The Cartan-Killing Form

1. The Cartan metric tensor

We begin by introducing a metric tensor and a scalar product on a real Lie algebra $\mathfrak{g}$. Suppose that $X, Y \in \mathfrak{g}$. We first define a linear operator, $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, such that

$$\mathrm{ad}_X(Y) = \big[X, Y\big] \,. \tag{1}$$

Then, the scalar product of two elements of the Lie algebra, called the Killing form (also called the Cartan-Killing form in the literature), is defined as

$$(X, Y) = \mathrm{Tr}\big(\mathrm{ad}_X \; \mathrm{ad}_Y\big) \,. \tag{2}$$

One can evaluate (X, Y) explicitly by choosing a basis for the Lie algebra, $\{\mathcal{A}_i\}$, which satisfies the commutation relations,

$$\big[\mathcal{A}_i, \mathcal{A}_j\big] = f_{ij}^k \mathcal{A}_k \,, \tag{3}$$

where the structure constants f_{ij}^k are real and $i, j, k = 1, 2, \ldots, \dim \mathfrak{g}$.[1] Any element of $X \in \mathfrak{g}$ is a real linear combination of the basis vectors, $\mathcal{A}_k$, i.e., $X = x^i \mathcal{A}_i$ with $x^i \in \mathbb{R}$. With respect to the basis, $\{\mathcal{A}_i\}$, the matrix elements of the linear operator $\mathrm{ad}_{\mathcal{A}_i}$ are easily obtained by noting that,

$$\mathrm{ad}_{\mathcal{A}_i}(\mathcal{A}_j) = \big[\mathcal{A}_i, \mathcal{A}_j\big] = f_{ij}^k \mathcal{A}_k \,, \tag{4}$$

from which it follows that the matrix elements of $\mathrm{ad}_{\mathcal{A}_i}$ are given by,[2]

$$(\mathrm{ad}_{\mathcal{A}_i})^k{}_j = f_{ij}^k \,. \tag{5}$$

Hence, it follows that $\mathrm{ad}_X(\mathcal{A}_j) = [X, \mathcal{A}_j] = x^i \big[\mathcal{A}_i, \mathcal{A}_j\big] = f_{ij}^k x^i \mathcal{A}_k$, which yields

$$(\mathrm{ad}_X)^k{}_j = f_{ij}^k x^i \,. \tag{6}$$

We can not compute the trace in eq. (2),

$$(X, Y) = \sum_{jk} (\mathrm{ad}_X)^k{}_j (\mathrm{ad}_Y)^j{}_k = f_{ij}^k f_{\ell k}^j x^i y^\ell \,. \tag{7}$$

[1] That is, the dimension of the Lie algebra $\mathfrak{g}$ (denoted by $\dim \mathfrak{g}$) is equal to the maximal number of linearly independent basis vectors spanning the Lie algebra.

[2] Recall that using bra and ket notation, one can write $T\,|j\rangle = \sum_k |k\rangle \langle k|\, T\,|j\rangle$. Equivalently, $T\,|j\rangle = T^k{}_j\,|k\rangle$ where the $T^k{}_j \equiv \langle k|\, T\,|j\rangle$ are the matrix elements of the operator T with respect to the basis $\{|i\rangle\}$ and there is an implicit sum over the repeated index k.

We now introduce the Cartan metric tensor of the Lie algebra, $g_{i\ell}$. For $X = x^i\mathcal{A}_i$ and $Y = y^i\mathcal{A}_i$, we define the metric tensor in terms of the scalar product in the usual way,

$$(X, Y) = g_{i\ell}x^i y^\ell \,. \tag{8}$$

Comparing eqs. (7) and (8) then yields an explicit expression for the Cartan metric tensor,

$$g_{i\ell} = f_{ij}^{k} f_{\ell k}^{j} \,. \tag{9}$$

Equivalently, one can write,

$$g_{i\ell} = (\mathcal{A}_i, \mathcal{A}_\ell) = \mathrm{Tr}\big(\mathrm{ad}_{\mathcal{A}_i}, \mathrm{ad}_{\mathcal{A}_\ell}\big)\,, \tag{10}$$

where the $\{\mathcal{A}_k\}$ are a basis for the Lie algebra $\mathfrak{g}$. Note that the metric tensor is a symmetric covariant tensor, since

$$g_{\ell i} = f_{\ell j}^{k} f_{ik}^{j} = f_{\ell k}^{j} f_{ij}^{k} = g_{i\ell}\,, \tag{11}$$

after relabeling $j \to k$ and $k \to j$.

The expression for the Cartan metric tensor is basis-dependent. Recall that given a Lie group G, the basis vectors of the corresponding Lie algebra $\mathfrak{g}$ are defined by

$$\mathcal{A}_i = \left.\frac{\partial A\big(\vec{\boldsymbol{a}}(t)\big)}{\partial a^i}\right|_{t=0}\,, \tag{12}$$

where the coordinates of the Lie group element, $A \in G$ are specified by $\vec{\boldsymbol{a}}$. The analytic curve, $\vec{\boldsymbol{a}}(t)$, passes through the identity element at $t = 0$, which corresponds to the origin of the coordinate system on the Lie group manifold. If we make a change of coordinates, $\vec{\boldsymbol{a}}' = \vec{\boldsymbol{a}}'(\vec{\boldsymbol{a}})$ on the Lie group manifold, then the basis vectors of $\mathfrak{g}$ are changed to

$$\mathcal{A}'_i = \left.\frac{\partial A\big(\vec{\boldsymbol{a}}'(t)\big)}{\partial a'^{\,i}}\right|_{t=0}\,, \tag{13}$$

Employing the chain rule,

$$\frac{\partial A\big(\vec{\boldsymbol{a}}'\big)}{\partial a'^{\,i}} = \frac{\partial A\big(\vec{\boldsymbol{a}}'(\vec{\boldsymbol{a}})\big)}{\partial a^j}\,\frac{\partial a^j}{\partial a'^{\,i}}\,. \tag{14}$$

Setting $t = 0$, it follows that

$$\mathcal{A}'_i = \left(\frac{\partial a^j}{\partial a'^{\,i}}\right)_{t=0} \mathcal{A}_j\,. \tag{15}$$

Using eq. (3) and the corresponding commutation relations of the transformed basis vectors, $\big[\mathcal{A}'_i, \mathcal{A}'_j\big] = f'^{\,k}_{ij}\mathcal{A}'_k$, it follows that

$$f'^{\,k}_{ij} = \left(\frac{\partial a'^{\,k}}{\partial a^n}\frac{\partial a^\ell}{\partial a'^{\,i}}\frac{\partial a^m}{\partial a'^{\,j}}\right)_{t=0} f_{\ell m}^{n}\,. \tag{16}$$

where $\partial a^\ell/\partial a'^{\,i}$ is the inverse Jacobian defined by,

$$\frac{\partial a'^{\,i}}{\partial a^n}\frac{\partial a^\ell}{\partial a'^{\,i}} = \delta_n^\ell\,. \tag{17}$$

