Solution to Homework Set #1, Problem #1.

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1 Relativistic Kinematics - a warm-up

Part (a)

A particle of mass m_1 decays into two particles of mass m_2 and m_3. Calculate the energy of the two final-state particles in the center of mass frame.

Solution

First off, we know that total momentum will be conserved. Let $\mathbf{p}_2$, $\mathbf{p}_2$ and $\mathbf{p}_3$ be the momenta of masses m_1, m_2 and m_3 respectively. Since we are in the center of mass frame, $\mathbf{p}_1 = 0$. Therefore, $\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3 = 0$ and thus, $\mathbf{p}_2 = -\mathbf{p}_3$. We can rotate our problem such that $\mathbf{p}_2$ and $\mathbf{p}_3$ are in the $\hat{x}$ direction. To determine the energies of masses m_2 and m_3, we will use both momentum and energy conservation. Using the fact that for a particle of mass m and momentum $\mathbf{p}$, $E^2 = |\mathbf{p}|^2 + m^2$, we find that the energies of m_1, m_2 and m_3 are

$$E_{m_1} = m_1 \tag{1}$$

$$E_{m_2} = \sqrt{m_2^2 + |\mathbf{p}_2|^2} \tag{2}$$

$$E_{m_2} = \sqrt{m_2^3 + |\mathbf{p}_3|^3} \tag{3}$$

Thus, $E_{m_1} = E_{m_2} + E_{m_3}$ implies that

$$m_1 = \sqrt{m_2^2 + |\mathbf{p}_2|^2} + \sqrt{m_2^3 + |\mathbf{p}_3|^3} \tag{4}$$

$$m_1 = \sqrt{m_2^2 + |\mathbf{p}_2|^2} + \sqrt{m_3^2 + |\mathbf{p}_2|^2} \tag{5}$$

where, in going from equation (4) to (5), we use the fact that $\mathbf{p}_2 = -\mathbf{p}_3$. Squaring both sides of equation (5), we find that,

$$m_1^2 = m_2^2 + |\mathbf{p}_2|^2 + m_3^2 + |\mathbf{p}_2|^2 + 2\sqrt{(m_2^2 + |\mathbf{p}_2|^2)(m_3^2 + |\mathbf{p}_2|^2)} \tag{6}$$

$$(m_1^2 - m_2^2 - m_3^2) - 2|\mathbf{p}_2|^2 = 2\sqrt{(m_2^2 + |\mathbf{p}_2|^2)(m_3^2 + |\mathbf{p}_2|^2)} \tag{7}$$

$$4|\mathbf{p}_2|^4 - 4|\mathbf{p}_2|^2(m_1^2 - m_2^2 - m_3^2) + (m_1^2 - m_2^2 - m_3^2)^2 = 4\left(m_2^2 + |\mathbf{p}_2|^2\right)\left(m_3^2 + |\mathbf{p}_2|^2\right) \tag{8}$$

Foiling out the right-hand side, we find

$$\begin{aligned} &4|\mathbf{p}_2|^4 - 4|\mathbf{p}_2|^2(m_1^2 - m_2^2 - m_3^2) + (m_1^2 - m_2^2 - m_3^2)^2 \\ &= 4m_2^2m_3^2 + 4(m_2^2 + m_3^2)|\mathbf{p}_2|^2 + 4|\mathbf{p}_2|^4 \end{aligned}$$

Simplifying, we find

$$-4|\mathbf{p}_2|^2 m_1^2 + (m_1^2 - m_2^2 - m_3^2)^2 = 4m_2^2m_3^2 \tag{9}$$

and hence

$$\begin{aligned} |\mathbf{p}_2| &= \frac{1}{2m_1}\sqrt{(m_1^2 - m_2^2 - m_3^2)^2 - 4m_2^2m_3^2} && (10) \\ &= \frac{1}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2} && (11) \\ &= |\mathbf{p}_3| && (12) \end{aligned}$$

Therefore,

$$\begin{aligned} E_{m_2}^2 &= \frac{1}{4m_1^2}\left(m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2\right) + m_2^2 && (13) \\ E_{m_3}^2 &= \frac{1}{4m_1^2}\left(m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2\right) + m_3^2 && (14) \end{aligned}$$

or

$$E_{m_2} = \frac{1}{2m_1}\left(m_1^2 + m_2^2 - m_3^2\right) \tag{15}$$

$$E_{m_3} = \frac{1}{2m_1}\left(m_1^2 - m_2^2 + m_3^2\right) \tag{16}$$

Part (b)

A positron of energy E pair-annihilates with a stationary electron producing two gamma rays. The mass of the positron is the same as the mass of the electron m, while photons are massless. Calculate the energy of the photons in the center of mass frame, as a function of the impinging positron energy E in the laboratory frame.

