

Solution to Homework Set #1, Problem #2.

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Part a

We recall the well-known relation regarding the composition of Dirac's function with other functions,

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \quad (1)$$

where the x_i 's are the zeros of g . For our case, we note,

$$\delta((k^0)^2 - (|\vec{k}|^2 + m^2)) = \frac{\delta(k^0 + \omega_k)}{2\omega_k} + \frac{\delta(k^0 - \omega_k)}{2\omega_k} \quad (2)$$

where $\omega_k \equiv \sqrt{|\vec{k}|^2 + m^2}$. I will now add a test function $g(k^0)$ to our integral in order to emphasize that the Dirac-delta function is formally defined as a distribution.

$$I = \frac{1}{2\omega_k} \int_{-\infty}^{\infty} dk^0 [\delta(k^0 + \omega_k) + \delta(k^0 - \omega_k)] \Theta(k^0) g(k^0) \quad (3)$$

Recall the fact that $\Theta(k^0)$ is only nonzero for $k^0 > 0$ (using the Schwartz variation on the Heaviside function), and equals 1 for $k^0 > 0$, and so only one of the Dirac deltas operate.

$$I = \frac{1}{2\omega_k} \int_{-\infty}^{\infty} dk^0 \delta(k^0 - \omega_k) \Theta(k^0) g(k^0) \quad (4)$$

$$I = \frac{1}{2\omega_k} \int_0^{\infty} dk^0 \delta(k^0 - \omega_k) g(k^0) \quad (5)$$

$$= \frac{1}{2\omega_k} g(\omega_k) \quad (6)$$

Then since we used a general function g it follows immediately that we can think of the integral over the distribution itself as $\frac{1}{2\omega_k}$:

$$\int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2) \Theta(k^0) = \frac{1}{2\omega_k} \quad (7)$$

Part b

To show that the measure is invariant we must recall how measures transform in the first place. It is derived in multivariate calculus that measures transform as the determinant of the Jakobian J ,

$$dn'_1 dn'_2 \cdots = |\det(J)| dn_1 dn_2 \cdots \quad (8)$$

where J is the matrix defined as follows. For some transformation $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$,

$$J_{mn} = \frac{\partial f_m}{\partial n_n}. \quad (9)$$

In the Lorentz case,

$$k'^{\mu} = \Lambda_{\nu}^{\mu} k^{\nu} \quad (10)$$

$$dk'^{\mu} = \Lambda_{\nu}^{\mu} dk^{\nu} \quad (11)$$

Or if you allow me to write this in a very suggestive way,

$$dk'^{\mu} = \Lambda_0^{\mu} dk^0 + \Lambda_1^{\mu} dk^1 \dots \quad (12)$$

$$= \frac{\partial k'^{\mu}}{\partial k^0} dk^0 + \frac{\partial k'^{\mu}}{\partial k^1} dk^1 \dots \quad (13)$$

it becomes evident that the Lorentz transformation is its own Jakobian,

$$\frac{\partial k'^{\mu}}{\partial k^{\nu}} = \Lambda_{\nu}^{\mu} \quad (14)$$

The Lorentz transformation has determinant one or minus one. For physical transformations we force the measure to stay positive (positive volume), as suggested by using the magnitude of the determinant of the Jakobian in equation 7, and this achieves for us true invariance.

$$d^4 k' = \left| \det \left(\frac{\partial k'^{\mu}}{\partial k^{\nu}} \right) \right| d^4 k \quad (15)$$

$$= |\det(\Lambda_{\nu}^{\mu})| d^4 k \quad (16)$$

$$= d^4 k \quad (17)$$

Part c

We want to show $\int \frac{d^3k}{2\omega_k}$ is invariant under Lorentz transformations. It will suffice to show that

$$\int d^4k \delta(k^2 - m^2) \Theta(k^0) g(k^0) \quad (18)$$

is invariant¹ Then we can just perform the k^0 integral as we did in part a and we'll have our answer.

From part b, we already have that the integral measure is invariant. Also, $k^2 - m^2$ being invariant Lorentz scalars forces $\delta(k^2 - m^2)$ to be invariant since it is evaluated at the same place in all reference frames.

The only tension we have here is this final part of the integrand, $\Theta(k^0)$. Clearly, k^0 is not invariant. We will have to see how it transforms and see if it does not affect the integration. Consider the transformation,

$$\int d^4k \delta(k^2 - m^2) \Theta(k^0) g(k^0) \rightarrow \int d^4k' \delta(k'^2 - m^2) \Theta(k'^0) g(k'^0). \quad (19)$$

