Solution to Homework Set #2, Problem #2 Part d.

Author: Kevin A. Hambleton

In this problem, we have two complex scalar fields with the same mass. We label the fields $\phi_a(x)$ where $a = 1, 2$. The lagrangian for these fields is a sum of the lagrangians of the individual scalar fields, which we may write as

$$\mathcal{L} = \partial_\mu \phi_a^* \partial^\mu \phi_a - m^2 \phi_a^* \phi_a$$

where the sum over the index a is implicit. First, let's calculate the equations of motion for the ϕ_a and the ϕ_a^*

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a^*)} - \frac{\partial \mathcal{L}}{\partial \phi_a^*} = (\partial^2 - m^2)\phi_a = 0$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = (\partial^2 - m^2)\phi_a^* = 0$$

Now we turn our attention to the symmetries of this Lagrangian where the index a goes from 1 to n for some general n.

Discussion Of $U(n)$ And $SU(n)$

Let us have a brief discussion about the unitary group, which is of great importance in Quantum Mechanics and Quantum Field Theory. Rewrite this Lagrangian by writing the fields in a column vector

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \Phi^\dagger = \begin{pmatrix} \phi_1^* & \phi_2^* \end{pmatrix}$$

so that the terms in the lagrangian become complex inner products of these vectors

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi$$

We can also recast the equations of motion using the vector Φ, in which case they tell us that

$$(\partial^2 - m^2)\Phi = (\partial^2 - m^2)\Phi^* = 0$$

Let's also be more general, and let the number of components in Φ be n. Note that the complex inner product has $U(n)$ symmetry where we multiply Φ by a matrix which satisfies $U^\dagger U = 1$

$$\Phi \to U\Phi,$$

$$\Phi^\dagger \Phi \to \Phi^\dagger U^\dagger U \Phi = \Phi^\dagger \Phi$$

Since U is not a function of x, we also have

$$\partial_\mu \Phi \to \partial_\mu (U\Phi) = U(\partial_\mu \Phi)$$

$$\partial_\mu \Phi^\dagger \partial^\mu \Phi \to (\partial_\mu \Phi)^\dagger U^\dagger U (\partial^\mu \Phi) = \partial_\mu \Phi^\dagger \partial^\mu \Phi$$

Now let the unitary transformation U be a function of some small, real parameters α_a. Let $U(0) = 1$, and Taylor expand $U(\alpha)$ about $\alpha = 0$ to first order

$$U(\alpha) = 1 - i\tau_a \alpha_a + O(\alpha^2)$$
$$U^\dagger(\alpha) = 1 + i\tau_a^\dagger \alpha_a + O(\alpha^2)$$

for some matrices τ_a, which we call the generators of $U(n)$. We want this to be unitary to first order, which means we have

$$U^\dagger(\alpha)U(\alpha) = 1 - i(\tau_a - \tau_a^\dagger)\alpha_a + O(\alpha^2) = 1$$

which means we have $\tau_a = \tau_a^\dagger$. The set of τ_a which satisfy this belong to $\mathfrak{u}(n)$, which we call the lie group of $U(n)$.

Now let α be finite. Applying the transformation $U(\alpha)$ is the same as applying one N^{th} of the transformation N times, and we let N go to infinity so that α/N is infinitesimal. Therefore we have

$$U(\alpha) = U\left(\frac{\alpha}{N}\right)^N = \left(1 - \frac{i\tau_a\alpha_a}{N}\right)^N = \exp\left(-i\tau_a\alpha_a\right)$$

Note that $\det(U^\dagger U) = \det(U)^* \det(U) = 1$, so $\det(U)$ must be some complex number with unit modulus. We will also be interested in the case in which $\det(U) = 1$, which specifies the group $SU(n)$. Now we use the fact that $\det(e^A) = e^{\text{tr}(A)}$ so we have

$$\det(U(\alpha)) = \exp\left(\text{tr}(-i\tau_a\alpha_a)\right) = 1$$

$$\Rightarrow \mathrm{tr}(\tau_a) = 0$$

Also, because we are free to choose a normalization, it is convention to choose a basis τ_a such that $\mathrm{tr}(\tau_a \tau_b) = \frac{1}{2}\delta_{ab}$. We will not prove this here, but it turns out that the number of generators of $\mathfrak{u}(n)$ is n^2, and the number of generators for $\mathfrak{su}(n)$ is $n^2 - 1$.

