13}$$

The solution to this first order differential equation is Abel's formula given in eq. (9).

The Wronskian also appears in the following application. Suppose that one of the two solutions of eq. (8), denoted by $y_1(x)$ is known. We wish to determine a second linearly independent solution of eq. (8), which we denote by $y_2(x)$. The following equation is an algebraic identity,

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2}\,,$$

after using the definition of the Wronskian W given in eq. (10). Integrating with respect to x yields

$$\frac{y_2}{y_1} = \int^x \frac{W(x)\,dx}{[y_1(x)]^2}\,.$$

Hence, it follows that*

$$y_2(x) = y_1(x) \int^x \frac{W(x)\,dx}{[y_1(x)]^2}\,. \tag{14}$$

Note that an indefinite integral always includes an arbitrary additive constant of integration. Thus, we could have written:

$$y_2(x) = y_1(x) \left\{ \int^x \frac{W(x)\,dx}{[y_1(x)]^2} + C \right\},$$

where C is an arbitrary constant. Of course, since $y_1(x)$ is a solution to eq. (8), then if $y_2(x)$ is a solution, then so is $y_2(x) + Cy_1(x)$ for any number C. Thus, we are free to choose any convenient value of C in defining the second linearly independent solution of eq. (8).

Finally, we note that the Wronskian also appears in solutions to inhomogeneous linear differential equations. For example, consider

$$y'' + a(x)y' + b(x)y = f(x)\,, \tag{15}$$

and assume that the solutions to the homogeneous equation [eq. (8)], denoted by $y_1(x)$ and $y_2(x)$ are known. Then the general solution to eq. (15) is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)\,,$$

where $y_p(x)$, called the particular solution, is determined by the following formula,

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)}\,dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)}\,dx\,. \tag{16}$$

This result is derived using the technique of variation of parameters. Namely, one writes

$$y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x)\,, \tag{17}$$

subject to the condition (which is chosen entirely for convenience):

$$v_1' y_1 + v_2' y_2 = 0\,. \tag{18}$$

With this choice, it follows that

$$y_p' = v_1 y_1' + v_2 y_2'\,, \tag{19}$$

$$y_p'' = v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2''\,. \tag{20}$$

*A second derivation of eq. (14) is given in Appendix B. This latter derivation is useful as it can be easily generalized to the case of an nth order linear differential equation.

Plugging eqs. (17), (19) and (20) into eq. (15), and using the fact that y_1 and y_2 satisfy the homogeneous equation [eq. (8)] one obtains,

$$v_1' y_1' + v_2' y_2' = f(x) . \qquad (21)$$

We now have two equations, eqs. (18) and (21), which constitute two algebraic equations for v_1' and v_2'. The solutions to these equations yield

$$v_1' = -\frac{y_2(x) f(x)}{W(x)} , \qquad\qquad v_2' = \frac{y_1(x) f(x)}{W(x)} ,$$

where $W(x)$ is the Wronskian. We now integrate to get v_1 and v_2 and plug back into eq. (17) to obtain eq. (16). The derivation is complete.

Reference:

Daniel Zwillinger, *Handbook of Differential Equations*, 3rd Edition (Academic Press, San Diego, CA, 1998).

APPENDIX A: Derivative of the determinant of a matrix

Recall that for any matrix A, the determinant can be computed by the cofactor expansion. The adjugate of A, denoted by adj A is equal to the transpose of the matrix of cofactors. In particular,

$$\det A = \sum_j a_{ij} (\text{adj } A)_{ji} , \qquad \text{for any fixed } i , \qquad (22)$$

where the a_{ij} are elements of the matrix A and $(\text{adj } A)_{ji} = (-1)^{i+j} M_{ij}$ where the minor M_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of A.

