Homework Set #4.

Due Date: Final Exam

Solve the following four exercises:

1. Show that the function

$$S = \frac{m\omega}{2}\left(q^2 + \alpha^2\right)\cot(\omega t) - m\omega q\alpha\csc(\omega t)$$

is a solution of the Hamilton-Jacobi equation for Hamilton's principal function for the linear harmonic oscillator with

$$H = \frac{p^2 + m^2\omega^2 q^2}{2m}.$$

Show that this function generates a correct solution to the motion of the harmonic oscillator.

2. A particle of mass m moves in one dimension q in a potential energy field $V(q)$ and is retarded by a damping force $-2m\gamma\dot{q}$ proportional to its velocity.

(a) Show that the equation of motion can be obtained from the Lagrangian

$$L = \exp(2\gamma t)\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)$$

and that the Hamiltonian is

$$H = \frac{p^2\exp(-2\gamma t)}{2m} + V(q)\exp(2\gamma t).$$

(b) For the generating function

$$F_2(q, P, t) = \exp(\gamma t) q P$$

find the transformed Hamiltonian $K(Q, P, t)$. For an oscillator potential

$$V(q) = \frac{1}{2}m\omega^2 q^2$$

show that the transformed Hamiltonian yields the constant of motion

$$K = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2 + \gamma QP.$$

(c) Obtain the solution $q(t)$ for the damped oscillator from the constant of motion in (b) in the under-damped case $\gamma < \omega$.

You may need the integral

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \sin^{-1} x.$$

3. A particle of mass m moves in one dimension subject to the potential

$$V = \frac{a}{\sin^2\left(\frac{x}{x_0}\right)}.$$

Obtain an integral expression for Hamilton's characteristic function. Under what conditions can action-angle variables be used? Assuming these are met, find the frequency of oscillation by the action-angle method. (Note: The integral for J can be evaluated by manipulating the integrand so that the square root appears in the denominator.) Check your result in the limit of oscillations of small amplitude.

4. For the point transformation in a system of two degrees of freedom,

$$Q_1 = q_1^2, \qquad Q_2 = q_1 + q_2,$$

find the most general transformation equations for P_1 and P_2 consistent with the overall transformation being canonical. Show that with a particular choice for P_1 and P_2 the Hamiltonian

$$H = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2$$

can be transformed to one in which both Q_1 and Q_2 are ignorable. By this means, solve the problem and obtain expressions for $q_1,\ q_2,\ p_1$ and p_2 as functions of time and their initial values.