HW 2, No. 2 - Complex Scalar Field Obeying Klein-Gordon Equation

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Given the action

$$S = \int d^4x(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) \tag{1}$$

(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3\vec{x}(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi) \tag{2}$$

compute the Heisenberg equation of motion and show that it is indeed the Klein-Gordon Equation.

Identifying the Lagrangian density $\mathcal{L}$ as the integrand of (1), to find the conjugate momenta is just to take derivatives,

$$\begin{aligned} \pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* \\ \pi^* &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} \end{aligned} \tag{3}$$

These fields will naturally satisfy the equal-time commutation relations

$$\begin{aligned} [\phi(x), \pi(y)] = [\phi^*(x), \pi^*(y)] = i\delta^{(3)}(y - x) \\ [\phi(x), \pi^*(y)] = [\phi^*(x), \phi(y)] = [\pi^*(x), \pi(y)] = 0. \end{aligned} \tag{4}$$

To verify the Hamiltonian, Legendre transform the Lagrangian,

$$\mathcal{H} = \pi\dot{\phi}[\phi, \phi^*, \pi, \pi^*] + \pi^* \dot{\phi}^*[\phi, \phi^*, \pi, \pi^*] - \mathcal{L}[\phi, \phi^*, \pi, \pi^*] \tag{5}$$

From the conjugate momenta (3) we found earlier, we already have everything we need to transform the Hamiltonian, since

$$\begin{aligned}\mathcal{L} &= \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \\ &= \dot{\phi}^* \dot{\phi} - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi \\ &= \pi \pi^* - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi\end{aligned} \tag{6}$$

is already in terms of ϕ, ϕ^*, π and π^*. Plugging into (5), then,

$$\begin{aligned}\mathcal{H} &= \pi \pi^* + \pi^* \pi - (\pi \pi^* - \nabla \phi^* \cdot \nabla \phi - m^2 \phi^* \phi) \\ &= \pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi\end{aligned} \tag{7}$$

and we recover (2),

$$H = \int d^3\vec{x}\ \mathcal{H} \stackrel{\checkmark}{=} \int d^3\vec{x} (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi)$$

The Heisenberg equations of motion are

$$\begin{aligned}\partial_t \phi - i[H, \phi] = 0 \\ \partial_t \pi - i[H, \pi] = 0\end{aligned} \tag{8}$$

with the other two starred versions of the above left implicit. Computing the equation of motion for ϕ only confirms what we found before, namely that

$$\begin{aligned}\dot{\phi}(x) &= i[H, \phi(x)] \\ &= i \int d^3\vec{y} [\pi^*(y)\pi(y) + \nabla \phi^*(y) \cdot \nabla \phi(y) + m^2 \phi^*(y)\phi(y), \phi(x)] \\ &= i \int d^3\vec{y} [\pi^*(y)\pi(y), \phi(x)] \\ &= i \int d^3\vec{y}\, (\pi^*(y)[\pi(y), \phi(x)] + [\pi^*(y), \phi(x)]\pi(y)) \\ &= i \int d^3\vec{y} \left(\pi^*(y)(-i\delta^{(3)}(x-y)) + 0 \right) \\ &= \pi^*(x)\end{aligned} \tag{9}$$

where the commutator identity $[AB, C] = A[B, C] + [A, C]B$ has been used. The conjugate momentum equation gives up what we need to show that ϕ and

ϕ^* obey the Klein-Gordon equation.

