Conformality Lost

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arXiv:0905.4752
Motivation: QCD at LARGE $N_c$ and $N_f$
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Define $x = N_f/N_c$, treat as a continuous variable

\[
\begin{aligned}
\text{asymptotic freedom} & \quad \langle \overline{\psi} \psi \rangle \neq 0 & \quad \text{conformal} & \quad 11/2 & \quad \text{trivial} & \quad x \\
0 & \quad x_c
\end{aligned}
\]
Motivation: QCD at LARGE $N_c$ and $N_f$

Colors  
Flavors

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\[
\alpha^*\quad \text{Banks-Zaks fixed point}
\]

\[
\text{Nucl.Phys.B196:189,1982}
\]
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Define $x = N_f/N_c$, treat as a continuous variable

What is the nature of this transition?

How does the IR scale appear as conformality is lost?

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\[ \alpha_* \]

Banks-Zaks fixed point


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How does the IR scale appear as conformality is lost?
I. A mechanism for vanishing conformal invariance
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II. The Berezinskii-Kosterlitz-Thouless (BKT) transition
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III. A quantum mechanics model: the $1/r^2$ potential
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V. Relativistic model: defect Yang-Mills
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II. The Berezinskii-Kosterlitz-Thouless (BKT) transition

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V. Relativistic model: defect Yang-Mills

VI. QCD with many flavors? A partner theory QCD* with a nontrivial UV fixed point?
A theory with an infrared conformal fixed point at $g=g^*$ has a zero in the beta function:

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**Example:** supersymmetric QCD is conformal for $3/2 \leq N_f/N_c \leq 3$
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"$\alpha$" = $N_f/N_c$, "$\alpha^*$" = 3/2, 3
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How is conformality lost?
Three ways to lose an infrared fixed point:
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#1: Fixed point runs to zero:

\[ \beta(g; \alpha) \]

\[ \alpha < \alpha_* \]

\[ \Rightarrow \]

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\[ \Rightarrow \alpha = \frac{N_f}{N_c}, \alpha_* = 3 \]
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Example: Supersymmetric QCD at large \( N_c \) and \( N_f \)

\[ \alpha = N_f / N_c, \quad \alpha_* = 3 \]

\[ N_f / N_c \lesssim 3 \Rightarrow \text{weak coupling Banks-Zaks conformal fixed point} \]

\[ N_f / N_c \gtrsim 3 \Rightarrow \text{trivial QED-like “free electric” theory} \]
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\[ F_E \sim \frac{g^2}{r^2 \ln (r \Lambda_{UV})} \]
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Possible example? SQCD again \( \rightarrow \) \( \alpha = \frac{N_f}{N_c}, \alpha_* = 3/2 \)

For \( \alpha \leq \alpha_* \) get “free magnetic phase” \[\text{[Seiberg]}\]
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For \( \alpha \leq \alpha_* \) get “free magnetic phase” \[ \text{[Seiberg]} \]

\( \Rightarrow \) electric theory dual to a QED-like magnetic theory:

\[ F_E \sim \frac{g^2 \ln (r \Lambda_{UV})}{r^2} \]

\[ F_M \sim \frac{g_M^2}{r^2 \ln (r \Lambda_{UV})} \]

\( g_M \sim 1/g \)
#3: UV and IR fixed points annihilate:

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A toy model:
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$\alpha \geq \alpha_*$: $g_{\pm} = g_* \pm \sqrt{\alpha - \alpha_*}$

**UV, IR fixed points**

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*UV, IR fixed points*

$\alpha = \alpha_* : \text{fixed points merge}$
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A toy model:

- \( \alpha \geq \alpha_* \): \( g_\pm = g_* \pm \sqrt{\alpha - \alpha_*} \)
  - **UV, IR fixed points**
- \( \alpha = \alpha_* \): fixed points merge
- \( \alpha < \alpha_* \): conformality lost
What happens just below the transition to nonconformal behavior?

\[ \beta(g; \alpha) \]

\[
\begin{align*}
g_{\text{UV}} & \quad g_* & \quad g_{\text{IR}} \\
\alpha & \lesssim \alpha_*
\end{align*}
\]
What happens just below the transition to nonconformal behavior?

\[ \beta(g; \alpha) \]

1. Start: \( g = g_{\text{UV}} < g_* \) in the UV
2. \( g \) grows, **stalling** near \( g_* \)
3. \( g \) strong at scale \( \Lambda_{\text{IR}} \)
What happens just below the transition to nonconformal behavior?

