Homework Set #2.

Due Date - Oral Presentation: Wednesday October 14, 2015
Due Date - Written Solutions: Wednesday October 21, 2015

1. **Yukawa Potential**

   (a) Calculate the equations of motion for a massive vector field A_μ from the Lagrangian
   $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2 - A_\mu J_\mu,$$
   where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Assuming $\partial_\mu J_\mu = 0$, use the equations to find a constraint on A_μ.

   (b) For J_μ the current of a point charge, show that the equation of motion for A_0 reduces to
   $$A_0(r) = \frac{e}{4\pi^2 ir}\int_{-\infty}^{\infty} \frac{kdk}{k^2+m^2}e^{ikr}.$$

   (c) Evaluate this integral with contour integration to get an explicit form for $A_0(r)$.

   (d) Show that as $m \to 0$ you reproduce the Coulomb potential

   (e) In 1935, Yukawa speculated that this potential might explain what holds protons together in nuclei. What qualitative feature does this Yukawa potential have, compared to a Coulomb potential, that make it a good candidate for the force between protons? What value for m might be appropriate (in MeV)?

2. **Complex Scalar Field** Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is
   $$S = \int d^4x(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi).$$
   It is easiest to analyze this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi).$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

(c) Rewrite the conserved charge

$$Q = \int d^3x \frac{i}{2}(\phi^*\pi^* - \pi\phi)$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

(d) [OPTIONAL] Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of part (c), and the other three given by

$$Q^i = \int d^3x \frac{i}{2}(\phi_a^*(\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i)_{ab}\phi_b).$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.