Physics 210 Homework #3

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1 Cyclotron Motion

Part (a) The Hamiltonian I don't need the speed of light c if I have magnetic field B in teslas and electric charge e in coulombs. In Cartesian coordinates, the Lagrangian is

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{eB}{2}(x\dot{y} - \dot{x}y) \tag{1.1}$$

Planar polar coordinates (r, θ) are defined such that $x = r\cos\theta$, $y = r\sin\theta$, so the velocities are

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \\ \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} \end{cases}$$

Substituting into (1.1),

$$\mathcal{L}(r, \dot{r}, \dot{\theta}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{eB}{2}r^2\dot{\theta} \tag{1.2}$$

To move over to the Hamiltonian formalism, I first compute the conjugate momenta,

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$
$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} - \frac{eB}{2}r^2$$

Then I take the Legendre transform of the Lagrangian (1.2),

$$\mathcal{H} = p_r\dot{r} + p_\theta\dot{\theta} - \mathcal{L} = \frac{m}{2}\dot{r}^2 + \frac{mr^2}{2}\dot{\theta}^2$$

The last step is to eliminate the velocities in favor of the momenta, and write the Hamiltonian as an explicit function of the phase space variables,

$$\boxed{\mathcal{H}(r, p_r, p_\theta) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{eBp_\theta}{2m} + \frac{e^2B^2r^2}{8m}}$$

Part (b) Circular Orbits For circular orbits the easiest thing to use is Newton's Second Law. Because the Lorentz force $F = evB = e\omega r_0 B$ provides the centripetal acceleration $a = \omega^2 r_0$, I immediately know the orbital frequency

$$\boxed{\omega = \frac{eB}{m}}$$

As for the orbital radius, in terms of the kinetic energy $E = (1/2)mv^2 = (1/2)m\omega^2 r_0^2$, I have

$$r_0 = \sqrt{\frac{2E}{m\omega^2}} = \boxed{\frac{\sqrt{2mE}}{eB}}$$

Part (c) Perturbations on Circular Orbits When the orbit is not circular is when the advantage of the Hamiltonian approach shows. For θ is ignorable and therefore p_θ is conserved. It helps then to consider an effective potential,

$$V^{\text{eff}}(r) = \frac{p_\theta^2}{2mr^2} + \frac{eBp_\theta}{2m} + \frac{e^2B^2r^2}{8m}$$

in which the radial coordinate r moves. As long as $p_\theta > 0$, V^{eff} always has a global minimum r_0 that corresponds to a stable circular orbit.

$$V'(r_0) = -\frac{p_\theta^2}{mr^3} + \frac{e^2B^2r}{4m} = 0 \quad \rightarrow \quad r_0 = \sqrt{\frac{2p_\theta}{eB}}$$

If the perturbation ρ is small, I may Taylor-expand the effective potential about the minimum,

$$V^{\text{eff}}(r_0+\rho) = V(r_0) + V'(r_0)\,\rho + \frac{1}{2}V''(r_0)\,\rho^2 + \mathcal{O}(\rho^3)$$

The first term is constant. The second term is zero. The third term is a harmonic oscillator potential with the "force constant"

$$V''(r_0) = \frac{3p_\theta^2}{mr_0^4} + \frac{e^2B^2}{4m} = \frac{e^2B^2}{m}$$

So the frequency of small oscillations is

$$\boxed{\omega_\rho = \sqrt{\frac{V''}{m}} = \frac{eB}{m}}$$

This is identical to the orbital frequency. So a small oscillation in the radial direction coupled with uniform circular motion causes a small overall translation of the circular orbit.

2 Spinning Disk

Part (a) Inertia Tensor I am given the inertia tensor with respect to the center of mass. To shift the reference point to the edge of the disk, I invoke the tensor form of the parallel-axis theorem,

$$\mathbf{I}^{\text{disk}} = \frac{MR^2}{4}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix} + M\begin{bmatrix}R^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & R^2\end{bmatrix} = \frac{MR^2}{4}\begin{bmatrix}5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 6\end{bmatrix}$$

Then I separately compute the bead's inertia tensor about the same reference point,

$$\mathbf{I}^{\text{bead}} = m\begin{bmatrix}R^2 & -R^2 & 0\\ -R^2 & R^2 & 0\\ 0 & 0 & 2R^2\end{bmatrix} = \frac{5MR^2}{4}\begin{bmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Knowing the moment of inertia is linear in mass distribution, I add up the bead to the disk.

$$\mathbf{I} = \mathbf{I}^{\text{disk}} + \mathbf{I}^{\text{bead}} = \boxed{MR^2\begin{bmatrix}5/2 & -5/4 & 0\\ -5/4 & 3/2 & 0\\ 0 & 0 & 4\end{bmatrix}}$$

Part (b) Principal Axes and Principal Moments Lamentably, the bead breaks enough symmetries that the x, y axes are no longer principal axes. The z axis, however, is still a principal axis, with principal moment

$$\boxed{I_3 = I_{33} = 4MR^2}$$

To find the other two axes, I only need to diagonalize the top-left 2×2 block of the inertia tensor.