Conserved Charges For $U(n)$ Invariant Lagrangians

Let us calculate the conserved current for general n, and then we will go back to the case in which $n = 2$. Since the lagrangian is invariant under $U(n)$, and there are n^2 generators for this group, there will be n^2 conserved currents. In order to calculate the conserved currents, note that

$$\frac{d\phi_a}{d\alpha_i} = \frac{dU(\alpha)_{ab}}{d\alpha_j}\phi_b = -i(\tau_j)_{ab}\phi_b$$

$$\frac{d\phi_a^*}{d\alpha_i} = \frac{dU^*(\alpha)_{ab}}{d\alpha_j}\phi_b^* = i(\tau_j)_{ab}^*\phi_b^* = i(\tau_j)_{ab}^T\phi_b^*$$

In the last line we used the fact that the generators of $U(n)$ are hermitian, so the complex conjugate is the same as the transpose. Now we can calculate the conserved currents using Noether's theorem

$$J_j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\frac{d\phi_a}{d\alpha_j} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a^*)}\frac{d\phi_a^*}{d\alpha_j}$$

$$= -i\left(\partial^\mu \phi_a^*(\tau_j)_{ab}\phi_b - \partial^\mu \phi_a(\tau_j)_{ab}^T\phi_b^*\right) = -i\left(\partial^\mu \Phi^\dagger \tau_j \Phi - \Phi^\dagger \tau_j \partial^\mu \Phi\right)$$

Now let us check that this current is conserved

$$\partial_\mu J_j^\mu = -i\left(\partial^2 \Phi^\dagger \tau_j \Phi + \partial^\mu \Phi^\dagger \tau_j \partial_\mu \Phi - \partial_\mu \Phi^\dagger \tau_j \partial^\mu \Phi - \Phi^\dagger \tau_j \partial^2 \Phi\right)$$

$$= i(m^2 \Phi^\dagger \tau_j \Phi - m^2 \Phi^\dagger \tau_j \Phi) = 0$$

where we used the equations of motion in the second line to get rid of the ∂^2 terms.

We can calculate the conserved charges by using the fact that $\pi_a = \partial_0 \phi_a^*$ and $\pi_a^* = \partial_0 \phi_a$. Then the conserved charges are

$$Q_j = \int d^3x\, J_j^0 = -i \int d^3x\, \left(\pi_a(\tau_j)_{ab}\phi_b - \phi_a^*(\tau_j)_{ab}\pi_b^*\right)$$

We can also use the canonical commutation relations to compute the commutator of the charges. The canonical commutation relations for general n are

$$[\phi_a(\vec{x}), \pi_b(\vec{y})] = i\delta_{ab}\delta^3(\vec{x}-\vec{y})$$
$$[\phi_a^*(\vec{x}), \pi_b^*(\vec{y})] = i\delta_{ab}\delta^3(\vec{x}-\vec{y})$$

with all other commutators being zero. In order to compute the commutator of the charges, note that

$$\pi_a(\vec{x})(\tau_i)_{ab}\phi_b(\vec{x})\,\pi_c(\vec{y})(\tau_j)_{cd}\phi_d(\vec{y})$$

$$= \pi_a(\vec{x})(\tau_i)_{ab}\pi_c(\vec{y})\phi_b(\vec{x})(\tau_j)_{cd}\phi_d(\vec{y}) + \pi_a(\vec{x})(\tau_i)_{ab}[\phi_b(\vec{x}), \pi_c(\vec{y})](\tau_j)_{cd}\phi_d(\vec{y})$$
$$= \pi_c(\vec{y})(\tau_j)_{cd}\pi_a(\vec{x})\phi_d(\vec{y})(\tau_i)_{ab}\phi_b(\vec{x}) + i\pi_a(\vec{x})(\tau_i\tau_j)_{ab}\phi_b(\vec{x})\delta^3(\vec{x}-\vec{y})$$
$$= \pi_c(\vec{y})(\tau_j)_{cd}\phi_d(\vec{y})\pi_a(\vec{x})(\tau_i)_{ab}\phi_b(\vec{x}) + i\pi_a(\vec{x})(\tau_i\tau_j - \tau_j\tau_i)_{ab}\phi_b(\vec{x})\delta^3(\vec{x}-\vec{y})$$

therefore

$$[\pi_a(\vec{x})(\tau_i)_{ab}\phi_b(\vec{x}), \pi_c(\vec{y})(\tau_j)_{cd}\phi_d(\vec{y})] = i\pi_a(\vec{x})[\tau_i, \tau_j]_{ab}\phi_b(\vec{x})\delta^3(\vec{x}-\vec{y})$$

Similarly

$$[\phi_a^*(\vec{x})(\tau_i)_{ab}\pi_b^*(\vec{x}), \phi_c^*(\vec{y})(\tau_j)_{cd}\pi_d^*(\vec{y})] = -i\phi_a^*(\vec{x})[\tau_i, \tau_j]_{ab}\pi_b^*(\vec{x})\delta^3(\vec{x}-\vec{y})$$

therefore we have that the commutator of the charges is

$$[Q_i, Q_j] = -\int d^3x\, d^3y\, \left([\pi_a(\vec{x})(\tau_i)_{ab}\phi_b(\vec{x}), \pi_c(\vec{y})(\tau_j)_{cd}\phi_d(\vec{y})] + [\pi_a(\vec{x})(\tau_i)_{ab}\phi_b(\vec{x}), \pi_c(\vec{y})(\tau_j)_{cd}\phi_d(\vec{y})]\right)$$
$$= -i\int d^3x\, \left(\pi_a(\vec{x})[\tau_i, \tau_j]_{ab}\phi_b(\vec{x}) - \phi_a^*(\vec{x})[\tau_i, \tau_j]_{ab}\pi_b^*(\vec{x})\right)$$

Since the generators of the lie group must be closed under taking commutators, we have that

$$[\tau_i, \tau_j] = if_{ijk}\tau_k$$

where f_{ijk} are called the structure constants of the lie group. Plugging this into the commutator of the charges, we see that the charges obey

$$[Q_i, Q_j] = if_{ijk}Q_k$$

so the charges adopt the commutation relations of the lie group itself.

