Solution to Homework Set #2, Problem #1.

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1) a. As a close variation of this question is solved in the text, some parts of the solution will be left out. Starting with the given Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2 - A_\mu J_\mu \tag{1}$$

Now applying the Euler-Lagrange equations, one obtains:

$$m^2 A_\nu - J_\nu + \partial_\mu F_{\mu\nu} = 0 \tag{2}$$

Now, expanding the last term on the RHS and rearranging:

$$\partial^2 A_\nu - \partial_\nu(\partial_\mu A_\mu) = J_\nu - m^2 A_\nu \tag{3}$$

Now taking the "divergence" of both sides...

$$\partial_\nu \partial^2 A_\nu - \partial_\nu \partial_\nu \partial_\mu A_\mu = \partial_\nu J_\nu - m^2 \partial_\nu A_\nu \tag{4}$$

The partial derivatives in the second term can be rearranged (as they commute), simplyifying things:

$$\partial_\nu \partial^2 A_\nu - \partial_\mu \partial^2 A_\mu = \partial_\nu J_\nu - m^2 \partial_\nu A_\nu \tag{5}$$

This gives:

$$0 = \partial_\nu J_\nu - m^2 \partial_\nu A_\nu \tag{6}$$

Which implies the Lorentz gauge choice. In this case it also means the source current is conserved:

$$\partial_\nu A_\nu = 0 \tag{7}$$

b.

$$\Box A_\nu = J_\nu - m^2 A_\nu \tag{8}$$

$$(\Box + m^2)A_\nu = J_\nu \tag{9}$$

$$A_\nu = \frac{J_\nu}{\Box + m^2} \tag{10}$$

Using the source current for a point charge: $J_0 = e\delta^3(x), J_{1,2,3} = 0$
The only non-zero component of A will be A_0...

$$A_0 = \frac{e\delta^3(x)}{\Box + m^2} = \int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m^2}e^{i\vec{k}\cdot\vec{x}} = \frac{e}{(2\pi)^2}\int_0^\infty \int_{-1}^1 \frac{k^2}{k^2+m^2}e^{ikrcos\theta}dcos\theta dk \tag{11}$$

After evaluating the theta integral one has:

$$= \frac{e}{(2\pi)^2}\int_0^\infty \frac{k^2}{k^2+m^2}\frac{e^{ikr}-e^{-ikr}}{ikr}dk \tag{12}$$

Note: the second term in the integrand is identical to the first upon a change of variables like k = -y with the integration limits being from 0 to -∞.

$$\frac{e}{(2\pi)^2 ir}\int_{-\infty}^\infty \frac{k}{k^2+m^2}e^{ikr}dk \tag{13}$$

This integral has poles at $k = \pm ik$. However, completing the contour in the upper plane means only the first pole's residue must be computed.

$$R(z = im) = \frac{ime^{-mr}}{2im} = \frac{e^{-mr}}{2} \tag{14}$$

This means......

$$A_0 = \frac{2\pi i e^{-mr}}{2(2\pi)^2 ir} = \frac{e}{4\pi r}e^{-mr} \tag{15}$$

c. This is great because in the limit is $m \to 0 : A_0 \to \frac{e}{4\pi r}$, which is the Coulomb potential from before.

d. The Yukawa potential is a good candidate for the force between protons because its extent is finite (governed by the exponential decay above) like the strong force and it is attractive. Since the range of the strong force is about the size of a nucleus ∼1fm the mass is dictated by that:

m $\sim \frac{\hbar c}{r} = 200 MeV$