In particular, the structure constants change if one performs a linear transformation,

$$a'^j = M^j{}_k a^k \,, \tag{18}$$

where M is a nonsingular matrix (j labels the rows and k labels the columns). In this case, it follows that

$$\mathcal{A}'_i = (M^{-1})^j{}_i \,\mathcal{A}_j \,, \qquad f'^{\,k}_{ij} = M^k{}_n (M^{-1})^\ell{}_i (M^{-1})^m{}_j f^n_{\ell m} \,. \tag{19}$$

Using eqs. (9) and (19), it follows that $g'_{i\ell} = f'^{\,k}_{ij} f'^{\,j}_{\ell k}$ is related to $g_{i\ell}$ as expected for a rank two covariant tensor,

$$g'_{i\ell} = (M^{-1})^j{}_i (M^{-1})^m{}_\ell \, g_{jm} \,. \tag{20}$$

If we define the matrix $S \equiv M^{-1}$, then we can rewrite eq. (20) in matrix form,

$$\mathcal{G}' = S^{\mathsf{T}} \mathcal{G} S \,, \tag{21}$$

where $\mathcal{G}$ is a real symmetric matrix whose matrix elements are given by g_{jm}. Using Sylvester's theorem (see Appendix A), a real invertible matrix S exists such that,

$$\mathcal{G}' = S^{\mathsf{T}} \mathcal{G} S = \text{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{-1, -1, \ldots, -1}_{s}, \underbrace{0, 0, \ldots, 0}_{t}) \,, \tag{22}$$

where r, s and t are non-negative integers such that $r + s + t = \dim \mathfrak{g}$. Moreover, if $\mathcal{G}$ is positive definite then $s = t = 0$ and $(S^{\mathsf{T}} \mathcal{G} S)_{ij} = \delta_{ij}$. Likewise, if $\mathcal{G}$ is negative definite then $r = t = 0$ and $(S^{\mathsf{T}} \mathcal{G} S)_{ij} = -\delta_{ij}$.

2. Properties of the Killing form

Recall that for $X, Y \in \mathfrak{g}$,[3]

$$(\exp \text{ad}_X) Y = e^X Y x^{-X} \in \mathfrak{g} \,. \tag{23}$$

We therefore introduce the notation,

$$\text{Ad}_g \equiv \exp \text{ad}_X \,, \quad \text{where } g \equiv e^X. \tag{24}$$

Since $g = e^X \in G$, this allows us to rewrite eq. (23) as,

$$\text{Ad}_g(Y) = g Y g^{-1} \,. \tag{25}$$

The following identity, which is equivalent to the Jacobi identity, is notable. For $X, Y \in \mathfrak{g}$,

$$\text{ad}_{[X,Y]} = [\text{ad}_X, \text{ad}_Y] \,, \tag{26}$$

To prove this result, consider the action of both sides of eq. (26) on $Z \in \mathfrak{g}$.

$$\text{ad}_{[X,Y]}(Z) = \big[[X, Y], Z\big] = \big[Z, [Y, X]\big] \,, \tag{27}$$

[3]See Theorem 1 in the class handout entitled, *Results for Matrix Exponentials and Logarithms.*

and

$$[\mathrm{ad}_X,\mathrm{ad}_Y](Z) = \bigl(\mathrm{ad}_X\,\mathrm{ad}_Y - \mathrm{ad}_Y\,\mathrm{ad}_X\bigr)(Z) = \mathrm{ad}_X([Y,Z]) - \mathrm{ad}_Y([X,Z])$$
$$= \bigl[X,[Y,Z]\bigr] - \bigl[Y,[X,Z]\bigr]$$
$$= -\bigl[X,[Z,Y]\bigr] - \bigl[Y,[X,Z]\bigr]\,, \tag{28}$$

after using the antisymmetry of the commutator. Hence, eqs. (26)–(28) yields,

$$\bigl[Z,[Y,X]\bigr] + \bigl[X,[Z,Y]\bigr] + \bigl[Y,[X,Z]\bigr] = 0\,, \tag{29}$$

which is the Jacobi identity. Note that if we put $X = \mathcal{A}_i$ and $Y = \mathcal{A}_j$ in eq. (26), we obtain

$$[\mathrm{ad}_{\mathcal{A}_i},\mathrm{ad}_{\mathcal{A}_j}] = \mathrm{ad}_{[\mathcal{A}_i,\mathcal{A}_j]} = f_{ij}^{k}\,\mathrm{ad}_{\mathcal{A}_k}\,, \tag{30}$$

after using eq. (3).[4] Hence, $\{\mathrm{ad}_X \mid X \in \mathfrak{g}\}$ constitutes the adjoint representation of $\mathfrak{g}$, with basis vectors $\{\mathrm{ad}_{\mathcal{A}_i}\}$.

Finally, we prove one additional identity,

$$\mathrm{ad}_{\mathrm{Ad}_g(X)} = \mathrm{Ad}_g\,\mathrm{ad}_X(\mathrm{Ad}_g)^{-1}\,. \tag{31}$$

First, we note that $\mathrm{Ad}_g\,\mathrm{Ad}_{g^{-1}}(Y) = Ad_g(g^{-1}Yg) = g(g^{-1}Yg)g^{-1} = Y$, which implies that

$$(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}\,. \tag{32}$$

Then, by using the definitions of the operators ad and Ad, it follows that for $X, Y \in \mathfrak{g}$ and $g \in G$,

$$\mathrm{Ad}_g\,\mathrm{ad}_X\,\mathrm{Ad}_{g^{-1}}(Y) = \mathrm{Ad}_g\,\mathrm{ad}_X(g^{-1}Yg) = Ad_g\bigl([X,g^{-1}Yg]\bigr) = g([X,g^{-1}Yg]g^{-1}$$
$$= gXg^{-1}Y - YgXg^{-1} = [gXg^{-1},Y] = \mathrm{ad}_{gXg^{-1}}(Y) = \mathrm{ad}_{\mathrm{Ad}_g(X)}(Y)\,. \tag{33}$$

Combining eqs. (32) and (33) yields eq. (31).

For $X, Y, Z \in \mathfrak{g}$ and $g \in G$, the Killing form, $(X,Y) = \mathrm{Tr}\bigl(\mathrm{ad}_X\;\mathrm{ad}_Y\bigr)$, satisfies the following four properties:

$$\text{1. bilinearity:}\qquad (\alpha X + \beta Y, Z) = \alpha(X,Z) + \beta(Y,Z)\,, \tag{34}$$
$$\text{2. symmetry:}\qquad (X,Y) = (Y,X)\,, \tag{35}$$
$$\text{3. antisymmetric:}\qquad \bigl(\mathrm{ad}_Z(X),Y\bigr) = -\bigl(X,\mathrm{ad}_Z(Y)\bigr)\,, \tag{36}$$
$$\text{4. orthogonal:}\qquad \bigl(\mathrm{Ad}_g(X),\mathrm{Ad}_g(Y)\bigr) = (X,Y)\,, \tag{37}$$

where $\alpha,\ \beta \in \mathbb{R}$.

