Table 7.1. Dynkin diagrams and root structure of the classical Lie algebras

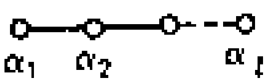
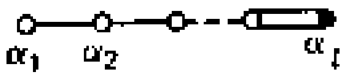
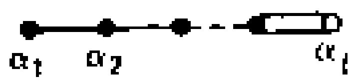
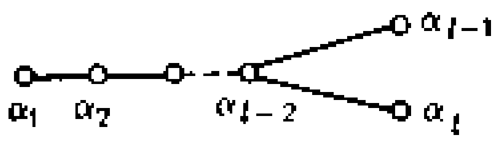
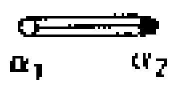
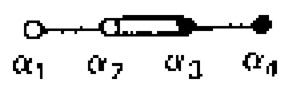
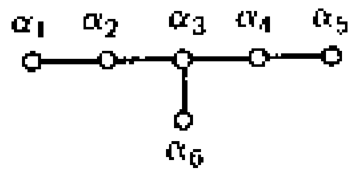
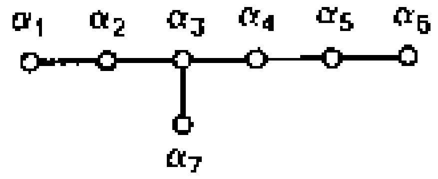
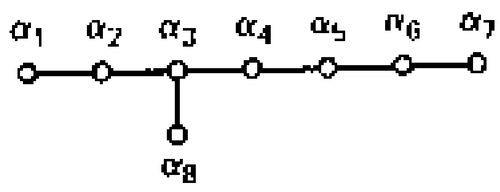
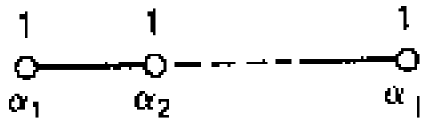
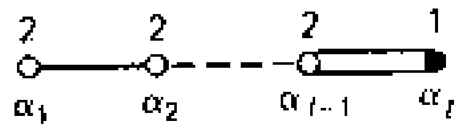
Order	Cartan label	Group label	Dynkin diagram	Roots
$l(l+2)$	A_l	$SU(l+1)$		$\mathbf{e}_i-\mathbf{e}_j\ (i,j=1,\ldots,l+1)$
$l(2l+1)$ $l\geqslant 2$	B_l	$SO(2l+1)$		$\pm\mathbf{e}_i$ and $\pm\mathbf{e}_i\pm\mathbf{e}_j\ \ (i,j=1,\ldots,l)$
$l(2l+1)$ $l\geqslant 3$	C_l	$Sp(2l)$		$\pm 2\mathbf{e}_i$ and $\pm\mathbf{e}_i\pm\mathbf{e}_j(i,j=1,\ldots,l)$
$l(2l-1)$ $l\geqslant 4$	D_l	$SO(2l)$		$\pm\mathbf{e}_i\pm\mathbf{e}_j(i,j=1,\ldots,l)$
14	G_2	G_2		$\mathbf{e}_i-\mathbf{e}_j(i,j=1,2,3;\ i\neq j)$ $\pm 2\mathbf{e}_i\mp\mathbf{e}_j\mp\mathbf{e}_k\,(i,j,k=1,2,3+b,i\neq j\neq k)$
52	F_4	F_4		As for B_4 plus the 16 roots $\frac{1}{2}(\pm\mathbf{e}_1\pm\mathbf{e}_2\pm\mathbf{e}_3\pm\mathbf{e}_4)$
78	E_6	E_6		As for A_5 plus the roots $\pm\sqrt{2}\,\mathbf{e}_7$ and $\frac{1}{2}(\pm\mathbf{e}_1\pm\mathbf{e}_2\pm\mathbf{e}_3\pm\mathbf{e}_4\pm\mathbf{e}_5\pm\mathbf{e}_6)\pm\mathbf{e}_7/\sqrt{2}$ (three signs + and three − in first fraction)
133	E_7	E_7		As for A_7, plus the roots $\frac{1}{2}(\pm\mathbf{e}_1\pm\mathbf{e}_2\pm\mathbf{e}_3\pm\mathbf{e}_4\pm\mathbf{e}_5\pm\mathbf{e}_6\pm\mathbf{e}_7\pm\mathbf{e}_8)$ with four signs + and four −.
248	E_8	E_8		As for D_8, plus the roots $\frac{1}{2}(\pm\mathbf{e}_1\pm\mathbf{e}_2\pm\mathbf{e}_3\pm\mathbf{e}_4\pm\mathbf{e}_5\pm\mathbf{e}_6\pm\mathbf{e}_7\pm\mathbf{e}_8)$ with the number of + signs even

