Solution to Homework Set #5, Problem #1.

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We are asked to find the differential cross section at tree-level for the elastic scattering of two massless scalar fields ϕ_1 and ϕ_2 obeying the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi_1\Box\phi_1 - \frac{1}{2}\phi_2\Box\phi_2 + \frac{\lambda}{2}\phi_1\left(\partial_\mu\phi_2\right)\left(\partial_\mu\phi_2\right) + \frac{g}{2}\phi_1^2\phi_2.$$

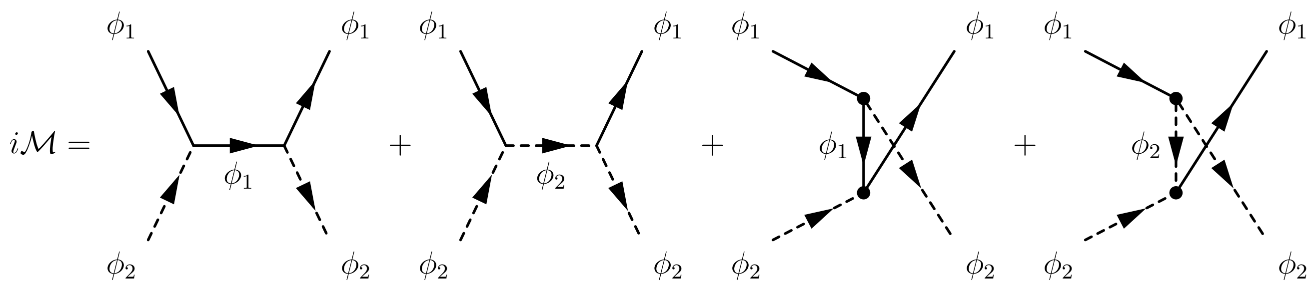
We can easily find the solution by utilizing the famous Feynman Problem-Solving Algorithm:

1. Write down the problem.
2. Think very hard.
3. Write down the answer.

Or we can do it the old-fashioned way:

This Lagrangian tells us that there are two possible interactions between ϕ_1 and ϕ_2, as dictated in the last two terms. The last term is just the familiar three-point interaction between ϕ_1 and ϕ_2, with ϕ_1 being the propagator. The second-to-last term also represents a three-point interaction, but includes derivative couplings. Since we are working in momentum space, this derivative coupling will lead to factors of $-ip^\mu$ for ϕ_2 particles being destroyed and ip^μ for ϕ_2 particles being created at any given vertex in our amplitude calculation.

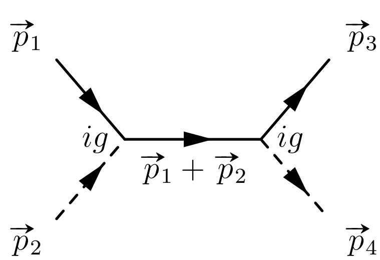
As we have seen in the text, there are three tree-level channels (denoted s, t, and u) for the $2 \to 2$ scattering of indistinguishable fields. In our case of two clearly distinguishable fields, though, the t and u channels are degenerate processes, and so we only need to consider one of them when calculating our total amplitude. We will arbitrarily choose the u-channel diagrams. Since the only difference is how we label the fields and their momenta, the same result would be obtained had we chosen to draw the t-channel diagrams instead. Thus, there will be a total of 4 tree-level diagrams to calculate, one for each scattering channel and one for each type of interaction. Pictorially, the total tree-level amplitude for $\phi_1\phi_2 \to \phi_1\phi_2$ scattering is given by



We will use the Feynman rules to calculate each term separately. But before we do that, it will be useful to recall the definitions of the Mandelstam variables, which, for massless particles, reduce to:

$$
\begin{aligned}
s &= 2p_1 \cdot p_2 \quad = 2p_3 \cdot p_3, \\
t &= -2p_1 \cdot p_3 = -2p_2 \cdot p_4, \\
u &= -2p_1 \cdot p_4 = -2p_2 \cdot p_3.
\end{aligned}
$$

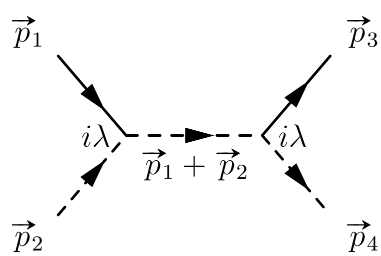
In the following calculations, we will omit the $i\epsilon$ in the propagators since they will have no effect at tree-level. Starting with the s-channel and conserving momentum, we get a factor of i times the coupling g for each vertex in addition to the propagator factor $\frac{i}{p_1^\mu + p_2^\mu}$:



$$= (ig)^2 \frac{i}{(p_1^\mu + p_2^\mu)^2}$$

$$= \frac{-ig^2}{s}.$$

Now we will consider the term arising from the derivative coupling in the Lagrangian. Moving from left-to-right in the diagram, we destroy a particle corresponding to ϕ_2 carrying momentum p_2^μ at the first vertex, and thus pickup a factor of $-ip_2^\mu$. At the same vertex, we create a ϕ_2 propagator, giving a factor of $i\left(p_1^\mu + p_2^\mu\right)$ due to the derivative coupling and momentum conservation. At the second vertex, we will similarly get factors of ip_4^ν and $-i\left(p_1^\nu + p_2^\nu\right)$ from the creation of ϕ_2 with momentum p_4^ν and the destruction of the propagator. In particular, the amplitude is given by

