Solution to Homework Set #3, Problem #2.

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1 Mandelstan Variables

We calculated that the $e^+e^- \to \mu^+\mu^-$ cross section had the form, in the CM frame,

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 E_{CM}^2}\left(1+\cos^2\theta\right) \ . \tag{1}$$

(a) Work out the Lorentz-invariant quantities

$$s = \left(p_{e^+} + p_{e^-}\right)^2 \qquad t = \left(p_{\mu^-} - p_{e^-}\right)^2 \qquad u = \left(p_{\mu^+} - p_{e^-}\right)^2 \tag{2}$$

known as Mandelstam variables, in terms of E_{CM} and $\cos\theta$ (still assuming $m_\mu = m_e = 0$).

Solution: In this system since we take the $m_e = m_\mu = 0$ then $E = |\vec{p}|$ and the four-momentum of the interaction are

$$p_{e^+} = p = E\left(1,0,0,1\right) \ , \tag{3}$$
$$p_{e^-} = p' = E\left(1,0,0,-1\right) \ , \tag{4}$$
$$p_{\mu^+} = k = E\left(1,0,-\sin\theta,-\cos\theta\right) \ , \tag{5}$$
$$p_{\mu^-} = k' = E\left(1,0,\sin\theta,\cos\theta\right) \ , \tag{6}$$

where E is the energy and θ is the angle between the path traced out by the incoming electron-positron and the outgoing muon-antimuon pairs (see Fig. 1). The total energy of this system in the center of mass frame is $E_{CM} = E_p + E_{p'} = E_k + E_{k'} = 2E$ such that $E = E_{CM}/2$. So we find the relations for s, t, and u to be

$$\begin{aligned}
s &= (p+p')^2 \\
&= p^\nu p_\nu + p'^\nu p'_\nu + 2p^\nu p'_\nu \\
&= \left[E^2 - E^2\right] + \left[E^2 - (-E)(-E)\right] + \left[2(E^2+E^2)\right] \\
&= 0 + 0 + 4E^2 \\
&= \boxed{E_{CM}^2},
\end{aligned} \tag{7}$$

$$\begin{aligned}
t &= (k'-p')^2 \\
&= k'^\nu k'_\nu + p'^\nu p'_\nu - 2k'^\nu p'_\nu \\
&= \left[E^2 - E^2\left(\sin^2\theta + \cos^2\theta\right)\right] \\
&+ [E^2 - (-E)(-E)] - 2[E^2 - (-E)(E\cos\theta)] \\
&= 0 + 0 - 2E^2(1+\cos\theta) \\
&= \boxed{-\frac{E_{CM}^2}{2}(1+\cos\theta)},
\end{aligned} \tag{8}$$

$$\begin{aligned}
u &= (k-p')^2 \\
&= k^\nu k_\nu + p'^\nu p'_\nu - 2k^\nu p'_\nu \\
&= \left[E^2 - E^2((-\sin\theta)^2 + (-\cos\theta)^2)\right] \\
&+ [E^2 - (-E)(-E)] - 2[E^2 - (-E)(-E\cos\theta)] \\
&= 0 + 0 - 2E^2(1-\cos\theta) \\
&= \boxed{-\frac{E_{CM}^2}{2}(1-\cos\theta)}.
\end{aligned} \tag{9}$$

(b) Derive a relationship between s, t, and u.

Solution: Using the results of part (a) it is readily apparent that $s+t+u = 0$ is a relation for s, t, and u. We will do a more general demonstration in part (d).

$$\begin{aligned}
s+t+u &= E_{CM}^2 - \frac{E_{CM}^2}{2}(1+\cos\theta) - \frac{E_{CM}^2}{2}(1-\cos\theta) \\
&= E_{CM}^2 - 2E_{CM}^2/2 = 0
\end{aligned} \tag{10}$$

(c) Rewrite $\frac{d\sigma}{d\Omega}$ in terms of s, t, and u. Solution: Recall that the differential cross section is worked out to be

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 E_{CM}^2}\left(1+\cos^2\theta\right) \ . \tag{11}$$

The first step is recognize that t^2, ut, and u^2 all produce a $\cos^2\theta$ term.

$$t^2 = \frac{E_{CM}^2}{4}\left(1-2\cos\theta+\cos^2\theta\right) \tag{12}$$

$$u^2 = \frac{E_{CM}^2}{4}\left(1+2\cos\theta+\cos^2\theta\right) \tag{13}$$

$$ut = tu = \frac{E_{CM}^2}{4}\left(1-\cos^2\theta\right) \tag{14}$$

We notice that $t^2+u^2 = (E_{CM}^4/2)\left(1+\cos^2\theta\right)$ and $s = E_{CM}^2$ so that $\left(1+\cos^2\theta\right) = 2(t^2+u^2)/s^2$. Thus the differential cross section in terms of s, t, and u is

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2}\frac{t^2+u^2}{s^3}} \ . \tag{15}$$

(d) Now assume m_μ and m_e are non-zero. Derive a relationship between s, t, and u and the masses.

Solution: Now we can show that the relation from part (b) in general is zero when the rest masses are vanishingly small and equals the sum of the rest masses squared when we take m_μ and m_e to be non-zero.

$$\begin{aligned} s+t+u &= (p+p')^2 + (k'-p')^2 + (k-p')^2 \\ &= (p^2+p'^2+2pp') + (k'^2+p'^2-2k'p') + (k^2+p'^2-2kp') \\ &= m_p^2 + m_{p'}^2 + m_k^2 + m_{k'}^2 + 2pp' + 2p'p' - 2kp' - 2k'p' \\ &= m_p^2 + m_{p'}^2 + m_k^2 + m_{k'}^2 + 2p'(p+p'-k-k') \end{aligned} \tag{16}$$

We note that the last term vanishes ($p+p'-k-k'=0$) due to conservation of four-momentum.

$$\boxed{s+t+u = m_p^2 + m_{p'}^2 + m_{k'}^2 + m_k^2} \tag{17}$$

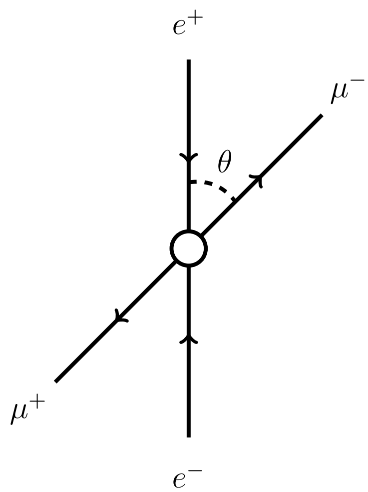

Figure 1: Electron-positron annihilation into a muon-antimuon pair.