APPENDIX III

GROUP ALGEBRA AND THE REDUCTION OF REGULAR REPRESENTATION

This appendix serves two purposes: (i) it gives a systematic introduction to the *group algebra* and the *regular representation*, to supplement the very brief treatment of Sec. 3.7; (ii) it provides the mathematical framework for the construction of irreducible representations of the symmetric groups studied in Chap. 5 and Appendix IV.

Given a finite group $G = \{g_i;\, i = 1, 2, \ldots, n_G\}$ and the group multiplication rule $g_i g_j = g_k$, we introduced the *regular representation* matrices Δ by $g_i g_j = g_m \Delta^m_{ij}$ (Sec. 3.7). The right-hand side was interpreted as a "formal sum", since the original definition of a group does not involve the operation of taking linear combinations. In order to explore the detailed structure of the regular representation, it is necessary to be more precise about what we do.

III.1 Group Algebra

Although in "pure" group theory one only deals with a single operation—the group multiplication—it is natural to introduce the idea of *linear combinations of group elements* in group representation theory because the representation operators $\{U(g)\}$ have such an algebraic structure. This can lead to powerful techniques for constructing irreducible representations.

Definition III.1: For a given finite group G, the *group algebra* $\tilde{G}$ is defined to consist of all formal linear combinations of g_i, $r = g_i r^i$, where $g_i \in G$ and $\{r^i\}$ are complex numbers. Linear combinations of elements of the algebra are formed according to the obvious rule: $\alpha r + \beta q = g_i(\alpha r^i + \beta q^i)$ where α, β are arbitrary complex numbers. In addition, multiplication of one element of the algebra (q) by another (r) is given by $rq = g_i g_j\, r^i q^j = g_k(\Delta^k_{ij} r^i q^j)$, where Δ^k_{ij} are determined by the group multiplication rule as indicated.

The group algebra so defined has the mathematical structure of a *ring*. With respect to the operation of taking linear combinations, a group algebra obviously constitutes a linear vector space. In order to make this aspect of its properties explicit, we sometimes adopt the Dirac vector notation for elements of the algebra: $|r\rangle \in \tilde{G}$. By definition, the original group elements $\{g_i\}$ form a basis on this vector space. It is possible to define an inner product on this vector space, $\langle r|q\rangle = r_i^* q^i$ for $r, q \in \tilde{G}$. With respect to this scalar product, the basis $\{g_i\}$ is by definition orthonormal. We shall not need to make use of this scalar product in what follows.

Let $G = \{g_i\}$, and $r \in \tilde{G}$; the element r induces a natural mapping on the group algebra space $\tilde{G}$ by the rule of group multiplication. This can be seen most clearly using the vector notation for elements of the algebra and interpreting the identity $rg_i = g_j g_i r^j = g_k r^j \Delta^k_{ji}$ [cf. Eq. (3.7-1)] as

$$r|g_i\rangle = |g_k\rangle r^j \Delta^k_{ji} \quad . \tag{III.1-1}$$

In general, for any $q \in \tilde{G}$,

$$r|q\rangle = r|g_i\rangle q^i = |g_k\rangle [r^j \Delta^k_{ji} q^i] \quad . \tag{III.1-2}$$

Therefore, every element of the group algebra r also plays the role of an *operator* on the vector space $\tilde{G}$.

The dual role played by the elements of the group algebra—as vectors *and* operators—is the key feature of the regular representation and is primarily responsible for its remarkable properties.

Definition III.2: A *representation of the group algebra* $\tilde{G}$ is a mapping from $\tilde{G}$ to a set of linear operators $\{U\}$ on a vector space V which preserves the group algebra structure (Def. III.1): if $q, r \in \tilde{G}$, and $U(q)$, $U(r)$ are their images, then $U(\alpha q + \beta r) = \alpha U(q) + \beta U(r)$, and $U(qr) = U(q)U(r)$. An *irreducible representation* of $\tilde{G}$ is one which does not have any non-trivial invariant subspace in V.

Because, by definition, elements of the group G form a basis in the group algebra $\tilde{G}$, it is straightforward to establish the following theorem.

Theorem III.1: (i) A representation of $\tilde{G}$ is also a representation of G, and *vice versa*; (ii) An irreducible representation of $\tilde{G}$ is also irreducible with respect to G, and *vice versa.*

The construction of irreducible representations of the group algebra is facilitated by the possibility of taking linear combinations of group elements to form the appropriate projection operators.

