HW # 3 SOLUTION TO PROBLEM # 1

IN POLAR COORDINATES $\vec{v} = \dot{r}\,\hat{r} + r\dot{\varphi}\,\hat{\varphi}$, SO

$$\vec{v}\cdot\vec{A} = \frac{B}{2}\, r^2 \dot{\varphi}$$

SO THE LAGRANGIAN READS

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{e}{c}\frac{B}{2}\, r^2 \dot{\varphi}$$

APPLY THE LEGENDRE TRANSFORM

$$P_r = m\dot{r} \qquad P_\varphi = m r^2 \dot{\varphi} - \frac{eB}{2c}\, r^2$$

SETTING $\omega = \frac{eB}{mc}$, WE HAVE $\dot{r} = \frac{P_r}{m}$ $\quad \dot{\varphi} = \frac{P_\varphi}{m r^2} + \frac{\omega}{2}$

AND

$$H = p_r \dot{r} + P_\varphi \dot{\varphi} - L = \frac{P_r^2}{2m} + \frac{P_\varphi^2}{2 m r^2} + \frac{\omega P_\varphi}{2} + \frac{m r^2 \omega^2}{8}$$

NOTE THAT φ IS CYCLIC, SO P_φ = CONSTANT

IF THE MOTION LIES ON A CIRCULAR ORBIT, $P_r = \dot{P}_r = 0$

THIS IS EQUIVALENT TO $\frac{\partial H}{\partial r} = 0$, OR $-\frac{P_\varphi^2}{m} r^{-3} + \frac{1}{4} m r \omega^2 = 0$

THE SOLUTION TO THIS EQUATION GIVES THE RADIUS OF THE ORBIT CORRESPONDING TO PARAMETERS P_φ, ω:

$$r_0 = \left(\frac{2 P_\varphi}{m\omega}\right)^{1/2} \qquad P_\varphi = \frac{m\omega r_0^2}{2}$$

CORRESPONDINGLY, WE HAVE

$$\dot{\varphi} = \frac{\partial H}{\partial P_\varphi} = \frac{P_\varphi}{m r_0^2} + \frac{\omega}{2} = \omega$$

SO ω IS THE ANGULAR VELOCITY OF CIRCULAR MOTION.

P_φ IS DETERMINED BY THE KINETIC ENERGY:

$$T = \frac{1}{2} m r_0^2 \dot{\varphi}^2 = \frac{P_\varphi^2}{2 m r_0^2} + \frac{1}{8} m r_0^2 \omega^2 + \frac{1}{2}\omega P_\varphi = \omega P_\varphi$$

TO STUDY STABILITY OF CIRCULAR MOTION OF RADIUS r_0, SET

$r = r_0 + \rho$, KEEP P_φ TO VALUE CORRESPONDING TO r_0

$$\frac{1}{r^2} = \frac{1}{r_0^2}\frac{1}{(1+\rho/r_0)^2} \simeq \frac{1}{r_0^2}\left(1 - 2\frac{\rho}{r_0} + 3\left(\frac{\rho}{r_0}\right)^2 + O(\rho^3)\right)$$

$$P_\rho = P_r$$

SUBSTITUTE IN H: $\left(P_\varphi = \frac{m\omega r_0^2}{2}\right)$

$$H = \frac{P_\rho^2}{2m} + \underbrace{\frac{m^2\omega^2 r_0^4}{8 m r_0^2}\left(1 - \frac{2\rho}{r_0} + \frac{3\rho^2}{r_0^2}\right)}_{\frac{P_\varphi^2}{2mr^2}} + \frac{\omega P_\varphi}{2} +$$

$$+ \frac{m\omega^2}{8}\left(r_0^2 + 2\rho r_0 + \rho^2\right) =$$

$$= \frac{P_\rho^2}{2m} + \frac{m\omega^2 r_0^2}{8} - \frac{m\omega^2 r_0^{\cancel{2}}}{4}\frac{\rho}{\cancel{r_0}} + \frac{3}{8} m\omega^2\rho^2 + \frac{m\omega^2 r_0^2}{4} + \frac{m\omega^2 r_0^2}{8} +$$

$$+ \frac{m\omega^2}{4}\rho r_0 + \frac{m\omega^2\rho^2}{8}$$

So $$H = \frac{P_\rho^2}{2m} + \frac{1}{2} m\omega^2\rho^2 + \underbrace{\frac{m\omega^2 r_0^2}{2}}_{\text{CONST}}$$

WHICH IS THE HAMILTONIAN FOR HARMONIC OSCILLATOR WITH FREQUENCY ω!