$$\begin{cases} \boxed{I_1 = \left(2 - \dfrac{\sqrt{29}}{4}\right) MR^2 \qquad \hat{x}' = \dfrac{(-2+\sqrt{29})\hat{x} + 5\hat{y}}{\sqrt{58 - 4\sqrt{29}}}} \\ \boxed{I_2 = \left(2 + \dfrac{\sqrt{29}}{4}\right) MR^2 \qquad \hat{y}' = \dfrac{(-2-\sqrt{29})\hat{x} + 5\hat{y}}{\sqrt{58 + 4\sqrt{29}}}} \end{cases}$$

Part (c) Force Needed for Uniform Rotation Given a constant angular velocity $\vec{\omega} = \omega\,\hat{y}$, I can decompose it in the body system,

$$\begin{cases} \omega_1' = \vec{\omega} \cdot \hat{x}' = \dfrac{5\omega}{\sqrt{58 + 4\sqrt{29}}} \\ \omega_2' = \vec{\omega} \cdot \hat{y}' = \dfrac{5\omega}{\sqrt{58 - 4\sqrt{29}}} \\ \omega_3' = \vec{\omega} \cdot \hat{z}' = 0 \end{cases}$$

The inertia tensor is constant and diagonal in the body frame, the diagonal elements being the principal moments I just computed. So the angular velocity in the body frame is

$$\boxed{\vec{L} = \mathbf{I}\,\vec{\omega} = I_1\omega_1'\,\hat{x}' + I_2\omega_2'\,\hat{y}'}$$

Although it appears constant, the body frame is rotating with angular velocity $\vec{\omega}$, so the rate of change of angular momentum in the inertial frame is

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{\omega} \times \vec{L} = (I_2 - I_1)\omega_1'\omega_2'\,\hat{z}'$$

However, $\hat{z}'$ is not constant either—it rotates about the y axis at angular velocity $\vec{\omega}$. Simple trigonometry tells me $\hat{z}' = \cos\omega t\,\hat{z} + \sin\omega t\,\hat{x}$. On the other hand, the torque exerted on the disk about point A is

$$\vec{N} = \vec{r}_B \times \vec{F} = 2RF_z\,\hat{x} - 2RF_x\,\hat{z}$$

The law of angular momentum says $\vec{N} = \mathrm{d}\vec{L}/\mathrm{d}t$, so I can equate

$$\begin{cases} F_x = -\dfrac{(I_2 - I_1)\omega_1'\omega_2'}{2R}\cos\omega t \\ F_z = \dfrac{(I_2 - I_1)\omega_1'\omega_2'}{2R}\sin\omega t \end{cases}$$

while F_y is undetermined. Plugging in the numbers now, I obtain

$$\boxed{\vec{F} = \frac{5}{8}M\omega^2 R(-\cos\omega t\,\hat{x} + \sin\omega t\,\hat{z})}$$

3 Bead on a Hoop

Part (a) Equations of Motion The stable equilibrium of the system is when the hoop hangs straight down, with the bead at the bottom of the hoop (Figure 1). Define angular coordinates θ_1, θ_2 that measure small deviations from equilibrium (Figure 2). The hoop's center of mass is at

$$\begin{cases} x_1 = a\sin\theta_1 \\ y_1 = -a\cos\theta_1 \end{cases}$$

and the bead is at

$$\begin{cases} x_2 = x_1 + a\sin\theta_2 \\ y_2 = y_1 - a\cos\theta_2 \end{cases} \tag{3.1}$$

Parallel-axis theorem tells me the hoop's rotational inertia about the pivot, $I = Ma^2 + Ma^2 = 2Ma^2$. Therefore the kinetic energy of the hoop is

$$T^{\text{hoop}} = \frac{I}{2}\dot{\theta}_1^2 = Ma^2\dot{\theta}_1^2$$

The kinetic energy of the bead is more complicated. I need to first differentiate (3.1),

$$\begin{cases} \dot{x}_2 = a\cos\theta_1\,\dot{\theta}_1 + a\cos\theta_2\,\dot{\theta}_2 \\ \dot{y}_2 = a\sin\theta_1\,\dot{\theta}_1 + a\sin\theta_2\,\dot{\theta}_2 \end{cases}$$

and then compute the quadratic form

$$T^{\text{bead}} = \frac{m}{2}(\dot{x_2}^2 + \dot{y_2}^2) = \frac{ma^2}{2}\left[\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1 - \theta_2)\,\dot{\theta}_1\dot{\theta}_2\right]$$

As for potential energy, since only gravity is acting,

$$\begin{aligned} V^{\text{hoop}} &= Mgy_1 = -Mga\cos\theta_1 \\ V^{\text{bead}} &= mgy_2 = -mga(\cos\theta_1 + \cos\theta_2) \end{aligned}$$