<table>
<thead>
<tr>
<th>$\beta(g; \alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{UV}$</td>
</tr>
</tbody>
</table>

\[ \alpha \lesssim \alpha_* \]

i. Start: $g = g_{UV} < g_*$ in the UV

ii. $g$ grows, **stalling** near $g_*$

iii. $g$ strong at scale $\Lambda_{IR}$

\[
\Lambda_{IR} \approx \Lambda_{UV} e^{- \int \frac{dg}{\beta(g)}}
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\[
= \Lambda_{UV} e^{- \frac{\pi}{\sqrt{|\alpha_* - \alpha|}}}
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(Not like 2\textsuperscript{nd} order phase transition: \( \Lambda_{IR} \quad \simeq \quad \Lambda_{UV} \sqrt{\mid \alpha_* - \alpha \mid} \) )
Analogue to “intermittency” in chaotic systems

Iterative maps: \( f(x) = \lambda x(1 - x) \)
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“conformal window”

intermittency

\( \lambda \)
\[ \Lambda_{\text{IR}} \approx \Lambda_{\text{UV}} e^{-\frac{\pi}{\sqrt{\alpha_* - \alpha}}} \]

Scaling behavior of toy model is reminiscent of the Berezinskii-Kosterlitz-Thouless (BKT) transition (an “infinite order” phase transition)
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Vortices in XY model

box size R, vortex core size a:
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box size $R$, vortex core size $a$:

\[
E = E_0 \ln \frac{R}{a}, \quad S = 2 \ln \frac{R}{a}
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\[
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Vortices condense for \( T > T_c = E_0 / 2 \); can show correlation length forms:

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RG analysis of the BKT transition

XY model = Coulomb gas
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Z = \mathcal{N} \sum_{N_+, N_-} \frac{z^{N_+} z^{N_-}}{N_+! N_-!} \int \frac{\prod \prod}{i=1 j=1} d^2 x_i d^2 y_j \int D\phi e^{-\int d^2 x \frac{T}{2} (\nabla \phi)^2 + i \sum_{i,j} (\phi(x_i) - \phi(y_j))}
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Coulomb field

$$= \mathcal{N} \int D\phi e^{-\int d^2 x \left( \frac{T}{2} (\nabla \phi)^2 - 2z \cos \phi \right)}$$

temp.  
fugacity
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\]

\[
= \mathcal{N} \int D \phi e^{- \int d^2 x \left[ \frac{T}{2} (\nabla \phi)^2 - 2z \cos \phi \right]}
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The XY model is equivalent to the Sine-Gordon model
Classical XY model BKT transition = zero temperature quantum transition in Sine-Gordon model:

\[ \mathcal{L} = \frac{T}{2} (\nabla \phi)^2 - 2z \cos \phi \]
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New variables:

\[ u = 1 - \frac{1}{8\pi T}, \quad v = \frac{2z}{T\Lambda^2} \]

Perturbative \( \beta \)-functions:

\[ \beta_u = -2v^2, \quad \beta_v = -2uv \]
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$\Lambda = $ UV cutoff at vortex core

~Dimensionful quantities in units of XY model interaction strength
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- $T < T_c$
- Bound vortices
- Trivially conformal
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Dimensionful quantities in units of XY model interaction strength

\( T < T_c \) - bound vortices

\( T > T_c \) - Coulomb gas

- trivially conformal

- screening length
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**Newer variables:**

\[ \tau = (u + v), \quad \alpha = u^2 - v^2 \]

\[ \beta_\tau = \alpha - \tau^2, \quad \beta_\alpha = 0 \]
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Correlation length in BKT transition:

$\beta_\tau \quad \alpha > 0$: Conformal (unbound vortices)

$\alpha < 0$: finite $\xi$ (bound vortices)

$T = T_c$
Correlation length in BKT transition:

For small negative $\alpha$, assume $\tau$ small & positive in UV

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$\tau$ blows up in RG time

$$t = \int \frac{d\tau}{\beta(\tau)} = -\frac{\pi}{2\sqrt{-\alpha}}$$
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...giving rise to an IR scale (like $\Lambda_{\text{QCD}}$) which sets the scale for the finite correlation length for $\alpha<0$:

$$\xi_{\text{BKT}} \sim \frac{1}{\Lambda} e^{\frac{\pi}{2\sqrt{-\alpha}}}$$
So far:
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• Mechanism of fixed point merger in general gives rise to “BKT scaling“:

