APPENDIX A

Proof of Bertrand's Theorem*

The orbit equation under a conservative central force, Eq. (3–34), may be written

$$\frac{d^2u}{d\theta^2} + u = J(u), \tag{A–1}$$

where

$$J(u) = -\frac{m}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right) = -\frac{m}{l^2u^2}f\left(\frac{1}{u}\right). \tag{A–2}$$

The condition for a circular orbit of radius $r_0 = u_0^{-1}$, Eq. (3–41), now takes the form

$$u_0 = J(u_0). \tag{A–3}$$

In addition, of course, the energy must satisfy the condition of Eq. (3–42). If the energy is slightly above that needed for circularity, and the potential is such that the motion is stable, then u will remain bounded and vary only slightly from u_0 and $J(u)$ can be expressed in terms of the first term in a Taylor series expansion about $J(u_0)$:

$$J(u) = u_0 + (u - u_0)\frac{dJ}{du_0} + O((u - u_0)^2). \tag{A–4}$$

As is customary, the derivative appearing in Eq. (A–4) is a shorthand symbol for the derivative of J with respect to u evaluated at $u = u_0$. If the difference $u - u_0$ is represented by x, the orbit equation for motion in the vicinity of the circularity conditions is then

$$\frac{d^2x}{d\theta^2} + x = x\frac{dJ}{du_0},$$

or

$$\frac{d^2x}{d\theta^2} + \beta^2x = 0, \tag{A–5}$$

* See Section 3–6.

where

$$\beta^2 = 1 - \frac{dJ}{du_0}. \tag{A–6}$$

In order for x to describe a bounded stable oscillation, β^2 must be positive definite. From the definition, Eq. (A–2), we have

$$\frac{dJ}{du} = \frac{2m}{l^2u^3} f\left(\frac{1}{u}\right) - \frac{m}{l^2u^2}\frac{d}{du} f\left(\frac{1}{u}\right) = -\frac{2J}{u} - \frac{m}{l^2u^2}\frac{d}{du} f\left(\frac{1}{u}\right).$$

In view of the circularity conditions, Eq. (3–41) or Eq. (A–3), it then follows that

$$\frac{dJ}{du_0} = -2 + \frac{u_0}{f_0}\frac{df}{du_0},$$

where, in addition to the convention employed for the derivatives, f_0 stands for $f(1/u_0)$. Hence β^2 is given by

$$\beta^2 = 3 - \frac{u_0}{f_0}\frac{df}{du_0} = 3 + \frac{r}{f}\frac{df}{dr}\bigg|_{r=r_0}, \tag{A–7}$$

which is the same as Eq. (3–46), and the stability condition, $\beta^2 > 0$, thus reduces to Eq. (3–43).

By a suitable choice of origin of θ, the solution to Eq. (A–5) for β^2 positive definite can be written

$$x = a\cos\beta\theta \tag{A–8}$$

(cf. Eq. (3–45)). In order for the orbit to remain closed when the energy and angular momentum are thus slightly disturbed from circularity, the quantity β must be a rational number. We are concerned with finding force laws such that for a wide range of initial conditions, i.e., for a wide range of u_0, the orbits deviating slightly from circularity remain closed. Under these circumstances, as argued in the main text, β must have the same value over the entire domain of u_0, and Eq. (A–7) can be looked on as a differential equation for $f(1/u)$ or $f(r)$. The desired force law must therefore conform to a dependence on r given in Eq. (3–48):

$$f(r) = -\frac{k}{r^{3-\beta^2}}, \tag{A–8}$$

where k is some constant and β is a rational number.

The force law of Eq. (A–8) still permits a wide variety of behavior for the force. However we seek more stringent conditions on the force law, by requiring that even when the deviations from circularity are considerable the orbit remain closed. We must therefore at least deal with deviations of u from u_0 so large that we must keep more terms than the first in the Taylor series expansion of $J(u)$. Equation (A–4) may therefore be replaced by

$$J(u) = u_0 + xJ' + \frac{x^2}{2}J'' + \frac{x^3}{6}J''' + O(x^4), \tag{A–9}$$

where it is understood the derivatives are evaluated at $u = u_0$. In terms of this expansion of $J(u)$ the orbit equation becomes

$$\frac{d^2x}{d\theta^2} + \beta^2 x = \frac{x^2 J''}{2} + \frac{x^3 J'''}{6}. \tag{A–10}$$

We seek to find the nature of the source law such that even when the deviation from the circular orbit, x, is large enough that the terms on the right cannot be neglected, the solution to Eq. (A–10) still represents a closed orbit. For small perturbations from circularity we know x has the behavior described by Eq. (A–8), which represents the fundamental term in a Fourier expansion in terms of $\beta\theta$. We seek therefore a closed-orbit solution by including a few more terms in the Fourier expansion:

$$x = a_0 + a_1 \cos \beta\theta + a_2 \cos 2\beta\theta + a_3 \cos 3\beta\theta. \tag{A–11}$$

