Solution to Homework Set #6

Some Particle Dark Matter Model Building

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In this problem, we consider the process of scalar electron (ϕ^-) scalar positron (ϕ^+) pair annihilation to two photons in scalar QED. The scalar QED Lagrangian is

$$\mathcal{L} = -F_{\mu\nu}F^{\mu\nu} - \phi(\partial_\mu \partial^\mu + m_e^2)\phi^* - ieA_\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + e^2A_\mu^2|\phi|^2 \quad (1)$$

Where m_e is the mass of the electron and of the positron. We are asked to calculate the tree-level matrix elements (Part (a)) and the cross section in the center of mass frame (Parts (b) and (c)) for the $e^-e^- \to \gamma\gamma$ in the case of a massless photon and for a massive photon. In part (d) we are asked to determine suitable combinations of the mass of said photon and a "dark matter" fine structure constant α_{dark} given specific electron/positron masses and cross sections.

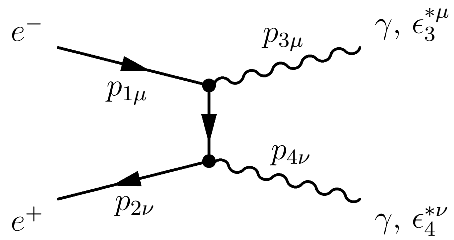
Part (a): Matrix Elements

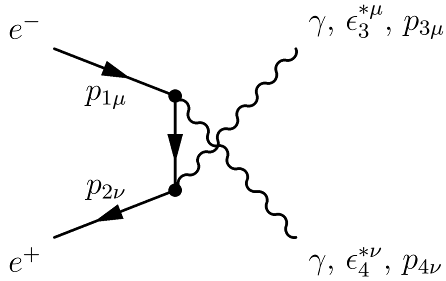
There are three tree-level diagrams for $e^-e^- \to \gamma\gamma$ in scalar QED for this process, where the fields are created and annihilated according to the scalar field equations for a particle and anti particle

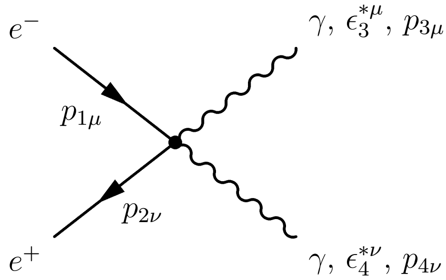
$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p}(a_p e^{-ipx} + b_p^\dagger e^{ipx}) \quad (2)$$

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p}(a_p^\dagger e^{ipx} + b_p e^{-ipx}) \quad (3)$$

where $b_p^\dagger$ and $a_p^\dagger$ create positrons and electrons respectively and their Hermitian conjugates annihilate them.The three tree-level diagrams are the u-channel, the t-channel, and a 4-point diagram (arising from the $e^2A_\mu^2|\phi|^2$ term in the Lagrangian density). These are shown, respectively, below:







where the momentum of the positron in each diagram points opposite the direction of the arrow. And each vertex gets a factor of $(ie)^2$, except the 4-point diagram which gets and $2e^2$ coming once again from the $e^2A_\mu^2$ and a symmetry factor for the two fields term. Using the momentum-space Feynman rules, we can find the matrix element for this process. For the t-channel, making sure to conserve momentum at each vertex, we obtain

$$i\mathcal{M}_t = (ie)^2 \frac{i(2p_{1\mu} - p_{3\mu})(p_{4\nu} - 2p_{2\nu})}{t - m_e^2} \epsilon_3^{*\mu} \epsilon_4^{*\nu} \tag{4}$$

Where $p_{1\mu}$ and $p_{2\nu}$ are the momenta of the electron and positron, respectively, and $p_{3\mu}$ and $p_{4\nu}$ are the momenta of the outgoing photons. However, the inner product of the polarization vectors with the corresponding photon's momentum is null by definition:

$$p_{3\mu}\epsilon_3^{*\mu} = p_{4\nu}\epsilon_4^{*\nu} = 0 \tag{5}$$

and equation (4) simplifies to

$$i\mathcal{M}_t = 4ie^2 \frac{p_{1\mu}p_{2\nu}}{t - m_e^2} \epsilon_3^{*\mu} \epsilon_4^{*\nu}$$

Similarly for the u-channel, we can take $p_{3\mu} \to p_{4\nu}$ simplify according to equation (5) and find

$$i\mathcal{M}_u = 4ie^2 \frac{p_{1\mu}p_{2\nu}}{u - m_e^2} \epsilon_3^{*\mu} \epsilon_4^{*\nu}$$

Lastly, we write the matrix element of the 4-point diagram as

$$i\mathcal{M}_{seagull} = 2ie^2 g_{\mu\nu} \epsilon_3^{*\mu} \epsilon_4^{*\nu}$$