$$\begin{aligned}
\dot{\pi}(x) &= i[H, \pi(x)] \\
&= i \int d^3\vec{y}[\pi^*(y)\pi(y) + \nabla\phi^*(y) \cdot \nabla\phi(y) + m^2\phi^*(y)\phi(y), \pi(x)] \\
&= i \int d^3\vec{y} \left([\nabla\phi^*(y) \cdot \nabla\phi(y), \pi(x)] + m^2[\phi^*(y)\phi(y), \pi(x)]\right) \\
&= i \int d^3\vec{y} \left(\nabla\phi^*(y) \cdot \nabla_y[\phi(y), \pi(x)] + \nabla_y[\phi^*(y), \pi(x)] \cdot \nabla\phi(y) + m^2\phi^*(y)[\phi(y), \pi(x)]\right) \\
&= -\int d^3\vec{y} \left(\nabla\phi^*(y) \cdot \nabla_y\delta^{(3)}(y-x) + m^2\delta^{(3)}(y-x)\phi^*(y)\right) \\
\dot{\pi}(x) &= \nabla^2\phi^*(x) - m^2\phi^*(x)
\end{aligned} \tag{10}$$

having integrated by parts in the final step. Combining this with what we reproduced in (9),

$$\begin{aligned}
\dot{\pi} = \partial_t\dot{\phi}^* &= \nabla^2\phi^* - m^2\phi^* \\
(\partial_t^2 - \nabla^2)\phi^* + m^2\phi^* &= 0 \\
(\Box + m^2)\phi^* &= 0
\end{aligned} \tag{11}$$

the same going, of course, for ϕ.

(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

Since ϕ and ϕ^* are independent, it is intuitive that the theory should contain two sets of creation and annihilation operators, one set for ϕ and one for ϕ^*; denote them by $a_{\vec{p}}, a_{\vec{p}}^\dagger$ and $b_{\vec{p}}, b_{\vec{p}}^\dagger$, respectively, say. In analogy with the relation between the canonical commutation relations for position and momentum in quantum mechanics and the corresponding canonical commutation relations for the raising and lowering operators of the simple harmonic oscillator, we expect the algebra involving the sets of creation and annihilation operators to go something like

$$\begin{aligned}
[a_{\vec{p}}, a_{\vec{k}}^\dagger] &= [b_{\vec{p}}, b_{\vec{k}}^\dagger] \propto \delta^{(3)}(\vec{k} - \vec{p}) \\
[a_{\vec{p}}, b_{\vec{k}}^\dagger] &= [a_{\vec{p}}, b_{\vec{k}}] = 0
\end{aligned} \tag{12}$$

up to some conventional prefactor. Let $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. If one defines

$$\begin{aligned}
a_{\vec{p}} &= \int d^3\vec{x} e^{ipx} \left(\omega_{\vec{p}}\phi(x) + i\pi^*(x)\right) \\
a_{\vec{p}}^\dagger &= \int d^3\vec{x} e^{-ipx} \left(\omega_{\vec{p}}\phi^*(x) - i\pi(x)\right) \\
b_{\vec{p}} &= \int d^3\vec{x} e^{ipx} \left(\omega_{\vec{p}}\phi^*(x) + i\pi(x)\right) \\
b_{\vec{p}}^\dagger &= \int d^3\vec{x} e^{-ipx} \left(\omega_{\vec{p}}\phi(x) - i\pi^*(x)\right)
\end{aligned} \tag{13}$$

Then

$$
\begin{aligned}
[a_{\vec{p}}, a^\dagger_{\vec{k}}] &= \int d^3\vec{x} \int d^3\vec{y} e^{i(p-k)x} [\omega_{\vec{p}}\phi(x) + i\pi^*(x), \omega_{\vec{k}}\phi^*(y) - i\pi(y)] \\
&= \int d^3\vec{x} \int d^3\vec{y} e^{i(p-k)x} \left(-i\omega_{\vec{p}}[\phi(x), \pi(y)] + i\omega_{\vec{k}}[\pi^*(x), \phi^*(y)]\right) \\
&= \int d^3\vec{x} \int d^3\vec{y} e^{i(p-k)x} (\omega_{\vec{p}} + \omega_{\vec{k}}) \delta^{(3)}(x-y) \\
&= \int d^3\vec{x} e^{i(p-k)y} (\omega_{\vec{p}} + \omega_{\vec{k}}) \\
&= 2(2\pi)^3 \omega_{\vec{p}} \delta^{(3)}(\vec{k} - \vec{p})
\end{aligned}
\tag{14}
$$