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So far:

- BKT transition = loss of conformality via fixed point merger
- Mechanism of fixed point merger in general gives rise to “BKT scaling”:

\[ \Lambda_{\text{IR}} \sim \Lambda_{\text{UV}} e^{-\frac{\pi}{\sqrt{\alpha^* - \alpha}}} \]

Next: other examples:

- QM with $1/r^2$ potential
- AdS/CFT
- Defect Yang-Mills
- QCD with many flavors
Example: QM in d-dimensions with $1/r^2$ potential

$$\left[-\nabla^2 + V(r) - k^2\right] \psi = 0 , \quad V(r) = \frac{\alpha}{r^2}$$
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$k=0$ solutions: \[ \psi = c_- r^{\nu^-} + c_+ r^{\nu^+} \]

$$\nu_{\pm} = -\left(\frac{d - 2}{2}\right) \pm \sqrt{\alpha - \alpha_*} \quad \alpha_* = -\left(\frac{d-2}{2}\right)^2$$
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\[
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\]

\[
\alpha_* = - \left( \frac{d-2}{2} \right)^2
\]

• valid for \( \alpha_* < \alpha < (\alpha_*+1) \)
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• valid for $\alpha_* < \alpha < (\alpha_* + 1)$

• $\alpha < \alpha_*: \nu_\pm$ complex, no ground state
Example: QM in d-dimensions with $1/r^2$ potential

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[ -\nabla^2 + V(r) - k^2 ] \psi = 0 , \quad V(r) = \frac{\alpha}{r^2}
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• valid for $\alpha_* < \alpha < (\alpha_*+1)$
  • $\alpha < \alpha_*$: $\nu_{\pm}$ complex, no ground state
  • $\alpha = \alpha_*$: $\nu_+ = \nu_-$
  • $\alpha > (\alpha_*+1)$: $r^{\nu_-}$ too singular to normalize
\[-\nabla^2 + V(r) - k^2\] \(\psi = 0\), \(V(r) = \frac{\alpha}{r^2}\)

**k=0 solutions:** \(\psi = c_- r^{\nu_-} + c_+ r^{\nu_+}\)

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\[ V(r) = \frac{\alpha}{r^2} - g \delta^d(r) \]
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- \( r^{\nu_+} \) corresponds to IR fixed point of \( g \)
- \( r^{\nu_-} \) corresponds to unstable UV fixed point of \( g \)
RG treatment of $1/r^2$ potential:  I. Perturbative

$\alpha_* \equiv -(d-2)^2/4$ so work in $d=2+\varepsilon$
RG treatment of $1/r^2$ potential: \textit{I. Perturbative}

$\alpha_* \equiv -(d-2)^2/4$ so work in $d=2+\varepsilon$

$$S = \int dt \, d^d x \left( i \psi^\dagger \partial_t \psi - \frac{\left| \nabla \psi \right|^2}{2m} + \frac{g \pi}{2} \psi^\dagger \psi^\dagger \psi \psi \right)$$

$$- \int dt \, d^d x \, d^d y \, \psi^\dagger(t, x) \psi^\dagger(t, y) \frac{\alpha}{|x - y|^2} \psi(t, y) \psi(t, x)$$
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\]

\textbf{propagator:} \( \frac{i}{\omega - p^2/2m} \)

\textbf{contact vertex:} \( i \pi g \mu^{-\varepsilon} \)

\textbf{“meson exchange”:} \( \frac{2\pi i \alpha}{\varepsilon} \frac{1}{|q|^\varepsilon} \)
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$\alpha_\star \equiv -(d-2)^2/4$ so work in $d=2+\varepsilon$

$$\begin{align*}
S &= \int dt\ d^d x \left(i\psi^\dagger \partial_t \psi - \frac{\left|\nabla \psi\right|^2}{2m} + \frac{g\pi}{2} \psi^\dagger \psi \psi \psi\right) \\
&\quad - \int dt\ d^d x\ d^d y\ \psi^\dagger (t, x) \psi^\dagger (t, y) \frac{\alpha}{|x-y|^2} \psi(t, y) \psi(t, x)
\end{align*}$$