The proofs of eqs. (34)–(37) are straightforward. Bilinearity of the Killing form follows from the linearity of the operator ad and the linearity of the trace. The symmetry of the Killing form follows from the identity $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Eq. (36) is a consequence of the Jacobi

[4] Observe that $\mathrm{ad}_{kX} = k\,\mathrm{ad}_X$ for $k \in \mathbb{R}$, since [cf. eq. (1)], $\mathrm{ad}_{kX}(Y) = [kX,Y] = k[X,Y] = k\,\mathrm{ad}_X(Y)$, for $X, Y \in \mathfrak{g}$ and $k \in \mathbb{R}$.

identity [cf. eqs. (26)–(29)]. To show this, consider the identity, $[A, BC] = [A, B]C + B[A, C]$. it then follows from eq. (26) that

$$\begin{aligned}[\mathrm{ad}_Z, \mathrm{ad}_X\, \mathrm{ad}_Y] &= [\mathrm{ad}_Z, \mathrm{ad}_X]\, \mathrm{ad}_Y + \mathrm{ad}_X [\mathrm{ad}_Z, \mathrm{ad}_Y] \\ &= \mathrm{ad}_{[Z,X]}\, \mathrm{ad}_Y + \mathrm{ad}_X\, \mathrm{ad}_{[Z,Y]}\ .\end{aligned} \tag{38}$$

Taking the trace of both sides of eq. (38) and using the fact that the trace of a commutator vanishes, one obtains

$$\big([Z, X], Y\big) + \big(X, [Z, Y]\big) = 0\,, \tag{39}$$

which is equivalent to eq. (36) in light of eq. (1). Finally, we use eq. (31) to compute,

$$\begin{aligned}\big(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)\big) &= \mathrm{Tr}\big\{\mathrm{ad}_{\mathrm{Ad}_g(X)}\, \mathrm{ad}_{\mathrm{Ad}_g(Y)}\big\} = \mathrm{Tr}\big\{\mathrm{Ad}_g\, \mathrm{ad}_X (\mathrm{Ad}_g)^{-1}\, \mathrm{Ad}_g\, \mathrm{ad}_Y (\mathrm{Ad}_g)^{-1}\big\} \\ &= \mathrm{Tr}\big(\mathrm{ad}_X\, \mathrm{ad}_Y\big) = (X, Y)\,,\end{aligned} \tag{40}$$

using the invariance of a trace under a cyclic permutation of its arguments. Thus, we have confirmed eq. (37).

It is instructive to examine eqs. (36) and (37) by expressing $X = x^k \mathcal{A}_k$ and $y = y^k \mathcal{A}_k$ with respect to a basis $\{\mathcal{A}_k\}$ of the Lie algebra $\mathfrak{g}$. We introduce the matrix elements of the operators Ad_g and ad_X, which are henceforth denoted by $A^j{}_k$ and $B^j{}_k$, respectively [cf. footnote 2],

$$\mathrm{Ad}_g(\mathcal{A}_k) = g\mathcal{A}_k g^{-1} = A^j{}_k \mathcal{A}_j\,, \tag{41}$$

$$\mathrm{ad}_X(\mathcal{A}_k) = [X, A_k] = B^j{}_k \mathcal{A}_j\,. \tag{42}$$

Writing $X = x^k \mathcal{A}_k \in \mathfrak{g}$, it follows that

$$\mathrm{Ad}_g(X) = x^k A^j{}_k \mathcal{A}_j\,, \qquad\qquad \mathrm{ad}(X) = x^k B^j{}_k \mathcal{A}_j\,. \tag{43}$$

Then, eqs. (8) and (37) yield,

$$g_{k\ell} x^k y^\ell = g_{ij} x^k y^\ell A^i{}_k A^j_\ell\,. \tag{44}$$

Since eq. (44) is valid for any choice of $X, Y \in \mathfrak{g}$, it follows that

$$g_{k\ell} = g_{ij} A^i{}_k A^j{}_\ell\,. \tag{45}$$

In matrix notation, where $\mathcal{G}$ is a real symmetric matrix whose matrix elements are given by g_{jm} and $A \equiv \mathrm{Ad}_g$, we can rewrite eq. (45) as

$$\mathcal{G} = A^{\mathsf{T}} \mathcal{G} A\,. \tag{46}$$

Similarly, eqs. (8) and (36) yield,

$$g_{ij} B^i{}_\ell x^\ell y^j = -g_{\ell k} x^\ell B^k{}_j y^j\,. \tag{47}$$

Since eq. (47) is valid for any choice of $X, Y \in \mathfrak{g}$, it follows that

$$g_{ij} B^i{}_\ell = -g_{\ell k} B^k{}_j\,. \tag{48}$$

In matrix form, eq. (48) can be written as $\mathcal{G}B = -(\mathcal{G}B)^{\mathsf{T}} = -B^{\mathsf{T}}\mathcal{G}$, after using $\mathcal{G} = \mathcal{G}^{\mathsf{T}}$. Equivalently, one can write,

$$B = -\mathcal{G}^{-1} B^{\mathsf{T}} \mathcal{G}\,. \tag{49}$$

3. Killing form of a real semisimple Lie algebra

In this section, we prove two theorems concerning the Killing form of a real semisimple Lie algebra $\mathfrak{g}$.

Theorem 1: If $\mathfrak{g}$ is a semisimple Lie algebra, then the Killing form is nondegenerate and conversely.

Proof: Suppose that $\mathfrak{g}$ is non-semisimple. Then, $\mathfrak{g}$ possesses a nonzero abelian ideal $\mathfrak{a}$. By definition, an ideal $\mathfrak{a}$ satisfies the condition that for all $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$, we have $[X, Y] \in \mathfrak{a}$. If $\{\mathcal{A}_n\}$ is a basis for $\mathfrak{g}$, then a subset of $\{\mathcal{A}_n\}$ will span the ideal $\mathfrak{a}$. Suppose that $\mathcal{A}_i \in \mathfrak{a}$ and $\mathcal{A}_j \in \mathfrak{g}$. Then, we can conclude that $[\mathcal{A}_i, \mathcal{A}_j] \in \mathfrak{a}$. Since $[\mathcal{A}_i, \mathcal{A}_j] = f_{ij}^k \mathcal{A}_k \in \mathfrak{a}$, it follows that $f_{ij}^k = 0$ unless $\mathcal{A}_k$ is in the set of basis vectors that span $\mathfrak{a}$. Moreover, if both $\mathcal{A}_i$ and $\mathcal{A}_j$ are in the set of basis vectors that span $\mathfrak{a}$ then $[\mathcal{A}_i, \mathcal{A}_j] \in \mathfrak{a}$ must vanish (since $\mathfrak{a}$ is abelian), which implies that $f_{ij}^k = 0$ for all k.