And the sum of the three diagrams is

$$i\mathcal{M} = i\mathcal{M}_t + i\mathcal{M}_u + i\mathcal{M}_{seagull} = 4ie^2 [\frac{1}{2} g_{\mu\nu} + p_{1\mu}p_{2\nu}(\frac{1}{t - m_e^2} + \frac{1}{u - m_e^2})] \epsilon_3^{*\mu} \epsilon_4^{*\nu} \tag{6}$$

However, since we are asked later to find the cross section in the center of mass frame we can simplify the second term in the brackets and the matrix element becomes

$$i\mathcal{M} = 4ie^2 [\frac{1}{2} g_{\mu\nu} + p_{1\mu}p_{2\nu}(\frac{u + t - 2m_e^2}{(t - m_e^2)(u - m_e^2)})] \epsilon_3^{*\mu} \epsilon_4^{*\nu}$$

and note that for this process $u + t + s = 2m_e^2$ and we see that

$$i\mathcal{M} = 4ie^2 [\frac{1}{2} g_{\mu\nu} + p_{1\mu}p_{2\nu}(\frac{-s}{(t - m_e^2)(u - m_e^2)})] \epsilon_3^{*\mu} \epsilon_4^{*\nu} \tag{7}$$

Part (b): Cross Section

In order to calculate the cross section for this process we must find the square of the matrix element and plug into the familiar equation for the differential cross section for a $2 \to 2$ process (in center of mass frame)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|p_f|}{|p_i|} |\mathcal{M}|^2 \tag{8}$$

To find $|\mathcal{M}|^2$, it makes sense to first write

$$\mathcal{M} = \epsilon_3^{*\mu} \epsilon_4^{*\nu} M_{\mu\nu}$$

then square and sum over the polarizations of each outgoing photon:

$$|\mathcal{M}|^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} |\epsilon_{3i}^{*\mu}\epsilon_{4j}^{*\nu}M_{\mu\nu}|^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} \epsilon_{3i}^{*\mu}\epsilon_{3i}^{\alpha}\epsilon_{4j}^{*\nu}\epsilon_{4j}^{\beta}\mathcal{M}_{\mu\nu}\mathcal{M}_{\alpha\beta} \tag{9}$$

In order to go any further with this, the Ward Identity will be used. In this process, take the initial e^- and e^+ to be traveling in opposite directions along the z-axis. In the center of mass frame, the initial four-momenta of the electron and positron are

$$p_{1\mu} = (E, 0, 0, \sqrt{E^2 - m_e^2}) \tag{10}$$

$$p_{2\nu} = (E, 0, 0, -\sqrt{E^2 - m_e^2})$$

respectively, and the center of mass energy $s = 4E^2$. There are two distinguisable polarization vectors for each photon, namely

$$\epsilon^\mu(1) = (0, 1, i, 0)$$

$$\epsilon^\mu(2) = (0, 1, -i, 0)$$

. In order to compute the sum, I will complete the sum for one outgoing photon and then generalize to N outgoing photons. For one outgoing photon, we can transform to a frame in which the four-momentum is (E,0,0,E) (this can be done since the total matrix element squared is Lorentz invariant), the Ward identity ($p_\mu\mathcal{M}^\mu = 0$) is utilized and it is found that

$$E^2\mathcal{M}_0 - E^2\mathcal{M}_3 \Rightarrow \mathcal{M}_0^2 - \mathcal{M}_3^2 = 0$$

Now we compute

$$\epsilon_\mu(1)\mathcal{M}^\mu = \frac{-1}{\sqrt{2}}(\mathcal{M}^1 + i\mathcal{M}^2)$$

$$\epsilon_\mu(2)\mathcal{M}^\mu = \frac{1}{\sqrt{2}}(i\mathcal{M}^2 - \mathcal{M}^1)$$

Combining this with what was found with the Ward Identity

$$|\mathcal{M}|^2 = \frac{1}{2}(2\mathcal{M}^1 + \mathcal{M}^2) = \mathcal{M}^1 + \mathcal{M}^2 + \mathcal{M}^3 - \mathcal{M}^0 = -g_{\mu\nu}\mathcal{M}^{\mu\nu} = -\mathcal{M}^\mu\mathcal{M}^*{}_\mu$$

This can be readily generalized to 2, 3,..., N outgoing photons as each new photon gives an independent sum that yields another factor of $-g_{\mu\nu}$. In general, this can be written as

$$\sum_{i_1 ... i_N} |\epsilon^{*\mu_1}_{i_1} ... \epsilon^{*\mu_N}_{i_N} \mathcal{M}_{\mu_1 ... \mu_N}|^2 = (-1)^N \mathcal{M}^{\mu_1 ... \mu_N} \mathcal{M}^*{}_{\mu_1 ... \mu_N}$$