III.2 Left Ideals, Projection Operators

The vector space of the regular representation D^R is the group algebra space $\tilde{G}$ itself. We know that every inequivalent irreducible representation D^μ is contained in D^R n_μ times, where n_μ is the dimension of the μ-representation [Theorem 3.8]. Therefore, G can be decomposed into a direct sum of irreducible invariant subspaces L^μ_a where $a = 1, 2, \ldots, n_\mu$. It is, in principle, possible to find basis vectors of $\tilde{G}$ such that the first one lies in L^1_1 (always of dimension 1), the next n_2 lie in $L^2_1, \ldots$ etc. With respect to such a basis, the regular representation matrices appear in block-diagonal form as shown, where all unfilled blocks consist of zero elements:

$$\begin{pmatrix} 1 & & & & & & \\ & D^2 & & & & & \\ & & D^2 & & & & \\ & & & \ddots & & & \\ & & & & D^{n_c} & & \\ & & & & & \ddots & \\ & & & & & & D^{n_c} \\ & n_2 \text{ blocks} & & \cdots & & n_c \text{ blocks} & \end{pmatrix}$$

In the group algebra space $\tilde{G}$, the subspaces discussed above are invariant under left multiplication, i.e. L consists of those elements $\{r\}$ such that $p|r\rangle \equiv |pr\rangle \in \mathrm{L}$ for all $p \in \tilde{\mathrm{G}}$ provided $|r\rangle \in \mathrm{L}$. Hence, they are also called *left ideals.* Left ideals which do not contain smaller left ideals are said to be minimal. Clearly, minimal left ideals correspond to irreducible invariant subspaces. If one can identify the minimal left ideals of the group algebra, all the inequivalent irreducible representations can be easily found.

A powerful method to identify the minimal left ideals is to find the corresponding projection operators. In Chap. 4 we discussed irreducible projection operators in general. However, the definition there requires knowledge of the irreducible representation matrices [cf. Theorem 4.2], hence it is not useful in the construction of these representations. We shall identify characteristic properties of projection operators on the group algebra space $\tilde{G}$, which can guide us in the construction of such operators for specific groups, such as the symmetric group S_n as discussed in Chap. 5.

If we denote the projection operator onto the minimal left ideal L_a^μ by P_a^μ, then we anticipate the following:

(i) $\mathrm{P}_a^\mu|r\rangle \in \mathrm{L}_a^\mu$ for all $r \in \tilde{G}$, in short, $\mathrm{P}_a^\mu \tilde{G} \doteq \mathrm{L}_a^\mu$

(ii) if $|q\rangle \in \mathrm{L}_a^\mu$, then $\mathrm{P}_a^\mu|q\rangle = |q\rangle$; hence

(iii) $\mathrm{P}_a^\mu r = r\,\mathrm{P}_a^\mu$ for all $r \in \tilde{G}$; and

(iv) $\mathrm{P}_a^\mu \mathrm{P}_b^\nu = \delta^{\mu\nu}\delta_{ab}\mathrm{P}_a^\mu$.

The commutativity condition (iii) can be established by applying each side of the equation to an arbitrary element of the algebra $|s\rangle \in \tilde{G}$ with $|s\rangle$ written in its fully decomposed form $|s\rangle = \sum_{\nu,b}|s_b^\nu\rangle$, $s_b^\nu \in \mathrm{L}_b^\nu$, and comparing the results. The other properties follow from the definition of projection operators.

In the following, we shall also denote the direct sum of all left ideals L_a^μ with the same μ by L^μ (recall that there are n_μ such minimal left ideals), and the corresponding projection operator by P^μ. We have, therefore, $\tilde{G} = \sum_\mu \mathrm{L}^\mu$ and $\mathrm{L}^\mu = \sum_a \mathrm{L}_a^\mu$.

III.3 Idempotents

The dual role of the group algebra elements as vectors and operators permits a particularly elegant realization of the projection operators discussed above. Let e be the identity element of the group G. Since $e \in \tilde{\mathrm{G}}$, it has a unique decomposition $e = \sum_\mu e_\mu$ where $e_\mu \in \mathrm{L}^\mu$.

Theorem III.2: The projection operator P^μ is realized by *right-multiplication* with e_μ, i.e. if we define $\mathrm{P}^\mu|r\rangle \equiv |re_\mu\rangle$ for all $r \in \tilde{\mathrm{G}}$, then P^μ has all the properties discussed in the previous section.

Proof: (i) One must first show that $\mathrm{P}^\mu|r\rangle = |re_\mu\rangle$ defines a linear operator. This is left as an exercise.