Consider the Cartan metric tensor, $g_{i\ell} = f_{ij}^k f_{\ell k}^j$, in the case where its indices i and ℓ run over values corresponding to the basis vectors of $\mathfrak{a}$. The argument presented above implies that because $\mathfrak{a}$ is an ideal, the index k runs only over values corresponding to the basis vectors of $\mathfrak{a}$ (since otherwise $f_{ij}^k = 0$). Finally, because $\mathfrak{a}$ is an abelian ideal, it follows that when the indices ℓ and k run over values corresponding to the basis vectors of $\mathfrak{a}$, then $f_{\ell k}^j = 0$. Hence, we conclude that $g_{i\ell} = 0$, when the indices i and ℓ run over values corresponding to the basis vectors of $\mathfrak{a}$. Consequently, $\det \mathcal{G} = 0$, or equivalently, the Killing form is degenerate.

Conversely, if the Killing form is degenerate, then $\{X \,|\, (X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$ is a nonzero solvable ideal of $\mathfrak{g}$. Thus, $\mathfrak{g}$ is non-semisimple.[5] Details can be found in the references.

Theorem 2: If $\mathfrak{g}$ is a compact semisimple Lie algebra, then for any nonzero $X \in \mathfrak{g}$, it follows that $(X, X) < 0$ and conversely.

Proof: Since Ad_g (for $g \in G$) is the adjoint representation of the Lie group G, it follows that the matrix elements of the adjoint representation satisfy eq. (46). Moreover, in light of eq. (22), one is always free to choose the coordinates on the Lie group manifold such that

$$\mathcal{G} = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{-1, -1, \ldots, -1}_{s}), \tag{50}$$

where $n = r + s$ is the dimension of the Lie algebra $\mathfrak{g}$. Note that in light of Theorem 1 above, the Killing form is nondegenerate, which implies that $\det \mathcal{G} \neq 0$ and hence $t = 0$ in the notation of eq. (22). Recall that the matrix Lie group $\mathrm{O}(r, s)$ is defined by[6]

$$\mathrm{O}(r, s) = \{A \in M_{r+s}(\mathbb{R}) \,|\, A^{\mathsf{T}} \mathcal{G} = \mathcal{G} A^{-1}\}, \tag{51}$$

[5]Given a Lie algebra $\mathfrak{a}$, one can define a sequence of derived ideals, $\mathfrak{a}^{(1)} = [\mathfrak{a}, \mathfrak{a}]$, $\mathfrak{a}^{(2)} = [\mathfrak{a}^{(1)}, \mathfrak{a}^1]$, etc. By induction and with the aid of the Jacobi identity, one can show that $\mathfrak{a}^{(n)}$ is an ideal of $\mathfrak{a}$ for any value of n. Then $\mathfrak{a}$ is solvable if a positive integer k exists such that $\mathfrak{a}^{(k)} = [\mathfrak{a}^{(k-1)}, \mathfrak{a}^{(k-1)}] = \{\vec{\mathbf{0}}\}$. It follows that $\mathfrak{a}^{(k-1)}$ is an abelian ideal of $\mathfrak{a}$. Hence, if $\mathfrak{g}$ is non-semisimple then it possesses a nonzero solvable ideal, which implies that it also possesses a nonzero abelian ideal.

[6]See, e.g., the class handout providing a table of the real Lie algebras corresponding to the classical matrix Lie groups, taken from J.F. Cornwell, *Group Theory in Physics: An Introduction* (Academic Press Inc., San Diego, CA, 1997).

where $M_{r+s}(\mathbb{R})$ is the set of all $(r+s)\times(r+s)$ real matrices and $\mathcal{G}$ is given in eq. (50). Thus, it follows from eq. (46) that the adjoint group corresponding to the matrices of the adjoint representation of G must be a subgroup of $\mathrm{O}(r,s)$.

Likewise, since ad_X is the adjoint representation of the Lie algebra $\mathfrak{g}$, it follows that the matrix elements of the adjoint representation satisfy eq. (49). Recall that the matrix Lie algebra $\mathfrak{so}(r,s)$ is defined by [cf. footnote 6],

$$\mathfrak{so}(r,s) = \{B \in M_{r+s}(\mathbb{R}) \mid B^{\mathsf{T}}\mathcal{G} = -\mathcal{G}B\}\,. \tag{52}$$

Thus, it follows from eq. (49) that the matrices of the adjoint representation of $\mathfrak{g}$ must be a subalgebra of $\mathfrak{so}(r,s)$.

Consider the special case where $\mathcal{G} = \pm\mathbf{I}$, where $\mathbf{I}$ is the $n\times n$ identity matrix. In this case, eqs. (46) and (49) yield,

$$A^{\mathsf{T}}A = \mathbf{I}\,, \qquad\qquad B^{\mathsf{T}} = -B\,. \tag{53}$$

In particular, the adjoint group corresponding to the matrices of the adjoint representation of G constitute a subgroup of $\mathrm{O}(n)$ and the matrices of the adjoint representation of $\mathfrak{g}$ constitute a subalgebra of $\mathfrak{so}(n)$. Moreover, for $X \in \mathfrak{g}$,[7]

$$(X,X) = \mathrm{Tr}\big(\mathrm{ad}_X\,\mathrm{ad}_X\big) = \mathrm{Tr}B^2 = -\mathrm{Tr}(BB^{\mathsf{T}}) = -\sum_{i,j} B_{ij}B_{ij} < 0\,, \tag{54}$$

unless $B = 0$ (or equivalently, unless $\mathrm{ad}_X = 0$). In particular, $(\mathcal{A}_i, \mathcal{A}_i) < 0$ [no sum over i], which in light of eq. (10) means that $g_{ii} < 0$. Thus, we conclude that in the special case under consideration, $\mathcal{G} = -\mathbf{I}$. Hence, a negative definite Killing form implies that the adjoint group corresponding to the matrices of the adjoint representation of G constitutes a closed subgroup of $\mathrm{O}(n)$, which implies that the Lie group G is compact. Conversely, an indefinite Killing form implies that the adjoint group constitutes a subgroup of $\mathrm{O}(r,s)$ where neither r nor s is zero. In this case, G is a noncompact group.

4. The Killing form of a complex semisimple Lie algebra

Although the Cartan metric and the Killing form defined in Section 1 were presented under the assumption that the Lie algebra $\mathfrak{g}$ was real, the same definitions can be applied to a complex Lie algebra. However, it is possible that some of the structure constants of a complex Lie algebra are complex. In such a case, the matrix $\mathcal{G}$ that appears in eq. (21) is a complex symmetric matrix, in which case Sylvester's theorem does not apply. Regarding the results of Section 3, Theorem 1 and its proof apply both to real and complex Lie algebras. In contrast, Theorem 2 is only relevant for real Lie algebras. In particular, in the case of a complex Lie group, we showed in class that any compact complex Lie group is abelian. Thus, any compact semisimple Lie group (and its corresponding Lie algebra) must be real.

