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THE INSTANTANEOUS MOTION OF A RIGID BODY

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1. Combination of instantaneous rigid motions. The assertion that a rigid body is rotating about the x-axis with a certain angular velocity, and rotating at the same time with another angular velocity about the y-axis, puts a strain on the imagination of a student meeting this form of statement for the first time. He is relieved to find that the statement is not one that needs to be interpreted literally, being merely a somewhat irresponsible substitute for a clear formulation in mathematical terms. What is meant is merely that if the body is regarded as a continuous distribution of matter, the vector velocity of each of its points is the geometric resultant of the velocity which *would be* associated with that point by the first rotation and the one which it would have in the second rotation.

The analysis of the most general instantaneous motion of a rigid body can be carried through in terms of ideas of corresponding simplicity. (The phrase "instantaneous motion" is understood for the purposes of this paper to be concerned throughout with the velocities of the points considered, not with their accelerations, which would present a more complex problem.) The notions involved are of course essentially vectorial. In particular, the theory offers notable concrete or semi-concrete exemplification of the significance of the distributive law for vector multiplication. The main features of the theory are presented below from this point of view.* The presentation lays no claim to novelty; its purpose is merely to give one possible arrangement of the details in order for consecutive reading.

* For a formulation in quite different language see for example R. S. Ball, The Theory of Screws, Dublin, 1876, pp. xix–xxiv; for further comparison see W. F. Osgood, Mechanics, New York, 1937, Chapter V.

The problem is that of characterizing the motion itself, without reference to the forces which produce or control it. The characterization is independent also of the size and shape of the body considered. If its motion or that of any three-dimensional portion of it is specified, the velocity which a particle at any point of space would possess if rigidly attached to it is determined. It is to be supposed for the purposes of the present study that *each point of space* has a definite vector velocity assigned to it, in a manner consistent with the *condition of rigidity* presently to be laid down. Any such distribution of velocities, or velocity field, will be called for brevity a *rigid motion.* For similar purposes of abbreviation any set of velocities which there is occasion to consider will be called a *motion*, whether subjected explicitly to the hypothesis of rigidity or not.

It will be seen that in a sense to be carefully defined *the most general rigid motion is either a translation or a rotation or the resultant of a translation and a rotation.*

The *condition of rigidity* is that *for every pair of points P_1, P_2, the vector velocity of P_1 and the vector velocity of P_2 have equal components along the line P_1P_2.* This expresses for instantaneous motion the property that the distance between any two points of the body remains invariable. As between the two opposite directions along the line, it is naturally to be understood that the equal components agree in direction as well as in magnitude.

The discussion here, as already remarked, is concerned exclusively with the motion itself, not with the conditions by which it is produced. It is of no consequence whether any material body to which the conclusions may be applied is *capable* of deformation or not, provided that the velocities which its particles actually possess under specified circumstances are such that the condition of rigidity is fulfilled.

If any two "motions," *i.e.* sets of velocities defined for the points of space, are denoted by M' and M'', their *resultant*, represented symbolically by $M'+M''$, is the motion in which the velocity of each point is the resultant or vector sum of the velocities assigned to that point by M' and M'' separately. It is an immediate consequence of the definitions that *the resultant of any two rigid motions is a rigid motion*, since for each pair of points P_1, P_2 the components of the resultant velocities along P_1P_2 are obtained by algebraic addition of components which are separately equal for the two points.

2. Simple rigid motions: translation and rotation. A *translation* is a motion in which all points have equal vector velocities; that is to say, in less technical but colloquially* more descriptive language, the velocities of all points are equal and parallel. It follows from the definition that a translation is a rigid motion.

Another fundamental type of motion, called a *rotation*, can be described as follows:

* In an endeavor to minimize technicality of expression, the word *velocity* will be used interchangeably for the vector velocity and for its magnitude, when no misunderstanding seems possible. If the reader desires to have the distinction appear in the record he can of course accomplish this with brevity by using the word *speed* on occasion for the magnitude of the vector velocity.

a) There is a straight line, called the *axis* of the rotation, all of whose points have zero velocity.

b) The velocity of any point not on the axis is perpendicular to the plane containing the point and the axis.

c) All points at equal distances from the axis have equal velocities (the word "equal" being used again as an abbreviation for "equal in magnitude").

d) Points at different distances from the axis have velocities proportional to those distances.

e) All velocities correspond to the same "sense" or direction of turning about the axis.

It will be seen presently that the content of all this verbal description can be condensed into a simple vector formula.