Let me now show the transformed integral is indeed the same as the original. Any Lorentz transformation can be written as a combination of *TBR* (time-reversal, boost and rotation). Working on the transformed integral,

$$\int d^4k' \delta(k'^2 - m^2) \Theta(k'^0) g(k'^0) = \int d^4k \delta(k^2 - m^2) \Theta(\gamma(\pm k^0 - vk^1)) g(\gamma(\pm k^0 - vk^1)) \quad (20)$$

where I have used the simplifications mentioned above. Also I used the fact that I can align my x axis with any possible boost and that rotations do not affect my k^0 term. Using the dirac delta trick again gives us,

$$\int d^4k \frac{1}{2\omega_k} (\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k)) \Theta(\gamma(\pm k^0 - vk^1)) g(\gamma(\pm k^0 - vk^1)). \quad (21)$$

So the question becomes: are the dirac deltas still going to work or is the new constraint on the theta function ruining it. In particular, we need only one dirac delta to be turned on

¹From now on, if the limits of integration are suppressed, then we are integrating over all possible values of k^μ .

and the other to be turned off by the theta function. The first dirac delta forces $k^0 = \omega_k = \sqrt{k^2 + m^2}$. Consider the inequality,

$$\omega_k = \sqrt{(k^1)^2 + (k^2)^2 + (k^3)^2 + m^2} \quad (22)$$

$$\geq vk_1 \quad (23)$$

since $v \in (-1, 1)$. Similarly,

$$-\omega_k = -\sqrt{(k^1)^2 + (k^2)^2 + (k^3)^2 + m^2} \quad (24)$$

$$\leq vk_1 \quad (25)$$

From these inequalities it becomes evident that only one dirac delta will operate for any given Lorentz transformation. Let me explicitly show it here:

$$I = \int d^4k \frac{1}{2\omega_k} \delta(k^0 - \omega_k) \Theta(\gamma(\pm\omega_k - vk^1)) g(\gamma(\pm\omega_k - vk^1)) + \quad (26)$$

$$\int d^4k \frac{1}{2\omega_k} \delta(k^0 + \omega_k) \Theta(\gamma(\pm - \omega_k - vk^1)) g(\gamma(\pm - \omega_k - vk^1)). \quad (27)$$

In the plus case, the theta function forces the bottom integral to be zero. In the minus case, the theta function forces the top integral to be zero. For both cases the integral simplifies to

$$\int \frac{d^3k}{2\omega_k} g(\omega'_k). \quad (28)$$

Notice that g need not be invariant and so we evaluate it at $\omega'_k \equiv \gamma(\omega_k - vk^1)$. Also notice the distribution forced ω'_k to be positive in g (suppressed the time reversal). In any case, we showed the the integral over the distribution is invariant.

$$\int \frac{d^3k}{2\omega_k} = \int \frac{d^3k'}{2\omega'_k} \quad (29)$$

The Canzano Consideration

N. Canzano pointed out in class a more efficient way to go about deriving part c. If we consider the transformed integral before changing the dirac delta,

$$I = \int d^3k \int_{-\infty}^{\infty} dk^0 \delta(k^2 - m^2) \Theta(k^0) g(k^0) \quad (30)$$

where we have used the invariance of the measure and delta function argument. The worst thing that can happen is that the argument in the theta function changes sign. Why? If $\text{sign}(k') = \text{sign}(k^0)$ for all k^0 the theta function is invariant and simply forces the integration to occur for the positive real axis as before, and we're done:

$$I = \int d^3k \int_0^{\infty} dk^0 \delta(k^2 - m^2) g(k^0) \quad (31)$$

So the danger is that it flips sign. But if the transformation flips sign, it *must* flip sign for all k^0 by the nature of the Lorentz transformation.

But one might argue, "alas, this seems like quite a bit to demand!" It is, if you are not on shell. It is not, if you are on shell. Recall the general Lorentz transformation on k^0 is as follows:

$$k'^0 = \Lambda_0^0 k^0 + \Lambda_1^0 k^1 + \Lambda_2^0 k^2 + \Lambda_3^0 k^3 \quad (32)$$

If you are not on shell, you can adjust k^i as you like. Indeed, we are integrating over all k^i . So we have no constraint on the sign of k'^0 depending on the sign of k^0 .

It is only once we are on shell (when we allow the dirac deltas to force k^0 to ω_k or $-\omega_k$) that we can talk about the sign of k' flipping or not, as noted in the part c derivation inequalities.

But if you have the foresight to note that the Dirac deltas are precisely there to keep you on shell, you can continue via the Canzano consideration and finish the derivation promptly.

$$I = \int d^3k \int_{-\infty}^0 dk^0 \delta(k^2 - m^2) g(k^0) = \int d^3k \int_0^{\infty} dk^0 \delta(k^2 - m^2) g(-k^0) \quad (33)$$

where I have used the delta function is even. Indeed, the Canzano consideration is very similar to that developed in part c, and even faster, because it uses the evenness of the delta function as opposed to actually using the delta functions. All you need is the foresight of what it means to be on shell to be able to use it.

However, there is an interesting subtlety that is missed using this method. While this argument holds for showing invariance it does not give us the extra constraint on g we got in the method presented first (that time reversal is suppressed).

“I find the work hard, thank God, & almost pleasant.”

- Oppenheimer, on physics.