Thus, equation 9 can be rewritten as

$$|\mathcal{M}|^2 = \mathcal{M}^{\mu\nu} \mathcal{M}^*{}_{\mu\nu} \tag{11}$$

Putting all of the previous results together,

$$\begin{aligned} |\mathcal{M}|^2 = 16e^4[[\frac{1}{2} - \frac{s^2}{4(t-m_e^2)(u-m_e^2)}]^2 \\ - [\frac{1}{2} - \frac{s^2}{4(t-m_e^2)(u-m_e^2)} + \frac{m_e^2 s}{(t-m_e^2)(u-m_e^2)}]^2 \\ - \frac{\frac{s}{2}(\frac{s}{4} - m_e^2)s^2}{(t-m_e^2)^2(u-m_e^2)^2} \\ - \frac{1}{2}] \end{aligned}$$

The algebra on the road to simplification of this equation was not fun and neither will be entering all 2 pages (and 3 days of mistakes) of algebra required. So, I will quote the result, after many cancellations and odd factorizations, the matrix element is

$$|\mathcal{M}|^2 = 4e^4[1 + (1 - \frac{2m^2 s}{(t-m_e^2)(u-m_e^2)})^2] \tag{12}$$

Plugging this back into equation 8, noting that $|p_f| = 2E^2 = \sqrt{s}$ and $|p_i| = 2\sqrt{E^2 - m_e^2} = \sqrt{s - 4m_e^2}$ we find the differential cross section to be

$$\frac{d\sigma}{d\Omega}(e^- e^- \to \gamma\gamma) = \frac{\alpha^2}{\sqrt{s(s-4m_e^2)}}[1 + (1 - \frac{2m^2 s}{(t-m_e^2)(u-m_e^2)})^2]$$

where $\alpha = \frac{e^2}{4\pi}$

Part (c): Cross Section with a Massive Photon

In the case of a massive photon, a few things change from of the sum over polarization vectors to the sum of the Mandelstam variables. We note that the sum over polarization vectors goes to 3 instead of 2 and there is now a term that is porportional to the photon momenta over m_γ^2. However, when taken with the total matrix element, the Ward identity implies that this term exactly cancels and we are left with the same matrix element squared, with one caveat. The sum of the Mandelstam variables is now

$$s + t + u = 2m_e^2 + 2m_\gamma^2$$

and the $p_{1\mu}p_{2\nu}$ term is now multiplied by $2m_\gamma^2 - s$ and $|p_f| = \sqrt{s - 4m_\gamma^2}$ and a guess at the new differential cross section is

$$\frac{d\sigma}{d\Omega}(e^- e^- \to \gamma\gamma) = \frac{\alpha^2}{s}\sqrt{\frac{s - 4m_\gamma^2}{s - 4m_e^2}}[1 + (1 + \frac{2m_e^2(2m_\gamma^2 - s)}{(t - m_e^2)(u - m_e^2)})^2] \tag{13}$$

Part (d): Dark Matter

Supposing that dark matter in the universe is made of scalar electron and scalar positrons of mass m who interact with massive photons of mass m_γ where the strength is given by α_{dark}, we can write the differential cross section for the process $e^- e^+ \to \gamma\gamma$ as the equation 13. To simplify the calculation of the total cross section, we will go to a frame where the outgoing photon's have zero momentum and we will look at the case where $\sqrt{s} \gg m_e > m_\gamma$. We can therefore right their momentum four-vectors as

$$p_{3\mu} = (m_\gamma, 0, 0, 0) = p_{4\nu}$$

and recalling the equations for $p_{1\mu}$ and $p_{2\nu}$ we can also rewrite the Mandelstam variables, t and s, as

$$t - m_e^2 = (p_1 - p_2)^2 - m_e^2 = (E - m_\gamma)^2 - (E^2 - m_e^2) - m_e^2 = m_\gamma^2 - 2Em_\gamma$$

$$u - m_e^2 = (p_1 - p_4)^2 - m_e^2 = m_\gamma^2 - 2Em_\gamma$$

Then, taking $E = \frac{\sqrt{s}}{2}$ the total cross section can be written as

$$\sigma = \frac{4\pi\alpha_{dark}^2}{s}\sqrt{\frac{s-4m_\gamma^2}{s-4m_e^2}}[1+(1-\frac{2m_e^2(s-2m_\gamma^2)}{(m_\gamma^2-\sqrt{s}m_\gamma)^2})^2]$$

$$= \frac{4\pi\alpha_{dark}^2}{s}\sqrt{\frac{s-4m_\gamma^2}{s-4m_e^2}}[1+(1-\frac{2m_e^2 s(1-\frac{2m_\gamma^2}{s})}{s(\frac{m_\gamma^2}{\sqrt{s}}-2m_\gamma)^2})]$$

And if approximation above ($\sqrt{s} \gg m_e > m_\gamma$) is made we find

$$\sigma \approx \frac{4\pi\alpha_{dark}^2}{s}[2+(\frac{m_e}{2m_\gamma})^4(1-\frac{2m_\gamma^2}{m_e^2})] \approx \frac{4\pi\alpha_{dark}^2}{s}[2+(\frac{m_e}{2m_\gamma})^4]$$

After attempting to find suitable combinations of α_{dark} and m_γ with a variety of reasonable center of mass energies for the given total cross sections, my attempts came at no avail, probably because I over simplified the problem and made an error (two?) on the way.