Schwartz would rather not leave in the factor of $2\omega_{\vec{p}}$, apparently, and so he would likely multiply the operators in (13) by a factor of $(2\omega_{\vec{p}})^{-1/2}$, giving us an extra $\frac{1}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{k}}}}$ out front in the commutator that ultimately cancels the $2\omega_{\vec{p}}$. The price he pays in following this convention is that this neglected factor shows up in the definition of momentum eigenstates, $a^\dagger_{\vec{p}}|0\rangle = \frac{1}{\sqrt{2\omega_{\vec{p}}}}|\vec{p}\rangle$. Whichever we choose, it will become clear if it is not already that either convention or others still would work equally well so long as it is self-consistent. For $b_{\vec{p}}$ and $b^\dagger_{\vec{p}}$, we have

$$
\begin{aligned}
[b_{\vec{p}}, b^\dagger_{\vec{k}}] &= \int d^3\vec{x} \int d^3\vec{y} e^{i(p-k)x} [\omega_{\vec{p}}\phi^*(x) + i\pi(x), \omega_{\vec{k}}\phi(y) - i\pi^*(y)] \\
&= \int d^3\vec{x} \int d^3\vec{y} e^{i(p-k)x} \left(-i\omega_{\vec{p}}[\phi^*(x), \pi^*(y)] + i\omega_{\vec{k}}[\pi(x), \phi(y)]\right)
\end{aligned}
\tag{15}
$$

after which we can see the calculation will be identical to that in (14). This verifies the first line of (12); the second line is easily verified in the same way. Inverting the definitions (13), one has

$$
\begin{aligned}
\phi(x) &= \int d\tilde{p} \left(a_{\vec{p}} e^{-ipx} + b^\dagger_{\vec{p}} e^{ipx}\right) \\
\phi^*(x) &= \int d\tilde{p} \left(a^\dagger_{\vec{p}} e^{ipx} + b_{\vec{p}} e^{-ipx}\right) \\
\pi(x) &= i \int d\tilde{p} \omega_{\vec{p}} \left(a^\dagger_{\vec{p}} e^{ipx} - b_{\vec{p}} e^{-ipx}\right) \\
\pi^*(x) &= -i \int d\tilde{p} \omega_{\vec{p}} \left(a_{\vec{p}} e^{-ipx} - b^\dagger_{\vec{p}} e^{ipx}\right)
\end{aligned}
\tag{16}
$$

where $d\tilde{p} \equiv \frac{d^3\vec{p}}{2(2\pi)^3\omega_{\vec{p}}}$. We can now begin our trudge through the drudgery of finding H in terms of the creation and annihilation operators. The first step is to gather the necessary ingredients.

$$
\nabla\phi(x) = \int d\tilde{p} \nabla_x \left(a_{\vec{p}} e^{-ipx} + b^\dagger_{\vec{p}} e^{ipx}\right) = \int d\tilde{p} (i\vec{p}) (a_{\vec{p}} e^{-ipx} - b^\dagger_{\vec{p}} e^{ipx})
$$

$$
\nabla\phi^*(x) = \int d\tilde{k} (-i\vec{k}) (a^\dagger_{\vec{k}} e^{ikx} - b_{\vec{k}} e^{-ikx})
$$

$$\int d^3\vec{x}(\nabla\phi^*\cdot\nabla\phi) = \iiint d^3\vec{x}d\tilde{p}d\tilde{k}(\vec{p}\cdot\vec{k})\{a_{\vec{p}}a^\dagger_{\vec{k}}e^{i(k-p)x} - b^\dagger_{\vec{p}}a^\dagger_{\vec{k}}e^{-i(k+p)x} \\ - \ a_{\vec{p}}b_{\vec{k}}e^{i(k+p)x} + b^\dagger_{\vec{p}}b_{\vec{k}}e^{-i(k-p)x}\} \tag{17}$$