A rotation is "obviously" a rigid motion, in the sense that intuition recognizes it as a kind of motion possible for a rigid body. It is another matter to show formally that it satisfies the definition of a rigid motion in terms of the velocities of an arbitrary pair of points. To give such a proof by the methods of elementary geometry would be a somewhat substantial exercise. It can be done in a few lines by means of vector algebra, for the reason that the geometric relations involved are precisely those which vector algebra recognizes as fundamental. Another reason for stressing this proof is that it appears in some way to form the backbone of the entire theory; simple as it is, all the other proofs to be given subsequently are so much simpler as to be scarcely more than a succession of "remarks."

By the specifications describing a rotation above, the ratio of the velocity of an arbitrary point P to its distance from the axis is the same for all points of space. This ratio is the *angular velocity* of the rotation. If the same unit of length is used for measuring velocity and for measuring distance, the angular velocity is measured in radians per unit of time. All the essential characteristics identifying a particular rotation are conveniently represented by a vector ω lying in the axis, with magnitude numerically equal to the angular velocity just defined, in the adopted scale of measurement, and pointing in the direction in which the rotation would carry a right-handed screw. This ω is called the *vector velocity of rotation*. Like a vector representing a force in the dynamics of a rigid body, it is to be thought of as lying in a definite line, but may be laid off from any point of that line as initial point.

Let O be an arbitrary point of the axis, and let ρ be the vector from O to an arbitrary point P of space. By a check of magnitude and direction it is seen at once that *the velocity of P is represented by the vector product* $\omega\times\rho$. For any specified ω this product defines a set of vectors throughout space having the characteristics of a rotation.

A rotation being given with ω as its vector representation, and a fixed point of reference O on its axis, let ρ_1 and ρ_2 be the vectors from O to a pair of arbitrary points P_1, P_2 anywhere in space. The vector velocities of P_1 and P_2 are $\omega\times\rho_1$ and $\omega\times\rho_2$. Their components in the direction from P_1 toward P_2, if the distance

P_1P_2 is denoted by D, are $D^{-1}\omega \times \rho_1 \cdot (\rho_2 - \rho_1)$ and $D^{-1}\omega \times \rho_2 \cdot (\rho_2 - \rho_1)$, and D times the difference between these components is

$$(\omega \times \rho_2 - \omega \times \rho_1) \cdot (\rho_2 - \rho_1) = \omega \times (\rho_2 - \rho_1) \cdot (\rho_2 - \rho_1) = 0,$$

because of the distributive law for the scalar product, the distributive law for the vector product, and the fact that $\omega \times (\rho_2 - \rho_1)$ is perpendicular to $\rho_2 - \rho_1$ (or, as an alternative formulation for the last step, the fact that a scalar triple product is zero if two of its factors are alike). The vanishing of the difference means that the condition of rigidity is fulfilled.

As a trivial special case, the assignment of zero velocities to all points will be regarded alternatively as defining a translation of zero magnitude or a rotation of zero magnitude about an arbitrary axis, or will be referred to simply as a zero motion.

3. Analysis of general rigid motion. A description of the most general rigid motion is now obtained through the following sequence of observations.

I. *A rigid motion in which three non-collinear points have zero velocities is a zero motion.* Let O_1, O_2, O_3 be three such points. Let P be any point outside their plane. By the condition of rigidity P can not have any velocity component along any of the lines O_1P, O_2P, O_3P. That is to say, all three lines must be perpendicular to the velocity of P, *if any*. Since they do not lie in one plane, this is impossible, and the words "if any" indicate a condition contrary to fact; the velocity of P must be zero. As for points in the plane $O_1O_2O_3$, let Q be any such point, and let O_4 be a point outside the plane. By the preceding proof the velocity of O_4 is zero, and repetition of the argument with O_4 in place of one of the three points originally given shows that Q has zero velocity.

II. *A rigid motion in which two distinct points have zero velocities is a rotation about the line of these points as axis.* Let O_1, O_2 be the given points. Let P_0 be a point outside their line. If P_0 has zero velocity, all points have zero velocity by the preceding paragraph, and the motion can be regarded as a zero rotation. If P_0 has a velocity, this velocity can have no component along O_1P_0 or O_2P_0, by the condition of rigidity, and so must be perpendicular to both lines and to the plane of the three points. Let v_0 be the magnitude of the velocity of P_0, and r the perpendicular distance of P_0 from the line O_1O_2. Let M denote the given rigid motion; of the two opposite rotations about O_1O_2 with angular velocity v_0/r, let R denote the one that gives to P_0 the velocity which it possesses in M, and $-R$ the other. Then the resultant of M and $-R$, for brevity $M-R$, is a rigid motion in which O_1, O_2, and P_0 have zero velocities, and so all points have zero velocities, by reference to the preceding paragraph once more. That is to say, M as a set of velocities for the points of space is identical with R.