Propagator:  $\frac{i}{\omega - p^2/2m}$

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Find $g$ runs:  $\begin{array}{c} \vdots \end{array} + \begin{array}{c} \infty \end{array}$

$\beta(g; \alpha) = \mu \frac{\partial g}{\partial \mu} = \left(\alpha + \frac{\varepsilon^2}{4}\right) - (g - \varepsilon)^2$

Same as toy model!  $\alpha_\star = -\varepsilon^2/4$,  $g_\star = \varepsilon$
RG treatment of $1/r^2$ potential:  

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Find $g$ runs: [ ] + \( \otimes \)

$$\beta(g; \alpha) = \mu \frac{\partial g}{\partial \mu} = \left( \alpha + \frac{\varepsilon^2}{4} \right) - (g - \varepsilon)^2$$

Same as toy model! $\alpha_* = -\varepsilon^2/4$, $g_* = \varepsilon$

$\alpha > \alpha_*$: conformal
$\alpha = \alpha_*$: critical
$\alpha < \alpha_*$: $g$ blows up in IR
RG treatment of $1/r^2$ potential:  

1. Perturbative  

$\alpha_* \equiv -(d-2)^2/4$  

so work in $d=2+\varepsilon$  

\[
S = \int dt \, d^dx \left( i \psi^\dagger \partial_t \psi - \frac{|\nabla \psi|^2}{2m} + \frac{g\pi}{2} \psi^\dagger \psi^\dagger \psi \psi \right) - \int dt \, d^dx \, d^d y \, \psi^\dagger(t, x) \psi^\dagger(t, y) \frac{\alpha}{|x - y|^2} \psi(t, y) \psi(t, x)
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$\delta$-function

propagator:  

\[
\frac{i}{\omega - p^2/2m}
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contact vertex:  

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$\alpha_* = -\varepsilon^2/4$,  

$g_* = \varepsilon$

$\alpha > \alpha_*$: conformal  

$\alpha = \alpha_*$: critical  

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$B \sim \left( \frac{\Lambda^2_{IR}}{m} \right) \sim \left( \frac{\Lambda^2_{UV}}{m} \right) e^{-2\pi / \sqrt{\alpha_* - \alpha}}$

bound state energy  

BKT scaling
RG treatment of $1/r^2$ potential:  II. Non-perturbative

regulate with square well:

\[ V(r) = \begin{cases} \alpha/r^2 & r > r_0 \\ -g/r_0^2 & r < r_0 \end{cases} \]

E=0 solution for $r>r_0$: \[ \psi = c_- r^{\nu_-} + c_+ r^{\nu_+} \]
RG treatment of $1/r^2$ potential:  

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Solve for $c_+/c_-$ (a physical dimensionful quantity) and require invariance:  
$$d(c+/c_-)/dr_0 = 0$$
RG treatment of $1/r^2$ potential: II. Non-perturbative

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Solve for $c_+/c_-$ (a physical dimensionful quantity) and require invariance: $d(c_+/c_-)/dr_0 = 0$:

Find exact $\beta$-function for $g$. Eg, for $d=3$:

$$\beta = \frac{2\sqrt{g} \left( \alpha + \sqrt{g} \cot \sqrt{g} - g \cot^2 \sqrt{g} \right)}{-\cot \sqrt{g} + \sqrt{g} \csc^2 \sqrt{g}}$$

$\alpha_* = -\frac{1}{4}, g_* \approx 1.36$
RG treatment of $1/r^2$ potential:  

II. Non-perturbative

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\[ V(r) = \begin{cases} 
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\( \alpha_* = -\frac{1}{4}, \; g_* \approx 1.36 \)
Even better: define

\[ \gamma = \left( \frac{\sqrt{g} \, J_{d/2}^{1/2}(\sqrt{g})}{J_{d/2-1}^{1/2}(\sqrt{g})} \right) \]

Condition \(d(c_+/c_-)/dr_0\) yields exact \(\beta\)-function in \(d\)-dimensions:

\[ \beta_\gamma = \frac{\partial \gamma}{\partial t} = (\alpha - \alpha_*) - (\gamma - \gamma_*)^2, \quad \gamma_* = \frac{d - 2}{2} \]