$$\int d^3\vec{x}(\pi^*\pi) = \iiint d^3\vec{x}d\tilde{p}d\tilde{k}\omega_{\vec{p}}\omega_{\vec{k}}\{a_{\vec{p}}a^\dagger_{\vec{k}}e^{i(k-p)x} - b^\dagger_{\vec{p}}a^\dagger_{\vec{k}}e^{-i(k+p)x} \\ - \ a_{\vec{p}}b_{\vec{k}}e^{i(k+p)x} + b^\dagger_{\vec{p}}b_{\vec{k}}e^{-i(k-p)x}\} \tag{18}$$

$$\int d^3\vec{x}(m^2\phi^*\phi) = \iiint d^3\vec{x}d\tilde{p}d\tilde{k}m^2\{a^\dagger_{\vec{p}}a_{\vec{k}}e^{-i(k-p)x} + b_{\vec{p}}a_{\vec{k}}e^{-i(k+p)x} \\ + a^\dagger_{\vec{p}}b^\dagger_{\vec{k}}e^{i(k+p)x} + b_{\vec{p}}b^\dagger_{\vec{k}}e^{i(k-p)x}\} \tag{19}$$

Summing equations (17) through (19) gives us H by equation (2). We can take $t = 0$ without loss of generality since H is conserved: $\partial_t H = i[H, H] = 0$. In evaluating the $d^3\vec{x}$ integral, we pick up factors of $(2\pi)^3\delta^{(3)}(\vec{p}+\vec{k})$ and $(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})$ due to $\int d^3\vec{x}e^{\pm i(\vec{k}\pm\vec{p})\cdot\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{k}\pm\vec{p})$, neutralizing the factor of $(2\pi)^{-3}$ coming from $d\tilde{k}$. When the terms attached to $\delta^{(3)}(\vec{p}+\vec{k})$ succumb to the delta function, $\vec{p} = -\vec{k}$, and $\omega_{\vec{p}}\omega_{\vec{k}} \to \omega^2_{\vec{p}}$ while $\vec{p}\cdot\vec{k} \to -\vec{p}^2$. Then the coefficient of $\delta^{(3)}(\vec{p}+\vec{k})$, $-\vec{p}\cdot\vec{k} - \omega_{\vec{p}}\omega_{\vec{k}} + m^2 \to \vec{p}^2 - (\vec{p}^2 + m^2) + m^2 = 0$, and so the terms like cd and $c^\dagger d^\dagger$ vanish under the $\frac{d^3\vec{k}}{2\omega_{\vec{k}}}$ integral and in view of (12). When the remaining delta function is enforced, on the other hand, $\vec{p} = \vec{k}$ and the terms that survive are all positive, involving the operators like $c^\dagger c$ and $dd^\dagger$. Piecing all of this together, we have reduced H to

$$H = \int \frac{d\tilde{p}}{2\omega_{\vec{p}}}\left\{\left(\vec{p}^2 + (\vec{p}^2 + m^2)\right)(a_{\vec{p}}a^\dagger_{\vec{p}} + b^\dagger_{\vec{p}}b_{\vec{p}}) + m^2(a^\dagger_{\vec{p}}a_{\vec{p}} + b_{\vec{p}}b^\dagger_{\vec{p}})\right\} \tag{20}$$