III. *A rigid motion in which a point O has zero velocity is a rotation about an axis through O.* The assertion that if there is a point at rest there must be a whole line of points at rest recalls the similarly striking fact in solid geometry that if two planes have a point in common they must have a whole line in common. It will be seen that one fact is a consequence of the other. In the formulation of the proof, trivial specializations which have the effect merely of bringing back the conditions of the preceding paragraphs will not be explicitly enumerated.

Let P_1 be a point distinct from O, and ϕ_1 its vector velocity. Since ϕ_1 must be perpendicular to OP_1, by the condition of rigidity, the plane through P_1 perpendicular to ϕ_1 contains O. Let p_1 denote this plane. Let P_1' be any point of p_1 outside the line OP_1. The velocity of P_1' must be perpendicular to OP_1' and to P_1P_1', and so perpendicular to p_1, since P_1 has no component of velocity along P_1P_1' and O has no velocity at all. A selected reference point in p_1 outside OP_1 may then be used to extend the conclusion to points of this line, in analogy with the final step in the proof of I. *All* points of p_1 have their velocities, if any, perpendicular to p_1.

Let P_2 be a point outside p_1, ϕ_2 its velocity, and p_2 the plane through P_2 perpendicular to ϕ_2. By reasoning similar to that just presented, p_2 passes through O, and the velocities of all points of p_2 are perpendicular to p_2.

Since p_1 and p_2 have the point O in common, they intersect in a line. If O_1 is another point of the line of intersection its velocity, if any, must be perpendicular to both planes. As this is impossible, O_1 has zero velocity, and reference to II completes the proof.

IV. *If O is an arbitrarily chosen point, the most general rigid motion is resultant of a rotation about an axis through O and a suitable translation.* (It is understood naturally that either the rotation or the translation may in particular be zero.) Let M be the given rigid motion, let ϕ be the vector velocity of O, and let T be the translation in which all points have this vector velocity, while $-T$ is the opposite translation, with velocity $-\phi$. Then $M-T$, interpreted as $M+(-T)$, is a rigid motion in which O has no velocity, and so by III is a rotation about an axis through O. If this rotation is denoted by R, M is the resultant of R and T.

It is to be noted that only a single point of the axis, not the whole axis, is arbitrary.

V. *The resultant of a rotation and a translation perpendicular to the axis of the rotation is a rotation of equal angular velocity about a parallel axis.* Let R denote the rotation, with ω as the vector representation of its angular velocity, and T the translation, with velocity ϕ; the hypothesis requires that $\omega \cdot \phi = 0$.

To demonstrate formally a fact which is obvious to geometric intuition, namely that there is a line of points to which R assigns the velocity $-\phi$, let $\psi = \omega \times \phi / \omega^2$, where ω^2 denotes the square of the magnitude of ω, let O be an arbitrary point of the axis of R, and let O' be the corresponding point such that the vector OO' is ψ. Then the velocity $\omega \times \psi$ which R gives to O' is in fact $-\phi$, as may be seen either by application of the rule for evaluating a vector triple product or by elementary interpretation of the successive operations of simple vector multiplication.

As O describes the axis of R, the point O' describes a parallel line, and the rigid motion $R+T$, giving zero velocity to all points of this line, is a rotation R' about it. The equality of the angular velocities is recognized by comparing the vector velocities of a pair of corresponding points O' and O in the respective rotations, the line joining them being perpendicular to both axes.

VI. *The most general rigid motion is resultant of a rotation and a translation parallel to the axis of the rotation.** On the basis of this analysis the motion is called a *screw motion.* Let O be an arbitrary point, and let the given motion M be expressed by IV as a rotation R about an axis through O plus a translation T. Let T be resolved into component translations T_0 and T_1, parallel and perpendicular to the axis of R. By V, R and T_1 can be combined into a rotation R_0 about a parallel axis, and M is then the sum of R_0 and T_0.

With the conclusions IV and VI the theory attains a certain stage of completeness. Some additional facts are nevertheless deserving of emphasis.

