Final Exam

Solve two of the following three problems
(extra points if you attempt a third problem)

1. The transformation equations between two sets of coordinates are:

$$Q = \ln\left(1 + q^{1/2}\cos p\right);$$

$$P = 2\left(1 + q^{1/2}\cos p\right) q^{1/2}\sin p.$$

(a) Show directly from these transformation equations that Q and P are canonical variables, if q and p are, using Poisson brackets.

(b) As in (a), but using the symplectic condition $J = MJM^T$, where M is the Jacobian of the transformation, M^T its transpose, and J the usual nonsingular, skew-symmetric matrix $((0,1),(-1,0))$.

(c) Show that the function

$$F = F(Q,p) = [\exp(Q) - 1]^2 \tan p$$

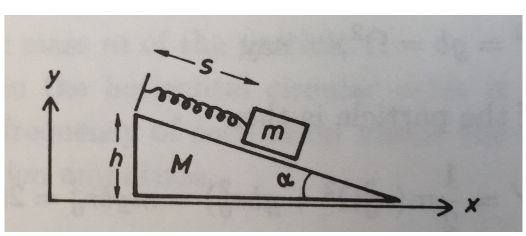
is a generating function (of the third type), i.e. that

$$q = \frac{\partial F}{\partial p}$$

$$P = \frac{\partial F}{\partial Q}.$$

[Hint: first solve the transformation equations for $q = q(Q,p)$ and $P = P(Q,p)$ and then verify the transformation conditions above]

2. A block of mass m is attached to a wedge of mass M by a spring with spring constant k. The inclined frictionless surface of the wedge makes an angle α to the horizontal. The wedge is free to slide on a horizontal frictionless surface, as in the figure

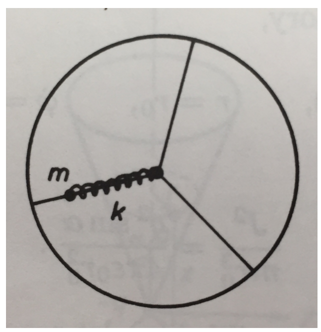


(a) Given the relaxed length of the spring along is d, find the value s_0 when both the block and the wedge are at rest.

(b) Find the Lagrangian for the system as a function of the x coordinate of the wedge and the length of the spring s. Write the equations of motion.

(c) Find the natural frequency of vibration

3. A disc of mass M and radius R rotates about its center on a horizontal plane. A mass m can slide freely along one of the radii of the disc, and is attached to the center by a spring of natural length l and force constant k, as shown in the figure.



(a) Find the moment of inertia of the disc about its center.

(b) Find an expression for the energy of the system in terms of r (the distance of the mass from the center), $\dot{r}$, and the total angular momentum J.

(c) Derive Lagrange's equations.

(d) Suppose the disc initially has a constant angular velocity Ω_0 and the spring has a steady extension $r = r_0$. Find r_0 as a function of Ω_0.

(e) Find the frequency of small oscillations around the equilibrium configuration.