Given a complex Lie algebra $\mathfrak{g}$, it is interesting to ask whether a basis exists in which all of the structure constants are real. If the answer is yes, then one can analyze the complex

[7] Note that when $g_{ij} \propto \delta_{ij}$, then there is no distinction between covariant and contravariant indices, and one can write all tensor quantities with lowered indices.

Lie algebra starting from this real basis,[8] in which case all the results of Section 1–3 apply. However, one can show that there exist complex Lie algebras that do not possess a real basis. For example, consider the following Lie algebra obtained by modifying one of the three commutators that defines the Lie algebra of the Euclidean group in two dimensions:

$$\big[\mathcal{A}_1\,,\,\mathcal{A}_2\big]=0\,,\qquad\qquad \big[\mathcal{A}_1\,,\,\mathcal{A}_3\big]=-\mathcal{A}_2\,,\qquad\qquad \big[\mathcal{A}_2\,,\,\mathcal{A}_3\big]=i\mathcal{A}_1\,. \tag{55}$$

Denoting the corresponding structure constants by f_{ij}^k, it is easy to show that no basis transformation of the form

$$\mathcal{A}'_j = M_j{}^k\mathcal{A}_k \tag{56}$$

exists such that the structure constants in the transformed basis satisfy $f'^k_{ij}\in\mathbb{R}$.

Nevertheless, there is a remarkable theorem that applies to complex semisimple Lie algebras.

Theorem 3: If $\mathfrak{g}$ is a complex semisimple Lie algebra, then a basis exists in which all the structure constants are real.

The proof of this theorem is rather involved. This theorem was originally obtained as a consequence of the classification or real simple Lie algebras by Cartan. Subsequently, Weyl provided a proof based on the detailed structure theory of semisimple Lie algebras. Proofs that avoid the more elaborate algebraic machinery employed by Cartan and Weyl can be found in Refs. 9 and 10. Note that this result implies that a basis exists in which the Cartan metric is real and symmetric, which ensures that a basis exists in which eq. (50) is satisfied.

5. Complex semisimple Lie algebras and their real forms

In general, the structure constants of a complex Lie algebra are complex. However, in light of Theorem 3 above, one can always transform the basis of a complex semisimple Lie algebra to a new basis in which the structure constants are real. Conversely, starting from eq. (3) [where the structure constants are real], one can complexify a real Lie algebra $\mathfrak{g}$ by writing $X=x^i\mathcal{A}_i$ with $x^i\in\mathbb{C}$. This process is called *complexification*. We shall denote the resulting complex Lie algebra by $\mathfrak{g}_{\mathbb{C}}$.

Given a complex semisimple Lie algebra, $\mathfrak{g}_{\mathbb{C}}$, with basis vectors satisfying $[\mathcal{A}_i,\mathcal{A}_j]=f_{ij}^k\mathcal{A}_k$ with $f_{ij}^k\in\mathbb{R}$, one can restrict $x^i\in\mathbb{R}$ to obtain a real semisimple Lie algebra, $\mathfrak{g}_{\mathbb{R}}$, with the same commutation relations.[9] The Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is an example of a *real form* of $\mathfrak{g}_{\mathbb{C}}$. More generally, a real form of $\mathfrak{g}_{\mathbb{C}}$ is defined as any real Lie algebra whose complexification yields $\mathfrak{g}_{\mathbb{C}}$. Note that $\dim_{\mathbb{C}}\,\mathfrak{g}_{\mathbb{C}}=\dim_{\mathbb{R}}\,\mathfrak{g}_{\mathbb{R}}$. That is, given a complex semisimple Lie algebra of complex dimension n, the corresponding real forms have real dimension n. In particular, the maximal number of linearly independent basis vectors is n for both $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{R}}$.

[8] Here, we have defined a real basis to mean a choice of generators for the complex Lie algebra in which all structure constants are real.

[9] In contrast, not every complex non-semisimple Lie algebra has a real form. For example, the Lie algebra defined by eq. (55) does not possess a real basis. Hence no real form can exist for this Lie algebra. In addition, see Example 5.1.24 on p. 88 of Ref. 7 or Example 5 on p. 19 of Ref. 8.

If $\mathfrak{g}_{\mathbb{C}}$ is semisimple, then its Killing form is non-degenerate (in light of Theorem 1). Hence, the corresponding real form, which has the same Cartan metric tensor, is also semisimple. Indeed, one can always transform the original basis of $\mathfrak{g}$ to a new basis in which the Cartan metric tensor is of the form given by eq. (50). However, this new basis is not unique. For example, one can always choose to multiply some of the basis vectors by i as long as all the structure constants remain real. In particular, it is always possible to find a basis for a semisimple $\mathfrak{g}_{\mathbb{C}}$ such that $\mathcal{G} = -\mathbf{I}$, in which case the corresponding real form is compact.

Indeed, there exists a simple algorithm to derive various possible real forms of $\mathfrak{g}_{\mathbb{C}}$. The algorithm consists of first choosing a basis for $\mathfrak{g}_{\mathbb{C}}$ such that the Cartan metric tensor is of the form given by eq. (50). Next, we consider all possible ways of multiplying a subset of the basis vectors $\{\mathcal{A}_k\}$ of $\mathfrak{g}_{\mathbb{C}}$ each by i such that $f_{ij}^{\,k} \in \mathbb{R}$. The resulting commutation relations will differ in some cases from the original one by some signs. Nevertheless, complex linear combinations of the new basis vectors, $x^i\mathcal{A}_i$ ($x^i \in \mathbb{C}$), still generate $\mathfrak{g}_{\mathbb{C}}$. After all possibilities for the $\{\mathcal{A}_i\}$ are considered, one now restricts the x^i to be real in each case in order to obtain the corresponding real forms. Note that among all possible real forms obtained in this way, only one of them is compact, corresponding to the basis in which $\mathcal{G} = -\mathbf{I}$. Other real forms obtained by the algorithm above necessarily possess an indefinite Killing form, and thus correspond to noncompact real Lie algebras.