(ii) Let $r \in \tilde{G}$, then

(III.3-1) $$r = \sum_\mu r_\mu \qquad \text{where } r_\mu \in L^\mu \quad .$$

We also have

(III.3-2) $$r = re = r\sum_\mu e_\mu = \sum_\mu re_\mu$$

where $re_\mu \in L^\mu$ because L^μ is a left ideal. Since the decomposition of r is unique, we conclude, $P^\mu r = re_\mu = r_\mu$. This coincides with the definition of the required projection operator.

(iii) Let us compare the two operators $P^\mu q$ and qP^μ for any $q \in \tilde{G}$ by observing their action on an arbitrary $|r\rangle \in \tilde{G}$,

$$P^\mu q|r\rangle \equiv P^\mu|qr\rangle \equiv |(qr)e_\mu\rangle = |qre_\mu\rangle$$

$$qP^\mu|r\rangle \equiv q|re_\mu\rangle \equiv |q(re_\mu)\rangle = |qre_\mu\rangle \quad .$$

Therefore $P^\mu q = qP^\mu$ for all $q \in \tilde{G}$.

(iv) Comparing the decomposition of e_ν, $e_\nu = 0 + \cdots + e_\nu + 0 + \cdots + 0$, with $e_\nu = e_\nu e = e_\nu \sum_\mu e_\mu = e_\nu e_1 + \cdots + e_\nu e_\nu + e_\nu e_{\nu+1} + \cdots$, and making use of the uniqueness of the decomposition again, we conclude $e_\nu e_\mu = \delta_{\nu\mu} e_\mu$. This condition implies that $P^\mu P^\nu = \delta^{\mu\nu} P^\mu$. QED

Definition III.3: Elements of the group algebra e_μ which satisfy the condition $e_\mu e_\nu = \delta_{\mu\nu} e_\mu$ are called *idempotents.* Those which satisfy the above relation up to an additional normalization constant are said to be *essentially idempotent.*

The above discussion only required use of the uniqueness of the decomposition into direct sums and the fact that L^μ are left ideals. Therefore, the theorem also applies to projection operators P^μ_a (for the minimal left ideals) defined as right-multiplication by the corresponding identity operators e^μ_a.

Definition III.4: An idempotent which generates a minimal left ideal is said to be a *primitive idempotent.*

How can we tell whether a given idempotent is primitive or not? The following theorem provides the answer.

Theorem III.3: An idempotent e_i is primitive if and only if $e_i r e_i = \lambda_r e_i$ for all $r \in \tilde{G}$, where λ_r is some number (which depends on r).

Proof: (i) Assume that e is a primitive idempotent. Then the left ideal $L = \{re; r \in \tilde{G}\}$ is a minimal ideal. Hence, the realization of the group algebra on L is irreducible. Now, define an operator R on $\tilde{G}$ by $R|q\rangle \equiv |qere\rangle$ for all $q \in \tilde{G}$. Clearly $R|q\rangle \in L$ and $Rs = sR$ for all $s \in \tilde{G}$. Therefore R represents a projection into L; and, according to Schur's Lemma, in the subspace L it must be proportional to the unit operator. We conclude that $ere = \lambda_r e$.

(ii) Assume $ere = \lambda_r e$ for all $r \in \tilde{G}$, and $e = e' + e''$ where e' and e'' are both idempotents. We shall prove this leads to a contradiction. First, we have, by definition, $ee' = e'$. Multiplying by e on the right, we get $ee'e = e'$, hence $e' = \lambda e$ by our assumption. Using the last result in the defining condition for an idempotent, we obtain $e' = e'e' = \lambda^2 ee = \lambda^2 e$. Therefore $\lambda^2 = \lambda$, which implies $\lambda = 0$ or $\lambda = 1$. If $\lambda = 0$, then $e = e''$; if $\lambda = 1$, then $e = e'$. In either case, e is not decomposable as assumed. QED

Finally, we need a criterion to distinguish among primitive idempotents those which generate inequivalent representations.

Theorem III.4: Two primitive idempotents e_1 and e_2 generate equivalent irreducible representations if and only if $e_1 r e_2 \neq 0$ for some $r \in \tilde{G}$.

Proof: Let L_1 and L_2 be the two minimal left ideals generated by e_1 and e_2, respectively.

(i) If $e_1 r e_2 = s \neq 0$ for some $r \in \tilde{G}$, then consider the linear transformation $q_1 \in L_1 \xrightarrow{S} q_2 = q_1 s \in L_2$. Clearly, for all $p \in \tilde{G}$, $Sp|q_1\rangle = S|pq_1\rangle = |(pq_1)s\rangle = |p(q_1 s)\rangle = p|q_1 s\rangle = pS|q_1\rangle$. Therefore, acting on L_1, $Sp = pS$ for all $p \in \tilde{G}$. According to Schur's Lemma, the two representations $D^1(\tilde{G})$ and $D^2(\tilde{G})$ (realized on L_1 and L_2 respectively) must be equivalent.