HW # 3 SOLUTION TO PROBLEM # 3

(a) FOR THE POINT MASS

$$I_{ij} = \left(\frac{5M}{4}\right)\left[r^2\delta_{ij} - x_i x_j\right] \quad \text{SO}$$

$$I_P = \frac{5M}{4}R^2\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & \\ 0 & 0 & 2 \end{pmatrix}$$

FOR THE DISC, APPLYING THE THEOREM OF PARALLEL AXES,

$$I_D = \frac{MR^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \frac{MR^2}{4}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \frac{MR^2}{4}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

SO THE MOMENT OF INERTIA IS

$$I = I_P + I_D = \frac{MR^2}{4}\begin{pmatrix} 10 & -5 & 0 \\ -5 & 6 & 0 \\ 0 & 0 & 16 \end{pmatrix}$$

(b) TO FIND PRINCIPAL AXES AND MOMENTS OF INERTIA, SOLVE SECULAR EQUATION.

$$\frac{MR^2}{4}\begin{vmatrix} 10-\gamma & -5 & 0 \\ -5 & 6-\gamma & 0 \\ 0 & 0 & 16-\gamma \end{vmatrix} = (16-\gamma)(\gamma^2 - 16\gamma + 35) = 0$$

$$\gamma_1 = 16 \quad \gamma_2 = 8 - \sqrt{29} \quad \gamma_3 = 8 + \sqrt{29}$$

SO $\quad I_1 = 4MR^2 \quad I_2 = \left(2 - \frac{\sqrt{29}}{4}\right)MR^2 \quad I_3 = \left(2 + \frac{\sqrt{29}}{4}\right)MR^2$

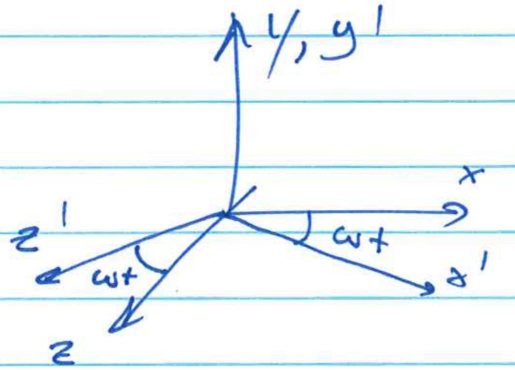
TO FIND ~~EIGENVALUES~~ EIGENVECTORS, STANDARD PROCEDURE . . .

(c) I WE FOUND ABOVE REFERS TO COORDINATE FRAME ATTACHED TO THE DISC.

IN THIS FRAME, $\vec{L} = I\vec{\omega}$, i.e.

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \frac{MR^2}{4} \begin{pmatrix} 10 & -5 & 0 \\ -5 & 6 & 0 \\ 0 & 0 & 16 \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = \frac{MR^2\omega}{4} \begin{pmatrix} -5 \\ 6 \\ 0 \end{pmatrix}$$

NOW, CONSIDER THE LAB FRAME (x', y', z'), SAME y-AXIS BUT ROTATING ON (x,z) PLANE WITH ANGULAR VELOCITY ω, SUCH THAT AXES COINCIDE AT $t=0$



SO: $\begin{cases} x' = x\cos\omega t + z\sin\omega t \\ y' = y \\ z' = -x\sin\omega t + z\cos\omega t \end{cases}$

TRANSFORMATION TENSOR IS $S = \begin{pmatrix} \cos\omega t & 0 & \sin\omega t \\ 0 & 1 & 0 \\ -\sin\omega t & 0 & \cos\omega t \end{pmatrix}$

SO THAT VECTORS TRANSFORM AS $\vec{V}' = S\vec{V}$

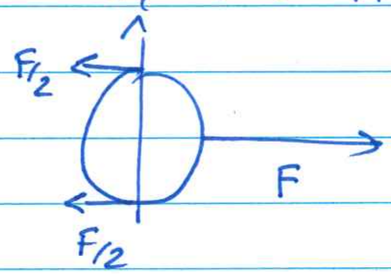
WE THUS FIND $\begin{pmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{pmatrix} = S\vec{L} = \frac{MR^2\omega}{4} \begin{pmatrix} -5\cos\omega t \\ 6 \\ 5\sin\omega t \end{pmatrix}$

NOTE THAT y-AXIS IS A PRINCIPAL AXIS FOR THE DISC ALONE, SO ROTATION ABOUT y DOESN'T PRODUCE ANY FORCE ON THE PIVOTS FROM THE DISC.

FORCES ARE THUS ONLY DUE TO THE POINT MASS

IN ROTATING FRAME, MASS POINT SUFFERS A CENTRIFUGAL FORCE

$F = \frac{5}{4} M R \omega^2$ BALANCED BY FORCES EXERTED ON THE DISC BY THE PIVOTS, $F = F/2 = \frac{5}{8} M R \omega^2$



THE FORCES @ PIVOTS ARE OPPOSITE TO THE CENTRIFUGAL FORCE, AND THUS ROTATE WITH ANGULAR FREQUENCY ω