A simple example demonstrates the procedure for finding real forms of $\mathfrak{g}_{\mathbb{C}}$. Consider the commutation relations of $\mathfrak{sl}(2,\mathbb{C})$,

$$[\mathcal{A}_i, \mathcal{A}_j] = \epsilon_{ijk}\mathcal{A}_k\,, \tag{57}$$

where the indices i, j, k run over 1,2 and 3. The corresponding Cartan metric tensor is negative definite,[10]

$$g_{i\ell} = \epsilon_{kij}\epsilon_{j\ell k} = -2\delta_{i\ell}\,. \tag{58}$$

If we restrict $x^i \in \mathbb{R}$, then we obtain the real compact Lie algebra, $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. Consider what happens if we redefine $\mathcal{A}_1 \to i\mathcal{A}_1$ and $\mathcal{A}_2 \to i\mathcal{A}_2$, while leaving $\mathcal{A}_3$ unchanged. The new commutation relations of $\mathfrak{sl}(2,\mathbb{C})$ are now given by,

$$[\mathcal{A}_1, \mathcal{A}_2] = -\mathcal{A}_3\,, \qquad [\mathcal{A}_2, \mathcal{A}_3] = \mathcal{A}_1\,, \qquad [\mathcal{A}_3, \mathcal{A}_1] = \mathcal{A}_2\,. \tag{59}$$

In this case, the Cartan metric is now indefinite,

$$g_{i\ell} = f_{ij}^{\,k} f_{\ell k}^{\,j} = \mathrm{diag}(2, 2, -2)\,. \tag{60}$$

Thus, if we now restrict $x^i \in \mathbb{R}$, the resulting real Lie algebra is noncompact. Indeed, the commutation relations given in eq. (59) correspond to those of $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1) \cong \mathfrak{so}(2,1)$. Note that any other redefinition of a subset of the basis vectors via multiplication by i, such that the structure constants remain real, will yield Lie algebras that are isomorphic to one of the two possible classes of real forms identified above.

It is straightforward to show that real forms of a simple Lie algebra are simple. However, the complexification of a real simple Lie algebra can yield either a simple or a semisimple real

[10] We can normalize the basis vectors appropriately such that $g_{i\ell} = -\delta_{i\ell}$ (although it is not necessary for this discussion).

algebra. For example, the complexification of the real simple Lie algebra $\mathfrak{so}(3,1)$ yields the semisimple complex Lie algebra $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

There is another way of obtaining a real Lie algebra from a complex Lie algebra called *realification*. Starting from the basis vectors, $\{\mathcal{A}_k\}$, of the complex Lie algebra, the elements of $\mathfrak{g}_{\mathbb{C}}$ are of the form $x^k \mathcal{A}_k$, with $x^k \in \mathbb{C}$. One can now construct a real Lie algebra that is given by real linear combinations of the basis vectors $\{\mathcal{A}_k, i\mathcal{A}_\ell\}$. We shall denote this real Lie algebra by $(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$.[11] Note that if $\dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} = n$, then $\dim_{\mathbb{R}} (\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}} = 2n$. As an example of this construction, the realification of $\mathfrak{sl}(2,\mathbb{C})$ yields $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \cong \mathfrak{so}(3,1)$. This isomorphism is important in the study of the Lie algebra of the Lorentz group.

One can now consider the complexification of the $2n$-dimensional real Lie algebra, $(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$, which will be denoted by $(\mathfrak{g}_{\mathbb{C}})^*_{\mathbb{R}}$. Note that $\dim_{\mathbb{C}} (\mathfrak{g}_{\mathbb{C}})^*_{\mathbb{R}} = 2n$. One can easily verify that,

$$(\mathfrak{g}_{\mathbb{C}})^*_{\mathbb{R}} \cong \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}_{\mathbb{C}}\,. \tag{61}$$

Thus, in our previous example, the complexification of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. Since the complexification of $\mathfrak{so}(3,1)$ is $\mathfrak{so}(4,\mathbb{C})$, it then follows that $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. In particular, $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ [or equivalently, $\mathfrak{so}(3,1)$] is one of real forms of $\mathfrak{so}(4,\mathbb{C})$.

6. A completely antisymmetric third rank tensor

Using the Cartan metric tensor, one can construct a completely antisymmetric third rank tensor that is related to the structure constants of the Lie algebra $\mathfrak{g}$,

$$f_{ijk} \equiv g_{k\ell} f_{ij}^{\ell}\,. \tag{62}$$

To prove that f_{ijk} is a completely antisymmetric third rank tensor, we employ eqs. (9) and (30) to write,

$$f_{ijk} = f_{ij}^{\ell}\,\mathrm{Tr}\big(\mathrm{ad}_{\mathcal{A}_k}\,\mathrm{ad}_{\mathcal{A}_\ell}\big) = \mathrm{Tr}\big(f_{ij}^{\ell}\,\mathrm{ad}_{\mathcal{A}_k}\,\mathrm{ad}_{\mathcal{A}_\ell}\big) = \mathrm{Tr}\big(\mathrm{ad}_{\mathcal{A}_k}[\mathrm{ad}_{\mathcal{A}_i}, \mathrm{ad}_{\mathcal{A}_j}]\big)\,, \tag{63}$$

after using the linearity of the trace. That is,

$$f_{ijk} = \mathrm{Tr}\big(\mathrm{ad}_{\mathcal{A}_k}\,\mathrm{ad}_{\mathcal{A}_i}\,\mathrm{ad}_{\mathcal{A}_j} - \mathrm{ad}_{\mathcal{A}_k}\,\mathrm{ad}_{\mathcal{A}_j}\,\mathrm{ad}_{\mathcal{A}_i}\big)\,, \tag{64}$$

which is manifestly antisymmetric under the interchange of any pair of indices i, j and k due to the invariance of the trace under a cyclic permutation of its arguments.

Note that in the special case where $\mathfrak{g}$ is a compact semisimple Lie algebra, we can choose a basis in which $g_{ij} = -\delta_{ij}$. For this basis choice, it follows that the structure constants of $\mathfrak{g}$, $f_{ij}^{k} = -f_{ijk}$, are completely antisymmetric under the interchange of any pair of indices.

[11] In the literature, the subscript $\mathbb{R}$ is sometimes omitted. One then says that the complex Lie algebra $\mathfrak{g}$ of complex dimension n can be regarded as a real Lie algebra of real dimension $2n$.

APPENDIX A: Sylvester's Theorem

In this appendix we prove the following theorem, often called Sylvester's law of inertia.[12]

Sylvester's Theorem: Consider an $n \times n$ hermitian matrix $M = M^\dagger$. Then there exists an invertible matrix S such that

$$S^\dagger MS = \mathrm{diag}(\underbrace{1,1,\ldots,1}_{r},\, \underbrace{-1,-1,\ldots,-1}_{s},\, \underbrace{0,0,\ldots,0}_{t})\,, \tag{65}$$

where r, s and t are non-negative integers such that $r+s+t=n$. That is, $S^\dagger MS$ is a diagonal matrix whose elements consist of 1 repeated r times, -1 repeated s times and 0 repeated t times along the diagonal. Moreover, if M is positive definite then $s=t=0$ and $(S^\mathsf{T} MS)_{ij} = \delta_{ij}$. Likewise, if M is negative definite then $r=t=0$ and $(S^\mathsf{T} MS)_{ij} = -\delta_{ij}$.