(ii) If the two representations are equivalent, then there exists a linear transformation S such that $SD^1(p) = D^2(p)S$ (or, as linear mappings from L_1 to L_2, $Sp = pS$) for all $p \in \tilde{G}$. Now, $|s\rangle \equiv S|e_1\rangle \in L_2$, and $S|e_1\rangle = S|e_1 e_1\rangle = Se_1|e_1\rangle = e_1 S|e_1\rangle = e_1|s\rangle = |e_1 s\rangle$. Therefore $s = e_1 s$. Since $s \in L_2$, we also have $s = se_2$. Combining the two, we obtain $e_1 s = se_2 = s$, hence $s = e_1 s e_2$. QED

Example: The Reduction of the Regular Representation of C_3.

This will turn out to be a long-winded way of deriving the irreducible representations of C_3. But it is useful to work through a concrete example to gain a firm grip on the general technique. Since $G = C_3$ is abelian, all the irreducible representations are one-dimensional, and each occurs in the regular representation just once. The three elements of the group are (e, a, a^{-1}) and the group multiplication table is given in Table 2.2 (with $b = a^{-1}$).

(i) The idempotent e_1 for the identity representation is, as always,

$$e_1 = \frac{1}{3}(e + a + a^{-1}) \quad .$$

It is straightforward to show that $ge_1 = e_1 g = e_1$ for any $g \in G$, $e_1 e_1 = e_1$, and hence $e_1 g e_1 = e_1$ for all $g \in G$. Thus if $r = g_i r^i \in \tilde{G}$, then $e_1 r e_1 = e_1 \sum_i r^i = \lambda_r e_1$.

(ii) Let $e_2 = xe + ya + za^{-1}$ be a second idempotent, then we must have $e_1 e_2 = e_1(x + y + z) = 0$, and $e_2 e_2 = e_2 = xe + ya + za^{-1} = (x^2 + 2yz)e + (2xy + z^2)a + (2zx + y^2)a^{-1}$. Therefore, (a) $x + y + z = 0$, (b) $x = x^2 + 2yz$, (c) $y = z^2 + 2xy$, and (d) $z = y^2 + 2zx$. Combining (a) and (b), we obtain $y^2 + z^2 + 4yz + y + z = 0$; whereas combining (d) and (c), we obtain $3(y - z)(y + z) + (y - z) = 0$.

Three solutions emerge:

(1) $x = 2/3 \quad y = z = -1/3$

(2) $x = 1/3 \quad y = (1/3)e^{i2\pi/3} \quad z = (1/3)e^{-i2\pi/3}$

(3) $x = 1/3 \quad y = (1/3)e^{-i2\pi/3} \quad z = (1/3)e^{i2\pi/3}$.

Do all these solutions correspond to primitive idempotents? Let us first check on $e' = (2e - a - a^{-1})/3$: $e'e = ee' = e'$, $e'a = ae' = (-e + 2a - a^{-1})/3$, $e'a^{-1} = a^{-1}e' = (-e - a + 2a^{-1})/3$, $e'ee' = e'$, and $e'ae' = (-ee' + 2ae' - a^{-1}e')/3 = e'a$. The last result indicates that e' is not a primitive idempotent (cf. Theorem III.3).

Next, let us try

$$e_+ = 1/3[e + ae^{i2\pi/3} + a^{-1}e^{-i2\pi/3}] \qquad ee_+ = e_+e = e_+$$

$$ae_+ = e_+a = [ee^{-i2\pi/3} + a + a^{-1}e^{i2\pi/3}]/3 = e^{-i2\pi/3}e_+$$

$$e_+a^{-1} = a^{-1}e_+ = [ee^{i2\pi/3} + ae^{-i2\pi/3} + a^{-1}]/3 = e^{i2\pi/3}e_+$$

$$e_+ee_+ = e_+ \qquad e_+ae_+ = e^{-i2\pi/3}e_+e_+ = e^{-i2\pi/3}e_+$$

$$e_+a^{-1}e_+ = e^{i2\pi/3}e_+e_+ = e^{i2\pi/3}e_+ \quad .$$

Thus e_+ is a primitive idempotent. Similarly, one can show that $e_- = (e + ae^{-i2\pi/3} + a^{-1}e^{i2\pi/3})/3$ is a primitive idempotent.