Suppose that the elements a_{ij} depend on a variable x. Then, by the chain rule,

$$\frac{d}{dx} \det A = \sum_{i,j} \frac{\partial \det A}{\partial a_{ij}} \, \frac{da_{ij}}{dx} . \qquad (23)$$

Using eq. (22), and noting that $(\text{adj } A)_{ji}$ does not depend on a_{ij} (since the ith row and jth column are removed before computing the minor determinant),

$$\frac{\partial \det A}{\partial a_{ij}} = (\text{adj } A)_{ji} .$$

Hence, eq. (23) yields *Jacobi's formula:*†

$$\frac{d}{dx} \det A = \sum_{i,j} (\text{adj } A)_{ji} \frac{da_{ij}}{dx} = \text{Tr} \left[(\text{adj } A) \frac{dA}{dx} \right] . \qquad (24)$$

†Recall that if $A = [a_{ij}]$ and $B = [b_{ij}]$, then the ij matrix element of AB are given by $\sum_k a_{ik} b_{kj}$. The trace of AB is equal to the sum of its diagonal elements, or equivalently

$$\text{Tr}(AB) = \sum_{jk} a_{jk} b_{kj} .$$

If A is invertible, then we can use the formula

$$A^{-1}\det A = \text{Adj } A\,,$$

to rewrite eq. (24) as[‡]

$$\frac{d}{dx}\det A = \det A\,\text{Tr}\left(A^{-1}\frac{dA}{dx}\right)\,, \tag{25}$$

which is the desired result.

Reference:

M.A. Goldberg, *The derivative of a determinant*, The American Mathematical Monthly, Vol. 79, No. 10 (Dec. 1972) pp. 1124–1126.

APPENDIX B: Another derivation of eq. (14)

Given a second order linear differential equation

$$y'' + a(x)y' + b(x)y = 0\,, \tag{26}$$

with a known solution $y_1(x)$, then one can derive a second linearly independent solution $y_2(x)$ by the method of variations of parameters.[§] In this context, the idea of this method is to define a new variable v,

$$y_2(x) = v(x)y_1(x) = y_1(x)\int w(x)dx\,, \tag{27}$$

where

$$v' \equiv w\,. \tag{28}$$

Then, we have

$$y_2' = vy_1' + wy_1\,, \qquad y_2'' = vy_1'' + w'y_1 + 2wy_1'\,.$$

Since y_2 is a solution to eq. (26), it follows that

$$w'y_1 + w[2y_1' + a(x)y_1] + v[y_1'' + a(x)y_1' + b(x)y_1] = 0\,,$$

Using the fact that y_1 is a solution to eq. (26), the coefficient of v vanishes and we are left with a first order differential equation for w

$$w'y_1 + w[2y_1' + a(x)y_1] = 0\,.$$

[‡]Note that Tr $(cB) = c\,\text{Tr } B$ for any number c and matrix B. In deriving eq. (25), $c = \det A$.

[§]This method is easily extended to the case of an nth order linear differential equation. In particular, if a non-trivial solution to eq. (3) is known, then this solution can be employed to reduce the order of the differential equation by 1. This procedure is called reduction of order. For further details, see pp. 352–354 of Daniel Zwillinger, *Handbook of Differential Equations*, 3rd Edition (Academic Press, San Diego, CA, 1998).

After dividing this equation by y_1, we see that the solution to the resulting equation is

$$\begin{aligned} w(x) &= c\,\exp\left\{-\int\left(\frac{2y_1'(x)}{y_1(x)}+a(x)\right)dx\right\} \\ &= c\,e^{-2\ln y_1(x)}\exp\left\{-\int a(x)dx\right\} \\ &= \frac{c}{[y_1(x)]^2}\exp\left\{-\int a(x)dx\right\} = \frac{W(x)}{[y_1(x)]^2}\,, \end{aligned} \tag{29}$$

after using eq. (9) for the Wronskian. The second solution to eq. (26) defined by eq. (27) is then given by

$$y_2(x) = y_1(x)\int \frac{W(x)}{[y_1(x)]^2}\,dx\,,$$

after employing eq. (29).