Proof: Since M is hermitian, it follows that the eigenvalues of M are real. Moreover, M is unitarily diagonalizable. That is, a unitary matrix U exists such that

$$U^\dagger MU = \mathrm{diag}(\lambda_1,\ldots,\lambda_r,\lambda_{r+1},\ldots,\lambda_{r+s},\underbrace{0,0,\ldots,0}_{t})\,, \tag{66}$$

where the eigenvalues $\lambda_1,\ldots,\lambda_r$ are all positive real numbers, the eigenvalues $\lambda_{r+1},\ldots,\lambda_{r+s}$ are all negative real numbers, and there are zero eigenvalues with multiplicity t. We can therefore define a diagonal matrix,

$$D \equiv \mathrm{diag}(\sqrt{\lambda_1},\ldots,\sqrt{\lambda_r},\sqrt{-\lambda_{r+1}},\ldots,\sqrt{-\lambda_{r+s}},\underbrace{1,1,\ldots,1}_{t})\,. \tag{67}$$

Setting $S = UD^{-1}$, it follows that,

$$S^\dagger MS = \mathrm{diag}(\underbrace{1,1,\ldots,1}_{r},\, \underbrace{-1,-1,\ldots,-1}_{s},\, \underbrace{0,0,\ldots,0}_{t})\,, \tag{68}$$

which completes the proof. Recall that a positive (negative) definite matrix is an hermitian matrix whose eigenvalues are positive (negative). Thus, if M is positive definite, it follows that $s=t=0$ and $(S^\mathsf{T} MS)_{ij} = \delta_{ij}$. Likewise, if M is negative definite then $r=t=0$ and $(S^\mathsf{T} MS)_{ij} = -\delta_{ij}$.

Corollary 1: Consider an $n\times n$ real symmetric matrix $G = G^\mathsf{T}$. Then there exists an invertible real matrix R such that

$$R^\mathsf{T} GR = \mathrm{diag}(\underbrace{1,1,\ldots,1}_{r},\, \underbrace{-1,-1,\ldots,-1}_{s},\, \underbrace{0,0,\ldots,0}_{t})\,, \tag{69}$$

where r, s and t are non-negative integers such that $r+s+t=n$. Moreover, if G is positive definite then $s=t=0$ and $(R^\mathsf{T} GR)_{ij} = \delta_{ij}$. Likewise, if G is negative definite then $r=t=0$ and $(R^\mathsf{T} GR)_{ij} = -\delta_{ij}$.

[12] See, e.g., Howard Eves, *Elementary Matrix Theory* (Dover Publications, Inc., Mineola, NY, 1980) pp. 237–245.

Proof: Since G is a real symmetric matrix, its eigenvalues are all real and it is diagonalizable by a real orthogonal matrix. That is, a real orthogonal matrix Q exists such that

$$Q^{\mathsf{T}}GQ = \mathrm{diag}(\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_{r+s}, \underbrace{0, 0, \ldots, 0}_{t}) , \tag{70}$$

where the eigenvalues, $\lambda_1, \ldots, \lambda_r$ $[\lambda_{r+1}, \ldots, \lambda_{r+s}]$, are all positive [negative] real numbers and the zero eigenvalues of G have multiplicity t. We again define the diagonal matrix D by eq. (67). Setting $R = QD^{-1}$ (which is a real matrix) then establishes eq. (69). Note that if G is is invertible then $\det(R^{\mathsf{T}}GR) \neq 0$, which implies that $t = 0$. If in addition G is positive [negative] definite, then $s = 0$ $[r = 0]$.

Corollary 2: Consider an $n \times n$ real symmetric matrix $G = G^{\mathsf{T}}$. Then there exists an invertible complex matrix S such that

$$S^{\mathsf{T}}GS = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{m}, \underbrace{0, 0, \ldots, 0}_{t}) , \tag{71}$$

where m and t are non-negative integers such that $m + t = n$. Moreover, if G is nonsingular (i.e., invertible) then $(S^{\mathsf{T}}GS)_{ij} = \delta_{ij}$.

Proof: Using the result of Corollary 1, it suffices to multiply eq. (69) on the left and on the right by the matrix $\mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{i, i, \ldots, i}_{s}, \underbrace{1, 1, \ldots, 1}_{t})$. That is,

$$\begin{aligned} &\mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{i, i, \ldots, i}_{s}, \underbrace{1, 1, \ldots, 1}_{t}) R^T GR \, \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{i, i, \ldots, i}_{s}, \underbrace{1, 1, \ldots, 1}_{t}) \\ &\qquad = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r+s}, \underbrace{0, 0, \ldots, 0}_{t}) . \end{aligned} \tag{72}$$

Thus, we have succeeded in deriving eq. (71), where $m = r + s$ and

$$S = R \, \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{r}, \underbrace{i, i, \ldots, i}_{s}, \underbrace{1, 1, \ldots, 1}_{t}) . \tag{73}$$

Note that if G is is invertible then $\det(S^{\mathsf{T}}GS) \neq 0$, which implies that $t = 0$ and $S^{\mathsf{T}}GS = \mathbf{I}$.

Remarkably, Corollary 2 is also valid for an $n \times n$ complex symmetric matrix. However this result requires a separate proof (which is provided in Appendix B), since not all complex symmetric matrices are diagonalizable via a similarity transformation.

APPENDIX B: An extension of Sylvester's Theorem

Theorem: Consider an $n \times n$ complex symmetric matrix $M = M^{\mathsf{T}}$. Then there exists an invertible complex matrix S such that

$$S^{\mathsf{T}}MS = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{m}, \underbrace{0, 0, \ldots, 0}_{t}) , \tag{74}$$

where m and t are non-negative integers such that $m + t = n$. Moreover, if M is nonsingular (i.e., invertible) then $(S^{\mathsf{T}}MS)_{ij} = \delta_{ij}$.

Proof: We shall employ the Autonne-Takagi factorization of a complex symmetric matrix, which states that a unitary matrix U exists such that

$$U^{\mathsf{T}} M U = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m, \underbrace{0, 0, \ldots, 0}_{t})\,, \tag{75}$$

where the *singular values* $\sigma_1, \ldots, \sigma_m$ are all positive real numbers, and there are zero singular values with multiplicity t. Note that

$$(U^{\mathsf{T}} M U)^\dagger (U^{\mathsf{T}} M U) = U^\dagger (M^\dagger M) U = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2, \underbrace{0, 0, \ldots, 0}_{t})\,. \tag{76}$$

That is, the singular values of M correspond to the positive square roots of the eigenvalues of $M^\dagger M$. A proof of the Autonne-Takagi factorization of a complex symmetric matrix is given in Appendix C.

We can therefore define a diagonal matrix,

$$D \equiv \mathrm{diag}(\sqrt{\sigma_1}, \sqrt{\sigma_2}, \ldots, \sqrt{\sigma_m}, \underbrace{1, 1, \ldots, 1}_{t})\,. \tag{77}$$

Setting $S = U D^{-1}$, it follows that,

$$S^{\mathsf{T}} M S = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{m}, \underbrace{0, 0, \ldots, 0}_{t})\,, \tag{78}$$

which completes the proof. Moreover, if M is invertible then $\det M \neq 0$, which implies that $\det(S^{\mathsf{T}} M S) \neq 0$. It then follows that $t = 0$, which implies that $S^{\mathsf{T}} M S = \mathbf{I}$.