Special Case Of Interest: $U(2)$

Now let us go back to the case when $n = 2$. We want to find a basis for $\mathfrak{u}(2)$. We know that we should have $2^2 - 1 = 3$ matrices which form a basis of $\mathfrak{su}(2)$, and one more which makes a basis for $\mathfrak{u}(2)$. Suppose we have a general two by two matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we want this matrix to be hermitian, so $b = c^*$ and $a^* = a$ and $d^* = d$. We also want this matrix to be traceless, so $a = -d$. This leaves us with

$$\begin{pmatrix} a & b \\ b^* & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \Re(b) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \Im(b) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Note that the basis vectors are proportional to the Pauli matrices. Imposing the condition that $\mathrm{tr}(\tau_a \tau_b) = \frac{1}{2}\delta_{ab}$ we have a basis for $\mathfrak{su}(2)$ which is $(\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_2, \frac{1}{2}\sigma_3)$. If we remove the condition that the matrix had to be traceless, then we get a basis for $\mathfrak{u}(2)$ which is $(\frac{1}{2}I, \frac{1}{2}\sigma_1, \frac{1}{2}\sigma_2, \frac{1}{2}\sigma_3)$ where I is the two by two identity matrix. Therefore we can write any unitary two by two matrix as

$$U(\alpha) = \exp\left(-\frac{i}{2}(\alpha_0 I + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3)\right) = \exp\left(-\frac{i}{2}\left(\alpha_0 I + \alpha_i\sigma_i\right)\right)$$

Plugging in our basis for $\mathfrak{u}(2)$ for τ_a in the formula for the conserved charge derived in the previous section, we have that the conserved charges are

$$Q_0 = \int d^3x\, J_j^0 = -\frac{i}{2}\int d^3x\, \left(\pi_a\phi_a - \phi_a^*\pi_a^*\right)$$
$$Q_j = \int d^3x\, J_j^0 = -\frac{i}{2}\int d^3x\, \left(\pi_a(\sigma_j)_{ab}\phi_b - \phi_a^*(\sigma_j)_{ab}\pi_b^*\right)$$

Now we use our fomula for the commutator of the conserved charges which we derived in the previous section. Because the identity matrix commutes with all other matrices, we have

$$[Q_0, Q_j] = 0$$

and since $[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}] = i\epsilon_{ijk}\frac{\sigma_k}{2}$ we have

$$[Q_i, Q_j] = i\epsilon_{ijk}Q_k$$

Another Special Case: $U(3)$

Now let us find a basis for $\mathfrak{u}(3)$. We know that, if we can find a basis for $\mathfrak{su}(3)$, then we just need to add on the three by three identity matrix to get a basis for $\mathfrak{u}(3)$. Therefore we need to find the $3^2 - 1 = 8$ matrices which form a basis for $\mathfrak{su}(3)$.

Let's say we have a general three by three matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$$

we want this matrix to be hermitian, so we have $b^* = d, c^* = g, f^* = h$ and $a^* = a, e^* = e, j^* = j$. Since we are looking for a basis for $\mathfrak{su}(3)$, we also want this to be traceless, so we have $j = -a - e$. We also choose to define $\alpha = \frac{1}{2}(a + e)$ and $\gamma = \frac{1}{2}(a - e)$ where α and γ are real numbers. This gives us

$$\begin{pmatrix} \alpha+\gamma & b & c \\ b^* & \alpha-\gamma & f \\ c^* & f^* & -2\alpha \end{pmatrix}$$

$$= \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \Re(b) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \Im(b) \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \Re(c) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \Im(c) \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} + \Re(f) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \Im(f) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

So we see that the basis for $\mathfrak{su}(3)$ is proportional to the Gell-Mann matrices. Therefore a suitable basis for $\mathfrak{u}(3)$ is $(I, \lambda_1, \lambda_2, \ldots, \lambda_8)$ where I is the three by three identity matrix and λ_i is the i^{th} Gell-Mann matrix. This basis has $3^2 = 9$ elements, just as it should.

The standard model has $U(1) \times SU(2) \times SU(3)$ symmetries, and the charges associated with these symmetries couple to vector bosons. In the standard model, the $3^2 - 1 = 8$ charges associated with $SU(3)$ couple the quarks to the 8 gluons. Likewise, the $2^2 - 1 = 3$ charges associates with $SU(2)$

couple the left handed fermions to three bosons, which we call W_1, W_2, and W_3. Also, the $U(1)$ symmetry couples the fermions to the B vector boson. Through the Higgs mechanism, the W_1, W_2, W_3, and B bosons become the massive $W_\pm$ and Z bosons, and the massless photon A.