Solution to Homework Set #3, Problem #1.

Author: Ian Chaffey

a)

The differential decay rate is given by,

$$d\Gamma = \frac{1}{2E_1}|\mathcal{M}|^2 d\Pi_{LIPS}$$

Assuming $m_{e^-} = m_{\nu_\mu} = m_{\overline{\nu}_e} = 0$ in the muon's rest frame we have

$$\begin{aligned} d\Gamma &= \frac{1}{2m}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(\sum p_i^\mu - \sum p_f^\mu)\frac{d^3\vec{p}_{\overline{\nu}_e}}{(2\pi)^3 2E}\frac{d^3\vec{p}_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}}\frac{d^3\vec{p}_{e^-}}{(2\pi)^3 2E_{e^-}} \\ &= \frac{1}{16(2\pi)^5 m}\frac{|\mathcal{M}|^2}{EE_{\nu_\mu}E_{e^-}}d^3\vec{p}_{\overline{\nu}_e}d^3\vec{p}_{\nu_\mu}d^3\vec{p}_{e^-}\delta(m - E - E_{\nu_\mu} - E_{e^-})\delta^{(3)}(\vec{p}_{\overline{\nu}_e} + \vec{p}_{\nu_\mu} + \vec{p}_{e^-}) \end{aligned}$$

Integrating over $d^3\vec{p}_{\nu_\mu}$ to get rid of the three dimensional delta function gives

$$d\Gamma = \frac{1}{16(2\pi)^5 m}\frac{|\mathcal{M}|^2}{EE_{\nu_\mu}E_{e^-}}d^3\vec{p}_{\overline{\nu}_e}d^3\vec{p}_{e^-}\delta(m - E - E_{\nu_\mu} - E_{e^-}) \qquad (1)$$

Where the three dimensional delta function enforces the condition

$$\begin{aligned} \vec{p}_{\nu_\mu} &= -(\vec{p}_{\overline{\nu}_e} + \vec{p}_{e^-}) \\ \implies |\vec{p}_{\nu_\mu}| &= |\vec{p}_{\overline{\nu}_e} + \vec{p}_{e^-}| \\ &= E_{\mu_\nu} \end{aligned}$$

Using the law of cosines gives

$$E_{\nu_\mu}^2 = E^2 + E_{e^-}^2 + 2EE_{e^-}\cos\theta$$

Where θ is the angle between $\vec{p}_{\overline{\nu}_e}$ and $\vec{p}_{e^-}$. Therefore

$$dE_{\nu_\mu} = \frac{1}{E_{\nu_\mu}}EE_{e^-}d(\cos\theta)$$

Also in spherical coordinates

$$d^3\vec{p}_{\overline{\nu}_e} = E^2 dEd(\cos\theta)d\phi$$

Also in the limit $\cos\theta = 1$, $E_{\nu_\mu} = E + E_{e^-}$. Similarly when $\cos\theta = -1$, $E_{\nu_\mu} = E - Ee^-$. Plugging all this into (1) and integrating gives

$$\begin{aligned} d\Gamma &= \frac{|\mathcal{M}|^2}{16(2\pi)^5 m} d^3\vec{p}_{e^-} \int \frac{EdEd(\cos\theta)d\phi}{E_{\mu_\nu}E_{e^-}} \delta(m - E - E_{\mu_\nu} - E_{e^-}) \\ &= \frac{|\mathcal{M}|^2}{16(2\pi)^5 m E_{e^-}^2} d^3\vec{p}_{e^-} dE \int_{E-E_{e^-}}^{E+E_{e^-}} dE_{\nu_\mu} \delta(m - E - E_{\nu_\mu} - E_{e^-}) \end{aligned} \tag{2}$$

In order for (2) to be nonzero it is required that

$$\begin{aligned} & E - E_{e^-} \leq m - E - E_{e^-} \leq E + E_{e^-} \\ & \implies E \leq m/2 \leq E + E_{e^-} \\ & \implies m/2 - E_{e^-} \leq E \leq m/2 \end{aligned}$$

Therefore

$$\begin{aligned} d\Gamma &= \frac{1}{16(2\pi)^4 m E_{e^-}^2} d^3\vec{p}_{e^-} \int_{m/2-E_{e^-}}^{m/2} dE|\mathcal{M}|^2 \\ &= \frac{2G_F^2}{(2\pi)^4 E_{e^-}^2} d^3\vec{p}_{e^-} \int_{m/2-E_{e^-}}^{m/2} dE(m^2 - 2mE)E \\ &= \frac{2G_F^2}{(2\pi)^4 E_{e^-}^2} d^3\vec{p}_{e^-} \left(\frac{1}{2}m^2E^2 - \frac{2}{3}mE^3\right)\Big|_{m/2-E_{e^-}}^{m/2} \\ &= \frac{2G_F^2}{(2\pi)^4 E_{e^-}^2} d^3\vec{p}_{e^-} \left(\frac{1}{4}m^2E_{e^-}^2\right) \\ &= \frac{G_F^2 m^2}{2(2\pi)^4} d^3\vec{p}_{e^-} \\ \implies \Gamma &= \frac{G_F^2 m^2}{2(2\pi)^4} 4\pi \int_0^{m/2} dE_{e^-} E_{e^-}^2 \\ &= \frac{G_F^2 m^2}{(2\pi)^3} \frac{1}{3} \left(\frac{m}{2}\right)^3 \\ &= \frac{G_F^2 m^5}{192\pi^3} \end{aligned}$$

b)

Given $G_F = 1.166 \times 10^{-5}$ GeV^{-2} and $m = 106$ MeV in natural units Γ has units of energy. In order to convert Γ to inverse seconds we must divide by $\hbar$. Therefore

$$\begin{aligned} \Gamma &= \frac{G_F^2 m^5}{192\pi^3\hbar} \\ \implies \tau &= \frac{192\pi^3\hbar}{G_F^2 m^5} \\ &= 2.15\mu s \end{aligned}$$

The discrepency from the observed $2.20\mu s$ is $\delta = 2.27\%$. Possible sources of error could be the assumption that all of the decay products are massless as well as the possibility of other decay modes that were not accounted for.