Do e_+ and e_- generate equivalent representations?

Applying Theorem III.4, we find: $e_+ee_- = e_+e_- = 0$, $e_+ae_- = e^{-i2\pi/3}e_+e_- = 0$, and $e_+a^{-1}e_- = e^{i2\pi/3}e_+e_- = 0$. Hence, e_+, e_- generate inequivalent representations. Now, let us evaluate the representations. The left ideal L_2 is spanned by e_+; hence $e|e_+\rangle = |e_+\rangle 1$, $a|e_+\rangle = |ae_+\rangle = |e_+\rangle e^{-i2\pi/3}$, and $a^{-1}|e_+\rangle = |a^{-1}e_+\rangle = |e_+\rangle e^{i2\pi/3}$. Thus, the representation elements corresponding to (e, a, a^{-1}) are $(1, e^{-i2\pi/3}, e^{i2\pi/3})$. Similarly, the left ideal L_3 spanned by e_- gives rise to the representation $(e, a, a^{-1}) \to (1, e^{i2\pi/3}, e^{-i2\pi/3})$. To summarize, we exhibit the results in Table III.1.

Table III.1 IRREDUCIBLE REPRESENTATIONS OF THE GROUP C_3

μ	e	a	a^{-1}
1	1	1	1
2	1	$e^{-i2\pi/3}$	$e^{i2\pi/3}$
3	1	$e^{i2\pi/3}$	$e^{-i2\pi/3}$

It is straightforward to verify that these representation elements satisfy the orthonormality and completeness relations [Theorems 3.5 and 3.6] as they should. We note in passing that the non-primitive idempotent e' encountered earlier is equal to $e_+ + e_-$. It is therefore indeed decomposable.

III.4 Complete Reduction of the Regular Representation

Let us summarize the situation: (i) the group algebra can be decomposed into left ideals L^μ with μ running over all inequivalent irreducible representations of the group; (ii) each L^{μ} is generated by right multiplication with an idempotent e which satisfies the conditions, $e_{\mu}e_{\nu} = \delta_{\mu\nu}e_{\mu}$, $\sum_{\mu} e_{\mu} = e$; (iii) each L^{μ} (and the corresponding e_{μ}) can be decomposed into n_{μ} minimal left ideals L^{μ}_{a}, $a = 1, \ldots, n_{\mu}$, with associated primitive idempotents e^{μ}_{a} which satisfy $e^{\mu}_{a} r e^{\mu}_{b} = \delta_{ab} \lambda_{r} e^{\mu}_{a}$ for all $r \in \tilde{G}$. Therefore, the problem of the complete reduction of the regular representation of a group G is reduced to that of identifying all the inequivalent primitive idempotents. In Chap. 5 and Appendix IV, this technique is applied to the symmetric group S_n to derive all the inequivalent irreducible representations.

In closing, we mention that the left ideals $L^{\mu}\left(= \sum_{a} L^{\mu}_{a}\right)$ which are associated with definite irreducible representations μ are, in fact, minimal two-sided ideals. A *two-sided ideal* T is a subspace of $\tilde{G}$ such that if $r \in T$ then $qrs \in T$ for all $q, s \in \tilde{G}$. A minimal two-sided ideal is one which does not contain smaller two-sided ideals. If T is a minimal two-sided ideal and it contains a minimal left ideal L^{μ}_{a}, then it contains all the other minimal left ideals corresponding to the same μ, and only these. This interesting property is a natural consequence of Theorem III.4 as can be seen from the following observation. If L^{μ}_{a} and L^{μ}_{b} correspond to equivalent irreducible representations then there exists an element of $Gs \neq 0$ such that $s = e^{\mu}_{a} s e^{\mu}_{b}$ (cf. part (ii) of the proof of Theorem III.4). Thus if $r \in L^{\mu}_{a}$ and L^{μ}_{a} is in T, then $rs = rse^{\mu}_{b}$ is both in L^{μ}_{b} and in T. It follows then that L^{μ}_{b} is in T. Conversely, if L^{μ}_{a} and L^{ν}_{b} are both in T, there exists an s such that $L^{\mu}_{b} s = L^{\nu}_{b}$ and they generate equivalent representations. (Show that if s does not exist, then T cannot be minimal.) We see, therefore, the complete reduction of the regular representation corresponds to decomposing $\tilde{G}$ first into minimal two-sided ideals L^{μ}, one for each inequivalent irreducible representation, and then reducing L^{μ} into minimal left ideals L^{μ}_{a}.