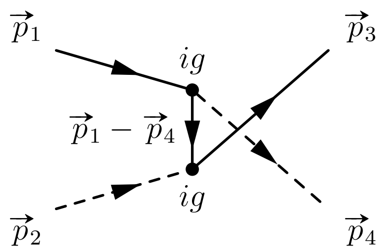


$$= (i\lambda)^2 \left(-ip_2^\mu\right)\left[i\left(p_1^\mu + p_2^\mu\right)\right]\frac{i}{\left(p_1^\mu + p_2^\mu\right)^2}\left(ip_4^\nu\right)\left[-i\left(p_3^\nu + p_4^\nu\right)\right]$$

$$\begin{aligned}
&= \frac{-i\lambda^2\left(p_2 \cdot p_1 + p_2^2\right)\left(p_3 \cdot p_4 + p_4^2\right)}{\left(p_1 + p_2\right)^2} \\
&= \frac{-i\lambda^2\left(p_2 \cdot p_1\right)\left(p_3 \cdot p_4\right)}{\left(p_1 + p_2\right)^2} \\
&= \frac{-i\lambda^2 s}{4},
\end{aligned}$$

where we have used $p^2 = 0$ for massless fields.

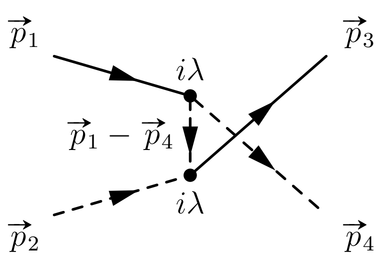
Calculating the amplitudes for the u-channel processes follows the exact same arguments. Conserving momentum, the regular three-point interaction term gives an amplitude of



$$= (ig)^2 \frac{i}{\left(p_1^\mu - p_4^\mu\right)^2}$$

$$= \frac{-ig^2}{u}.$$

For the diagram arising from the derivative coupling, we calculate



$$= (i\lambda)^2 \left(-ip_2^\mu\right)\left[i\left(p_1^\mu - p_4^\mu\right)\right] \frac{i}{\left(p_1^\mu - p_4^\mu\right)^2} \left(ip_4^\nu\right)\left[-i\left(p_1^\nu - p_4^\nu\right)\right]$$

$$\begin{aligned}
&= (i\lambda)^2 \left(-ip_2^\mu\right)\left[i\left(p_3^\mu - p_2^\mu\right)\right] \frac{i}{\left(p_1^\mu - p_4^\mu\right)^2} \left(ip_4^\nu\right)\left[-i\left(p_1^\nu - p_4^\nu\right)\right] \\
&= \frac{-i\lambda^2\left(p_2 \cdot p_3 - p_2^2\right)\left(p_1 \cdot p_4 - p_4^2\right)}{\left(p_1 - p_4\right)^2} \\
&= \frac{-i\lambda^2\left(p_2 \cdot p_3\right)\left(p_1 \cdot p_4\right)}{\left(p_1 - p_4\right)^2} \\
&= \frac{-i\lambda^2 u}{4}.
\end{aligned}$$

Thus, noting that $u + t + s = 0$ for massless fields, the total tree-level amplitude for $\phi_1\phi_2 \to \phi_1\phi_2$ scattering is given by

$$\begin{aligned}
\mathcal{M} &= -\left(\frac{g^2}{s} + \frac{g^2}{u} + \frac{\lambda^2 s}{4} + \frac{\lambda^2 u}{4}\right) \\
&= -\left[g^2\left(\frac{1}{s} + \frac{1}{u}\right) - \frac{\lambda^2 t}{4}\right] \\
&= \frac{g^2 t}{su} + \frac{\lambda^2 t}{4}.
\end{aligned}$$

We have seen that the differential cross section for $2 \to 2$ scattering in the center-of-momentum frame reduces nicely for equal particle masses:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{1}{64\pi^2 s}\left|\mathcal{M}\right|^2.$$

Plugging in our calculated amplitude, we finally obtain the differential cross section at tree-level:

$$\frac{d\sigma}{d\Omega}\left(\phi_1\phi_2 \to \phi_1\phi_2\right) = \frac{1}{64\pi^2 s}\left(\frac{g^2 t}{su} + \frac{\lambda^2 t}{4}\right)^2.$$

Note that while we calculated a differential cross section in the center-of-momentum frame, our answer only depends on the manifestly Lorentz invariant quantities s, t, and u. Therefore, the differential cross section itself is Lorentz invariant, as it should be.

Further Reading:

Gotta have my orange juice.