Solution

Suppose the positron is moving in the $\hat{x}$ direction with momentum $\mathbf{p} = p\hat{x}$. The four momentum of the system prior to the collision is given by

$$p_b^\mu = (E + m, p_x, 0, 0) \tag{17}$$

Now let's boost into the center of mass frame. This results in the following new four-momentum:

$$(p_b^\mu)' = (E', 0, 0, 0) \tag{18}$$

We know that $(p_b^\mu)'\,(p_{\mu,b})' = (p_b^\mu)\,(p_{\mu,b})$. Thus,

$$(E + m)^2 - p_x^2 = (E')^2 \tag{19}$$

Using the fact that $E^2 = p_x^2 + m^2$, we find that

$$(E + mc^2)^2 - (E^2 - m^2) = (E')^2 \tag{20}$$

$$\sqrt{2Em + 2m^2} = E' \tag{21}$$

Thus, the boosted four momentum of the system prior to the collision is:

$$(p_b^\mu)' = \left(\sqrt{2Em + 2m^2}, 0, 0, 0\right) \tag{22}$$

Assume that the photons move in the $\hat{x}$ direction after the annihilation. The four-momentum after the collision while thus be:

$$p_a^\mu = (E_1 + E_2, p_{x1} - p_{x2}, 0, 0) \tag{23}$$

Where E_1, p_{x1} and E_2, p_{x2} are the energies and momentums of photons 1 and 2. We require that the four-momentums be equal. Thus, $p_{1x} = p_{2x}$. Using the fact that $E = |\mathbf{p}|$ for a photon, this means that $E_1 = E_2$. Therefore,

$$2E_1 = 2E_2 = \sqrt{2Em + 2m^2} \tag{24}$$

or

$$E_1 = E_2 = \sqrt{\frac{Em + m^2}{2}} \tag{25}$$

Part (c)

Suppose one of the two photons is detected (in the laboratory frame) in the opposite direction to the incident positron: calculate the photon energy as a function of E and its limite for $E/m \gg 1$.

Solution

As mention is part (b), the four-momentum of the system prior to the annihilation is:

$$p_b^\mu = (E + m, p_x, 0, 0) \tag{26}$$

and after the annihilation

$$p_b^\mu = (E_1 + E_2, p_{1x} - p_{2x}, 0, 0) \tag{27}$$

Where $p_{1x} = E_1$ and $p_{1x} = E_2$. Then, equating the four-momenta, we find that

$$E_1 + E_2 = E + m \tag{28}$$

and using $E^2 = p_x^2 + m^2$,

$$E_1 - E_2 = p_x = \sqrt{E^2 - m^2} \tag{29}$$

Subtracting equation (29) from (28), we find that the energy of the detected photon is

$$E_2 = \frac{1}{2}\left(E + m - \sqrt{E^2 - m^2}\right) \tag{30}$$

and the energy of the other photon is

$$E_1 = \frac{1}{2}\left(E + m + \sqrt{E^2 - m^2}\right) \tag{31}$$

Now, in the limit of $E/m^2 \gg 1$, we have

$$E_2 = \frac{1}{2}\left(E + m - \sqrt{E^2 - m^2}\right) \tag{32}$$

$$= \frac{E}{2}\left(1 + \frac{m}{E} - \left(1 - \left(\frac{m}{E}\right)^2\right)^{-1/2}\right) \tag{33}$$

Using the binomial approximation, we have that

$$E_2 = \frac{E}{2}\left(1 + \frac{m}{E} - \left(1 - \left(\frac{m}{E}\right)^2\right)^{-1/2}\right) \tag{34}$$

$$= \frac{E}{2}\left(1 + \frac{m}{E} - 1 - \frac{1}{2}\left(\frac{m}{E}\right)^2\right) \tag{35}$$

$$= \frac{E}{2}\left(\frac{m}{E} - \frac{1}{2}\left(\frac{m}{E}\right)^2\right) \tag{36}$$

Keeping only terms first order in m/E, we find that the energy of the detected photon is $E_2 = m/2$. Additionally, we would get $E_2 = E + m/2$.

Part (d)

Suppose one of the two photons is detected in the orthogonal direction to the original positron direction: calculate the energy of this photon.

Solution

As stated in the previous two problems, the four-momentum of the system prior to the collision is

$$p_b^\mu = \left(\frac{E + mc^2}{c}, p_x, 0, 0\right) \tag{37}$$

The four-momentum after the collision is

$$p_a^\mu = \left(\frac{E_1 + E_2}{c}, p_2 \cos\theta, p_1 - p_1 \sin\theta, 0\right) \tag{38}$$

where p_1 is the momentum of the photon detected in the orthogonal direction of the positron, p_2 is the momentum of the second photon and θ is the angle the second photon makes with the direction of the positron. Since the two four-momenta must be equal, we find

$$E_1 + E_2 = E + mc^2 \tag{39}$$

$$p_x = p_2 \cos\theta \qquad \Longrightarrow \qquad \sqrt{E^2 - m^2c^4} = E_2 \cos\theta \tag{40}$$

and

$$p_1 - p_2 \sin\theta = 0 \qquad \Longrightarrow \qquad E_1 = E_2 \sin\theta \tag{41}$$

Squaring (40) and (41) and adding the resulting equation, we find that

$$E_2^2 = E_1^2 + E^2 - m^2c^4 \tag{42}$$

Subtracting E_1 from both sides of equation (39) and squaring both sides, we find that

$$E_2^2 = E_1^2 - 2E_1(E + mc^2) + (E + mc^2)^2 \tag{43}$$

Equating equations 42 and 43 and solving for E_1, we find that

$$E_1^2 - 2E_1(E + mc^2) + (E + mc^2)^2 = E_1^2 + E^2 - m^2c^4 \tag{44}$$

$$2E_1(E + mc^2) = (E + mc^2)^2 + m^2c^4 - E^2 \tag{45}$$

$$2E_1(E + mc^2) = 2Emc^2 + 2m^2c^4 \tag{46}$$

$$E_1(E + mc^2) = mc^2\left(E + mc^2\right) \tag{47}$$

$$E = mc^2 \tag{48}$$