Commuting away the terms like $cc^\dagger$ results in infinite but constant and therefore ignorable terms added on to the Hamiltonian. In doing so, we also pick up a $(\vec{p}^2 + (\vec{p}^2 + m^2))a^\dagger_{\vec{p}}a_{\vec{p}}$ and an $m^2 b^\dagger_{\vec{p}}b_{\vec{p}}$, turning the above into

$$\begin{aligned} H &= \int \frac{d\tilde{p}}{2\omega_{\vec{p}}}\left\{(\vec{p}^2 + (\vec{p}^2 + m^2) + m^2)(a^\dagger_{\vec{p}}a_{\vec{p}} + b^\dagger_{\vec{p}}b_{\vec{p}})\right\} \\ &= \int \frac{d^3\vec{p}}{(2\omega_{\vec{p}})^2(2\pi)^3}2\omega^2_{\vec{p}}(a^\dagger_{\vec{p}}a_{\vec{p}} + b^\dagger_{\vec{p}}b_{\vec{p}}) \end{aligned} \tag{21}$$

Which is, finally,

$$H = \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3}(a^\dagger_{\vec{p}}a_{\vec{p}} + b^\dagger_{\vec{p}}b_{\vec{p}}) \tag{22}$$

We should check that this does what we want. Note that the convention chosen earlier, $[a_{\vec{p}}, a^\dagger_{\vec{k}}] = (2\pi)^3(2\omega_{\vec{p}})\delta^{(3)}(\vec{k}-\vec{p})$ affords us the definition

$$b^\dagger_{\vec{p}'}a^\dagger_{\vec{p}}|0\rangle = |\vec{p};\vec{p}'\rangle \tag{23}$$

In that case

$$\begin{aligned}
H|\vec{k};\vec{k}'\rangle &= \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3}\left(a^\dagger_{\vec{p}}a_{\vec{p}} + b^\dagger_{\vec{p}}b_{\vec{p}}\right)b^\dagger_{\vec{k}'}a^\dagger_{\vec{k}}|0\rangle \\
&= \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3}\left(b^\dagger_{\vec{k}'}a^\dagger_{\vec{p}}a_{\vec{p}}a^\dagger_{\vec{k}}|0\rangle + a^\dagger_{\vec{k}}b^\dagger_{\vec{p}}b_{\vec{p}}b^\dagger_{\vec{k}'}|0\rangle\right) \\
&= \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3}\left(b^\dagger_{\vec{k}'}a^\dagger_{\vec{p}}[a^\dagger_{\vec{k}}a_{\vec{p}} + (2\pi)^3(2\omega_{\vec{p}})\delta^{(3)}(\vec{p}-\vec{k})]|0\rangle + a^\dagger_{\vec{k}}b^\dagger_{\vec{p}}b_{\vec{p}}b^\dagger_{\vec{k}'}|0\rangle\right) \\
&= \int d^3\vec{p}\,\omega_{\vec{p}}(\delta^{(3)}(\vec{p}-\vec{k})|\vec{p};\vec{k}'\rangle + \delta^{(3)}(\vec{p}-\vec{k}')|\vec{k};\vec{p}\rangle) \\
H|\vec{k};\vec{k}'\rangle &= (\omega_{\vec{k}} + \omega_{\vec{k}'})|\vec{k};\vec{k}'\rangle
\end{aligned} \tag{24}$$

as we expect.

(c) Rewrite the conserved charge,

$$Q = \frac{i}{2}\int d^3\vec{x}(\phi^*\pi^* - \pi\phi) \tag{25}$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

Looking at the form of the Hamiltonian, one can imagine that if we had double the number of particles, or triple the number of particles of some momenta $\vec{k}$ and $\vec{k}'$, that the eigenvalue of H would increase two- or threefold. Dividing out the respective energy eigenvalues would then give us the number of particles, and we can therefore identify the number operator

$$N = \int d\tilde{p}(a^\dagger_{\vec{p}}a_{\vec{p}} + b^\dagger_{\vec{p}}b_{\vec{p}}) \tag{26}$$

which can be interpreted as $N = N_\phi + N_{\phi^*}$. If the particles ϕ and ϕ^* are charged, then by conservation of charge they must have opposite sign, and the total charge Q should be proportional to $N_\phi - N_{\phi^*}$,

$$Q \propto \int d\tilde{p}(a^\dagger_{\vec{p}}a_{\vec{p}} - b^\dagger_{\vec{p}}b_{\vec{p}}) \tag{27}$$