Of these perhaps the most striking relates to the combination of rotations about intersecting axes. It follows at once from III that the resultant of two such rotations is a rotation about an axis through the point of intersection, since this point has zero velocity in the resultant motion. More specifically, let O be the intersection, let ω_1 and ω_2 be the vectors representing the given rotations, let P be an arbitrary point of space, and let ρ be the vector OP. The resultant of the velocities given to P by the two rotations separately is

$$\omega_1 \times \rho + \omega_2 \times \rho = (\omega_1 + \omega_2) \times \rho,$$

which is the same as the velocity corresponding to a single rotation represented by the vector $\omega_1+\omega_2$. *Instantaneous rotations about intersecting axes can be added vectorially, as an immediate consequence of the distributive law for vector multiplication.*

This fact is the basis for the resolution of an instantaneous rotation into component rotations about a set of coordinate axes.

In the combination of a rotation with a *non-vanishing* translation parallel to its axis, the velocity given to any point by the rotation, having no component parallel to the axis, can not cancel the velocity due to the translation; there is no point with zero velocity, and the resultant motion is not equivalent to any single rotation alone.

Suppose a given rigid motion is resolved in any way into a rotation R_1 and a translation T_1, and again into a rotation R_2 and a translation T_2. Then one may write the equations.

$$R_1 + T_1 = R_2 + T_2, \qquad R_2 = R_1 + T_1 - T_2.$$

(Such manipulation does not involve the setting up of any new type of algebra; it is merely a symbolism for representing comprehensively the corresponding elementary combinations of the vector velocities for the various points of space.) Let the translation $T_1 - T_2$ be resolved into components T', T'' perpendicular and parallel to the axis of R_1. By V, the resultant of the rotation R_1 and the

* Discovery of this theorem is ascribed to G. Mozzi, 1763; see Encyklopädie der mathematischen Wissenschaften, vol. 4: 1, article IV 2, H. E. Timerding, Geometrische Grundlegung der Mechanik eines starren Körpers, pp. 125–189; p. 143.

translation T' is a rotation R' with equal angular velocity about a parallel axis. In the equation

$$R_2 = R' + T''$$

it follows from the preceding paragraph that T'' must vanish. *The rotations* R_1 *and* R_2 *have equal angular velocities about parallel axes; the translations* T_1 *and* T_2 *have equal components in the common direction of these axes.* To restate a part of this conclusion in different words, *a given rigid motion has a vectorial angular velocity* ω *which is determined in magnitude and direction by the rigid motion itself, and is independent of any choice of a particular point of reference as origin.*

If in particular it is supposed that neither T_1 nor T_2 has any transverse component, T' vanishes, R_2 is the same as R_1, and T_2 is the same as T_1; *the resolution given by* VI *is uniquely determined.*

4. Supplementary notes. By way of additional comment, attention may be directed to certain facts with regard to translations which were not needed in the main body of the discussion. The obvious fact that a translation satisfies the condition of rigidity was noted at an early stage. It is almost as easy to see that a rigid motion in which all the velocities are parallel is necessarily a translation. For if P_1, P_2 are any two points such that the line joining them is not perpendicular to the common direction of the velocities, equality of the components along P_1P_2 implies that the total velocities of P_1 and P_2 are equal; from an equation $v_1 \cos \theta = v_2 \cos \theta$ it follows that $v_1 = v_2$, if $\cos \theta \neq 0$. If the line P_1P_2 is perpendicular to the direction of translation, the velocities of P_1 and P_2 are equal to that of any third point P_3 outside the plane through P_1 and P_2 perpendicular to that direction, and so again equal to each other.

Less obvious perhaps at the outset, but easily recognized when the general theorems have been established, is the proposition that if the velocities in a rigid motion are all equal they must be parallel, and the motion is a translation once more. For if the motion is resultant of a translation and a rotation in which the latter is not zero, the magnitudes of the velocities are certainly not all equal.

In summary, of the properties of rigidity, parallelism, and equality of magnitude, any two imply the third.

The reader may be interested to show as an "exercise" that *the most general rigid motion is either a rotation or* (in an infinite variety of ways) *resultant of two rotations about non-intersecting axes.*

A more extensive exercise, which is of importance in itself and will serve to throw much light in retrospect on the theory that has been developed, is to carry through a corresponding discussion in two dimensions, that is to say, with consideration only of points in one plane, and with the assumption that all velocities lie in that plane. A noteworthy difference between two dimensions and three is found in connection with the theorem numbered VI: *Every plane rigid motion is either a translation or a rotation.*