APPENDIX C: The Autonne-Takagi factorization of a complex symmetric matrix

Theorem: Given a complex symmetric matrix M, the Autonne-Takagi factorization states that a unitary matrix U exists such that[13]

$$U^{\mathsf{T}} M U = D\,, \tag{79}$$

where D is a diagonal matrix with nonnegative entries. The diagonal elements of D correspond to the singular values of M, which are defined to be the positive square roots of the eigenvalues of $M^\dagger M$.

Proof of the Autonne-Takagi factorization of a complex symmetric matrix M: Since $M^\dagger M$ is positive semidefinite, there exists a unitary matrix V such that

$$V^\dagger (M^\dagger M) V = D\,, \tag{80}$$

[13] A comprehensive treatment of the Autonne-Takagi factorization of a complex symmetric matrix can be found in Ref. 11. The derivation provided in this appendix is based on proof provided in Ref. 12.

where D is a diagonal matrix with real nonnegative entries. Consider the matrix

$$B \equiv V^{\mathsf{T}} M V\,. \tag{81}$$

Since M is a complex symmetric matrix, it follows that B is a complex symmetric matrix as well ($B^{\mathsf{T}} = B$). Moreover, $B^\dagger B = (V^{\mathsf{T}} M V)^\dagger (V^{\mathsf{T}} M V) = V^\dagger (M^\dagger M) V = D$ is a real diagonal matrix with nonnegative entries. Next, we define the hermitian matrices B_R and B_I,

$$B_R \equiv \tfrac{1}{2}(B + B^\dagger)\,, \qquad\qquad B_I \equiv -\tfrac{1}{2}i(B - B^\dagger)\,. \tag{82}$$

Note that $B = B_R + iB_I$. Moreover, since $B^{\mathsf{T}} = B$, it follows that $B^\dagger = B^*$. Hence, B_R and B_I are also real symmetric matrices. Moreover, B_R and B_I commute, since[14]

$$D = B^\dagger B = (B_R + iB_I)^\dagger (B_R + iB_I) = B_R^2 + B_I^2 + i(B_R B_I - B_I B_R)\,. \tag{83}$$

But D, B_R and B_I are real matrices, which implies that $\operatorname{Im}(B^\dagger B) = B_R B_I - B_I B_R = 0$.

Since B_R and B_I are commuting real symmetric matrices (and hence diagonalizable), we can simultaneously diagonalize B_R and B_I. In particular, there exists a real orthogonal matrix W such that $W^{\mathsf{T}} B_R W$ and $W^{\mathsf{T}} B_I W$ are both diagonal. Hence,

$$\mathcal{D} \equiv W^{\mathsf{T}} (B_R + iB_I) W = W^{\mathsf{T}} B W \tag{84}$$

is a diagonal matrix. We can now define the unitary matrix $X = VW$ to obtain:

$$X^{\mathsf{T}} M X = W^{\mathsf{T}} (V^{\mathsf{T}} M V) W = W^{\mathsf{T}} B W = \mathcal{D}\,, \tag{85}$$

after making use of eqs. (81) and (84). The most general form for $\mathcal{D}$ is given by:

$$\mathcal{D} = \operatorname{diag}(\sigma_1 e^{i\theta_1}, \sigma_2 e^{i\theta_2}\ \ldots, \sigma_m e^{i\theta_m}\,, \underbrace{0, 0, \ldots, 0}_{t})\,, \tag{86}$$

where the σ_i are real positive numbers and $0 \leq \theta_i < 2\pi$. Thus, we can define the unitary matrix,

$$U = X \operatorname{diag}(e^{-i\theta_1/2}, e^{-i\theta_2/2}, \ldots, e^{-i\theta_m/2}\,, \underbrace{1, 1, \ldots, 1}_{t})\,. \tag{87}$$

Then, eqs. (85) and (86) yield the Autonne-Takagi factorization of the complex symmetric matrix M,

$$U^{\mathsf{T}} M U = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m, \underbrace{0, 0, \ldots, 0}_{t})\,, \tag{88}$$

where the σ_i are identified as the nonzero singular values of M in light of eq. (76).

---

[14] It is tempting to avoid the introduction of the complex symmetric matrix B. Instead, suppose one defines $M_R \equiv \frac{1}{2}(M + M^\dagger)$ and $M_I = -\frac{1}{2}i(M - M^\dagger)$. Since M is symmetric, M_R and M_I are real symmetric matrices. However, in general $M^\dagger M$ is *not* a real matrix, in which case M_R and M_I do not commute [cf. eq. (83)]. It follows that M_R and M_I are not simultaneously diagonalizable. In contrast, the complex symmetric matrix B is diagonalizable as shown in eq. (84), which can then be used to establish the Autonne-Takagi factorization of M.

References

I have found the following references useful in preparing these notes.

1. Robert Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (Dover Publications, Inc., Mineola, NY, 2005).

2. J.F. Cornwell, *Group Theory in Physics, Volume 2* (Academic Press, London, UK, 1984).

3. Alexey P. Isaev and Valery A. Rubakov, *Theory of Groups and Symmetries* (World Scientific Publishing Company, Singapore, 2018).

4. D.H. Sattinger and O.L. Weaver, *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics* (Springer-Verlag, New York, NY, 1986). In particular, see Chapter 9.

5. Barry Simon, *Representations of Finite and Compact Groups* (American Mathematical Society, Providence, RI, 1996). In particular, see Chapter VIII.2.

6. Karin Erdmann and Mark J. Wildon, *Introduction to Lie Algebras* (Springer-Verlag, London, UK, 2006).

7. Joachim Hilgert and Karl-Hermann Neeb, *Structure and Geometry of Lie Groups* (Springer Science, New York, NY, 2012).

8. Arkady L. Onishchik, *Lectures on Real Semisimple Lie Algebras and their Representations* (European Mathematical Society, Zurich, Switzerland, 2004).

9. Roger Richardson, *Compact real forms of a complex semi-simple Lie algebra*, J. Differential Geometry **2** (1968) 411–419.

10. Simon K. Donaldson, *Lie Algebra Theory without Algebra*, in *Algebra, Arithmetic, and Geometry: In Honor of Yu. I. Manin*, Volume I, edited by Yuri Tschinkel and Yuri Zarhin (Springer Science, New York, 2009) pp. 549–566 [arXiv:math/0702016].

11. Roger A. Horn and Charles R. Johnson, *Matrix Analysis*, 2nd edition (Cambridge University Press, Cambridge, UK, 2013).

12. Nathaniel Johnston, *Advanced Linear and Matrix Algebra* (Springer Nature Switzerland AG, Cham, Switzerland, 2021).