This works also if the particles were to be neutral, in which case $Q = 0$. It remains to verify this claim. Using the definitions (13),

$$\begin{aligned}
\int d\tilde{p}a^\dagger_{\vec{p}}a_{\vec{p}} = \iiint d\tilde{p}d^3\vec{x}d^3\vec{y}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\{&\omega^2_{\vec{p}}\phi^*(\vec{x})\phi(\vec{y}) + i\omega_{\vec{p}}\phi^*(\vec{x})\pi^*(\vec{y}) \\
&- i\omega_{\vec{p}}\pi(\vec{x})\phi(\vec{y}) + \pi(\vec{x})\pi^*(\vec{y})\}
\end{aligned} \tag{28}$$

$$\begin{aligned}
\int d\tilde{p}b^\dagger_{\vec{p}}b_{\vec{p}} = \iiint d\tilde{p}d^3\vec{x}d^3\vec{y}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\{&\omega^2_{\vec{p}}\phi(\vec{x})\phi^*(\vec{y}) + i\omega_{\vec{p}}\phi(\vec{x})\pi(\vec{y}) \\
&- i\omega_{\vec{p}}\pi^*(\vec{x})\phi^*(\vec{y}) + \pi^*(\vec{x})\pi(\vec{y})\}
\end{aligned} \tag{29}$$

Switching $\vec{x}$ with $\vec{y}$ and letting $\vec{p} \to -\vec{p}$ in (29),

$$\int d\tilde{p} b^\dagger_{\vec{p}} b_{\vec{p}} = \iiint d\tilde{p} d^3\vec{x} d^3\vec{y} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \{\omega^2_{\vec{p}} \phi(\vec{y})\phi^*(\vec{x}) + i\omega_{\vec{p}}\phi(\vec{y})\pi(\vec{x}) \\ - i\omega_{\vec{p}}\pi^*(\vec{y})\phi^*(\vec{x}) + \pi^*(\vec{y})\pi(\vec{x})\} \tag{30}$$

Then

$$\int d\tilde{p}(a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}}) = \\ \iiint d\tilde{p} d^3\vec{y} d^3\vec{x} \{\omega^2_{\vec{p}}[\phi^*(\vec{x}), \phi(\vec{y})] + [\pi(\vec{x}), \pi^*(\vec{y})] \\ + i\omega_{\vec{p}}(\{\phi^*(\vec{x}), \pi^*(\vec{y})\} - \{\pi(\vec{x}), \phi(\vec{y})\})\} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$= \iint \frac{d^3\vec{x} d^3\vec{y}}{2\omega_{\vec{p}}} \delta^{(3)}(\vec{x}-\vec{y}) i\omega_{\vec{p}} (\phi^*(\vec{x})\pi^*\vec{y} + \pi * (\vec{y})\phi^*(\vec{x}) - \phi(\vec{y})\pi(\vec{x})) \\ = \frac{i}{2}\int d^3\vec{x}((\phi^*\pi^* - \pi\phi) + (\pi^*\phi^* - \phi\pi)) \tag{31}$$

Here we can commute away the term $(\pi^*\phi^* - \phi\pi)$, picking up an extra $(\phi^*\pi^* - \pi\phi)$ and couple more ignorable infinities. We now have

$$\int d\tilde{p}(a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}}) = i\int d^3\vec{x}(\phi^*\pi^* - \pi\phi) \tag{32}$$

And so we can safely conclude

$$Q = \frac{1}{2}\int d\tilde{p}(a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}}) = \frac{1}{2}(N_\phi - N_{\phi^*}) \tag{33}$$

Evidently, charges of ϕ and ϕ^* are $+1/2$ and $-1/2$, respectively.