The amplitudes a_0 and a_2 must be of smaller magnitude than a_1 because they vanish faster than a_1 as circularity is approached. As will be seen a_3 must be of even lower order of magnitude than a_0 or a_2, which is why terms in $\cos 4\beta\theta$ and so on can be neglected. Consequently in the x^2 term on the right of Eq. (A–10) the factors in $\cos 3\beta\theta$ are dropped, and in the x^3 term only factors in $\cos \beta\theta$ are kept. In evaluating the right-hand side powers and products of the cosine functions are reduced by means of such identities as

$$\cos \beta\theta \cos 2\beta\theta = \tfrac{1}{2}(\cos \beta\theta + \cos 3\beta\theta)$$

and

$$\cos^3 \beta\theta = \tfrac{1}{4}(3 \cos \beta\theta + \cos 3\beta\theta).$$

Consistently keeping terms through the order of a_1^3 in this manner, Eq. (A–10) with the solution (A–11) can be reduced to

$$\begin{aligned} \beta^2 a_0 - 3\beta^2 a_2 \cos 2\beta\theta - 8\beta^3 a_3 \cos 3\beta\theta \\ = \frac{a_1^2}{4} J'' + \left[\frac{2a_1 a_0 + a_1 a_2}{2} J'' + \frac{J''' a_1^3}{8}\right] \cos \beta\theta \\ + \frac{a_1^2}{4} J'' \cos 2\beta\theta + \left[\frac{a_1 a_2}{2} J'' + \frac{J''' a_1^3}{24}\right] \cos 3\beta\theta. \end{aligned} \tag{A–12}$$

In order for the solution to be valid, the coefficient of each cosine term must separately vanish, leading to four conditions on the amplitude and the derivatives of J:

$$a_0 = \frac{a_1^2 J''}{4\beta^2}, \tag{A–13a}$$

$$a_2 = -\frac{a_1^2 J''}{12\beta^2}, \tag{A–13b}$$

$$0 = \frac{2a_1 a_0 + a_1 a_2}{2} J'' + \frac{J''' a_1^3}{8}, \tag{A–13c}$$

$$a_3 = -\frac{1}{8\beta^3}\left[\frac{a_1 a_2}{2} J'' + \frac{J''' a_1^3}{24}\right]. \tag{A–13d}$$

It should be remembered that we already have shown, on the basis of slight deviations from circularity, that for closed orbits the force law must have the form of Eq. (A–8) or that $J(u)$ is given by

$$J = +\frac{mk}{l^2} u^{1-\beta^2}. \tag{A–14}$$

Keeping in mind the circularity condition, Eq. (A–3), the various derivatives at u_0 can be evaluated as

$$J'' = \frac{\beta^2(1-\beta^2)}{u_0} \tag{A–15a}$$

and

$$J''' = \frac{-\beta^2(1-\beta^2)(1+\beta^2)}{u_0^2}. \tag{A–15b}$$

Equations (A–13a, b) thus say that a_0/a_1 and a_2/a_1 are of the order of a_1/u_0, which is by supposition a small number. Further, Eq. (A–13d) shows that a_3/a_1 is of the order of $(a_1/u_0)^2$, which justifies the earlier statement that a_3 is of lower order of magnitude than a_0 or a_2.

Equation (A–13c) is a condition on β only, the condition that in fact is the principal conclusion of Bertrand's theorem. Substituting Eqs. (A–13a, b) and (A–15) into Eq. (A–13c) yields the condition

$$\beta^2(1-\beta^2)(4-\beta^2) = 0. \tag{A–16}$$

For deviations from a circular orbit, that is, $\beta \neq 0$, the only solutions are

$$\beta^2 = 1, \qquad f(r) = -\frac{k}{r^2} \tag{A–17a}$$

and

$$\beta^2 = 4, \qquad f(r) = -kr. \tag{A–17b}$$

Thus the only two possible force laws consistent with the solution are either the gravitational inverse-square law or Hooke's law!

We started out with orbits that were circular. These are possible for all attractive force laws over a wide range of l and E, whose values in turn fix the orbital radius. The requirement that the circular orbit be *stable* for all radii already restricts the form of the force law through the inequality condition of $\beta^2 > 0$ (Eq. (3–48)). If we further seek force laws such that orbits that deviate only slightly from a circular orbit are still closed, no matter what the radius of the reference orbit, then the force laws are restricted to the discrete set given by Eq. (A–8) with rational values of β. In order for the orbits to remain closed for larger deviations from circularity, no matter what the initial conditions of the reference orbit, only two of these rational values are permitted: $|\beta| = 1$ and $|\beta| = 2$. Since we know that these attractive force laws do in fact give closed orbits for all E and l leading to bounded motion, they must be the *only* force laws leading to closed orbits for all bounded motion.