Finally, the Lagrangian of the hoop-bead system is

$$\begin{aligned} \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) &= T^{\text{hoop}} + T^{\text{bead}} - V^{\text{hoop}} - V^{\text{bead}} \\ &= \left(M + \frac{m}{2}\right)a^2\dot{\theta}_1^2 + \frac{ma^2}{2}\dot{\theta_2}^2 + ma^2\cos(\theta_1 - \theta_2)\,\dot{\theta}_1\dot{\theta}_2 \\ &+ (M+m)g\cos\theta_1 + mga\cos\theta_2 \end{aligned}$$

The Euler-Lagrange equations are

$$\begin{cases} \boxed{\ddot{\theta}_1 = -\frac{m}{2M+m}\left[\sin(\theta_1 - \theta_2)\,\dot{\theta}_2^2 + \cos(\theta_1 - \theta_2)\,\ddot{\theta}_2\right] - \frac{M+m}{2M+m}\frac{g}{a}\sin\theta_1} \\ \boxed{\ddot{\theta}_2 = \sin(\theta_1 - \theta_2)\,\dot{\theta}_1^2 + \cos(\theta_1 - \theta_2)\,\ddot{\theta}_1 - \frac{g}{a}\sin\theta_2} \end{cases}$$

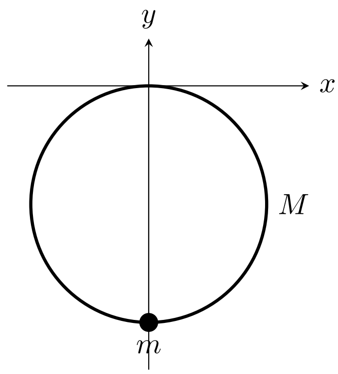

Figure 1: Stable Equilibrium

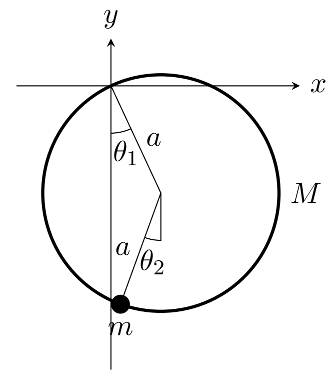

Figure 2: Generalized Coordinates

Part (b) Small Oscillations For $\theta_1, \theta_2 \ll 1$, the equations of motion are linear.

$$\begin{cases} \ddot{\theta}_1 = -\dfrac{m}{2M+m}\,\ddot{\theta}_2 - \dfrac{M+m}{2M+m}\dfrac{g}{a}\theta_1 + \mathcal{O}(\theta^2) \\ \ddot{\theta}_2 = \ddot{\theta}_1 - \dfrac{g}{a}\theta_2 + \mathcal{O}(\theta^2) \end{cases}$$

To discover normal modes, I take a Fourier transform, and put the linear, algebraic equations in matrix form,

$$\begin{bmatrix} \omega^2 - \frac{M+m}{2M+m}\frac{g}{a} & \frac{m}{2M+m}\,\omega^2 \\ \omega^2 & \omega^2 - \frac{g}{a} \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = 0 \tag{3.2}$$

For non-trivial solutions $\hat{\theta}_1$, $\hat{\theta}_2$ to exist, the matrix of coefficients $\mathbf{A}$ must be singular:

$$\det \mathbf{A} = \frac{2M}{2M+m}\,\omega^4 - \frac{3M+2m}{2M+m}\frac{g}{a}\,\omega^2 + \frac{M+m}{2M+m}\left(\frac{g}{a}\right)^2 = 0$$

This equation is quadratic in ω^2, so the solutions are given by the quadratic formula, and I discover the normal-mode frequencies

$$\boxed{\omega = \sqrt{\left(1+\frac{m}{M}\right)\frac{g}{a}} \quad \text{or} \quad \sqrt{\frac{g}{2a}}}$$

For the higher frequency $\omega = \sqrt{(1+(m/M))(g/a)}$, equation (3.2) becomes

$$\begin{bmatrix} \frac{(M+m)^2}{M(2M+m)}\frac{g}{a} & \frac{(M+m)m}{M(2M+m)}\frac{g}{a} \\ \frac{M+m}{M}\frac{g}{a} & \frac{m}{M}\frac{g}{a} \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = 0$$

By inspection $\hat{\theta}_1 = m$, $\hat{\theta}_2 = -(M+m)$ is a solution. In this mode the hoop and the bead oscillate out of phase, and the bead always has larger amplitude. For the lower frequency $\omega = \sqrt{(g/2a)}$, the equation to solve is

$$\begin{bmatrix} -\frac{m}{2(2M+m)}\frac{g}{a} & \frac{m}{2(2M+m)}\frac{g}{a} \\ \frac{g}{2a} & -\frac{g}{2a} \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = 0$$

Obviously $\hat{\theta}_1 = 1$, $\hat{\theta}_2 = 1$ is a solution. In this mode the hoop and the bead oscillate in phase with equal amplitude.