Table 7.2. Scalar Products (α_i, α_i)

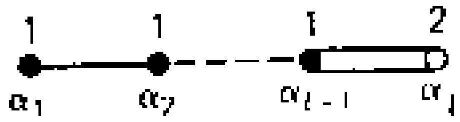
A_l



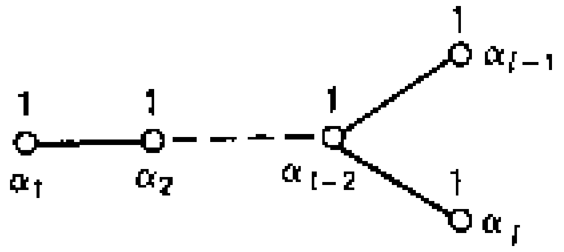
B_l



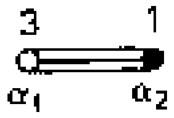
C_l



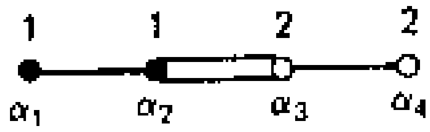
D_l



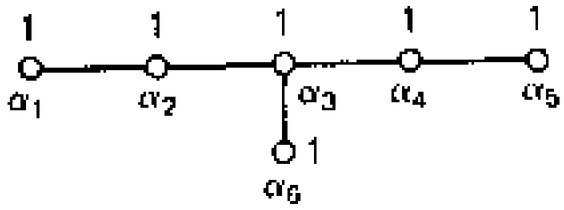
G_2



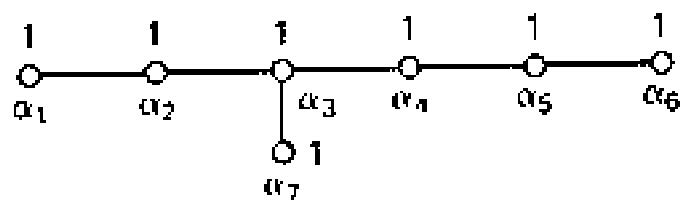
F_4



E_6



E_7



E_8

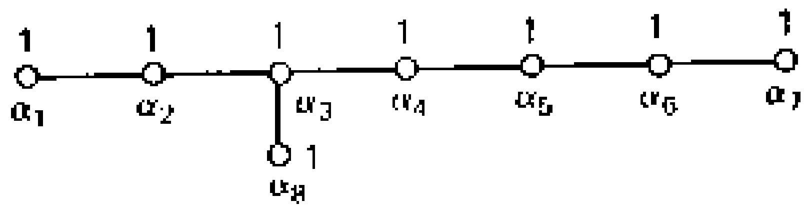


Table 7.3. Cartan Matrices for the Classical Lie Algebras

$$A_l: \begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -1 & 2 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 2 & -1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -1 & 2 \end{bmatrix} \qquad F_4: \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$B_l: \begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -1 & 2 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 2 & -1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -2 & 2 \end{bmatrix}$$

$$C_l: \begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -1 & 2 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 2 & -2 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -1 & 2 \end{bmatrix}$$

Table 7.3 (Continued)

$$E_6: \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

$$D_l: \begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -1 & 0 & 2 \end{bmatrix}$$

$$E_7: \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$G_2: \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$E_8: \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$