• Toy model is exact!
• \(\gamma\) is a periodic function of \(g\), \(\gamma = \pm \infty\) equivalent
• Limit cycle behavior for \(\alpha < \alpha_*\): explains “Efimov states” for trapped atoms at Feschbach resonance
Comments inserted by the author:
Universal spectrum of three-body states

(V. Efimov, Phys. Lett. 33B (1970) 563)

\[
E_n = \frac{1}{515.03^n}, \quad n \to \infty
\]

**Limit cycle behavior**

\[
E_{n+1}/E_n = 1/515.03, \quad n \to \infty
\]

\(a = \text{atom-atom scattering length}\)

A=atom  
D=dimer  
T=trimer

\[1/a \to 0\]

Discrete scale invariance for fixed angle

Geometrical spectrum

Manifestation in scattering observables

\[\log\text{-periodic dependence on } a\]

\[\text{indirect observation of the Efimov effect}\]

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Limit Cycle: Efimov Effect

(V. Efimov, Phys. Lett. 33B (1970) 563)

\[ E_{n+1}/E_n = 1/515.03, \quad n \to \infty \]

Limit cycle behavior

\[ (m|E|)^{1/2} \]

\[ l/a \]

\[ K \]

\[ E \]

\[ 1/a \]

A = atom
D = dimer
T = trimer

\[ a = \text{atom-atom scattering length} \]

Experimental evidence for Efimov states in \(^{133}\text{Cs}\)
(Kraemer et al. (Innsbruck), Nature 440 (2006) 315)
Conformal phases: measure correlations, not $\beta$-functions!
Look at operator scaling dimensions:
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From Nishida & Son, 2007:

- Replace $V(r_1 - r_2) \rightarrow V(r_1 - r_2) + \frac{1}{2} \omega^2 |r_1^2 + r_2^2|$
- Compute 2-particle ground state energy $E_0$
- Operator dimension of $\psi\psi$ is $\Delta_{\psi\psi} = E_0/\omega$
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As the two conformal theories merge when $\alpha \rightarrow \alpha_*$, operator dimensions in the two CFTs merge
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For \( 1/r^2 \) potential -- find for the two conformal theories:

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\Delta_{\psi\psi} = (d + \nu_{\pm}) = \left( \frac{d+2}{2} \right) \pm \sqrt{\alpha - \alpha_*}
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"+" = UV fixed point
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Conformal phases: measure correlations, not \( \beta \)-functions! Look at operator scaling dimensions:

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“+” = UV fixed point
“-” = IR fixed point

Note: \( (\Delta_+ + \Delta_-) = (d+2) \): scaling dimension of nonrelativistic spacetime.
Analog in AdS/CFT:
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AdS: \[ ds^2 = \frac{1}{z^2} \left( dz^2 + \sum_{i=1}^{d} dx_i^2 \right) \]
Analog in AdS/CFT:

**AdS:** \( ds^2 = \frac{1}{z^2} \left( dz^2 + \sum_{i=1}^{d} dx_i^2 \right) \)

Massive scalar in the bulk

two solutions to eq. of motion:

\[ \phi = c_+ z^{\Delta^+} + c_- z^{\Delta^-} \]

\[ \Delta_{\pm} = \frac{d}{2} \pm \sqrt{m^2 + \left( \frac{d}{2} \right)^2} \equiv \frac{d}{2} \pm \sqrt{m^2 - m^2_*} \]
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- \((\Delta^+_\psi \psi + \Delta^-_\psi \psi) = (d+2) = \) conformal wt. of nonrelativistic d-space+time
**Analog in AdS/CFT:**

AdS:
\[ ds^2 = \frac{1}{z^2} \left( dz^2 + \sum_{i=1}^{d} dx_i^2 \right) \]

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- \((\Delta_+ + \Delta_-) = d = \text{spacetime dim of CFT}\)
- when \(m^2 = m_*^2 = -d^2/4\), \(\Delta_\pm = d/2\)
- \((\Delta_+^\psi\psi + \Delta_-^\psi\psi) = (d+2) = \text{conformal wt. of nonrelativistic d-space+time}\)
- \(\alpha = \alpha_* = -(d-2)^2/4 \Rightarrow \Delta_\pm = (d+2)/2\)
Analog in AdS/CFT:

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\[ \varphi = c_+ z^{\Delta+} + c_- z^{\Delta-} \]

\[ \Delta_\pm = \frac{d}{2} \pm \sqrt{m^2 + \left(\frac{d}{2}\right)^2} = \frac{d}{2} \pm \sqrt{m^2 - m_{*}^2} \]

- \((\Delta_+ + \Delta_-) = d = \text{spacetime dim of CFT}\)
- when \(m^2 = m_{*}^2 = -d^2/4\), \(\Delta_\pm = d/2\)
- Instability (no AdS or CFT) for \(m^2 < m_{*}^2\) (B-F bound)

QM:

- \((\Delta_+ \varphi \varphi + \Delta_- \varphi \varphi) = (d+2) = \text{conformal wt. of nonrelativistic d-space+time}\)
- \(\alpha = \alpha_{*} = -(d-2)^2/4 \Rightarrow \Delta_\pm = (d+2)/2\)
- Conformality lost for \(\alpha < \alpha_{*}\)
Analog in AdS/CFT:

**AdS**:
\[ ds^2 = \frac{1}{z^2} \left( dz^2 + \sum_{i=1}^{d} dx_i^2 \right) \]

Massive scalar in the bulk:

Two solutions to eq. of motion:

\[ \varphi = c_+ z^{\Delta_+} + c_- z^{\Delta_-} \]

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**AdS**

- \((\Delta_+ + \Delta_-) = d = \text{spacetime dim of CFT}\)
- When \(m^2 = m_*^2 = -d^2/4\), \(\Delta_\pm = d/2\)
- Instability (no AdS or CFT) for \(m^2 < m_*^2\) (B-F bound)
- Lower bound on \(\Delta_-\)

**QM**

- \((\Delta^+_{\psi\psi} + \Delta^-_{\psi\psi}) = (d+2) = \text{conformal wt. of nonrelativistic d-space+time}\)
- \(\alpha = \alpha_* = -(d-2)^2/4 \Rightarrow \Delta_\pm = (d+2)/2\)
- Conformality lost for \(\alpha < \alpha_*\)
- Lower bound on \(\Delta^-_{\psi\psi}\)
As with QM example, 2 different solutions $\Rightarrow$ 2 different CFTs
AdS/CFT cont’d:

As with QM example, 2 different solutions ⇒ 2 different CFTs

\[ \varphi = \varphi_0 z^{\Delta_+} : \quad Z_{\text{grav.}} |_{\varphi \to \varphi_0 z^{\Delta_+}} = Z_{\text{CFT}}[\varphi_0] \]
AdS/CFT cont’d:

As with QM example, 2 different solutions $\Rightarrow$ 2 different CFTs

$$\varphi = \varphi_0 z^{\Delta^+} : \quad Z_{\text{grav.}} \bigg|_{\varphi \rightarrow 0^+ \varphi_0 z^{\Delta^+}} = Z_{\text{CFT}}[\varphi_0]$$

$$S = S_{\text{CFT}} + \int d^d x \phi_0 \mathcal{O}$$
As with QM example, 2 different solutions \( \Rightarrow \) 2 different CFTs

\[
\begin{align*}
\phi &= \phi_0 z^{\Delta^+} : \\
&= \frac{Z_{\text{grav.}}}{Z_{\text{CFT}}} \bigg|_{\phi \to \phi_0} = Z_{\text{CFT}}[\phi_0] \\
S &= S_{\text{CFT}} + \int d^d x \phi_0 \mathcal{O} \\
\phi &= J z^{\Delta^-} : \\
&= \frac{Z_{\text{grav.}}}{Z_{\text{CFT}}} \bigg|_{\phi \to J} = Z_{\text{CFT}}[J] \\
&= \int D\phi Z_{\text{CFT}}[\phi] e^{i \int d^d x J \phi}
\end{align*}
\]
AdS/CFT cont’d:

As with QM example, 2 different solutions $\Rightarrow$ 2 different CFTs

\[ \varphi = \varphi_0 z^\Delta^+ : \quad Z_{\text{grav.}}_{\varphi \xrightarrow{z \to 0} \varphi_0 z^\Delta^+} = Z_{\text{CFT}}[\varphi_0] \]

\[ \varphi = J z^\Delta^- : \quad Z_{\text{grav.}}_{\varphi \xrightarrow{z \to 0} J z^\Delta^-} = Z_{\text{CFT}}[J] \]

$S = S_{\text{CFT}} + \int d^d x \varphi_0 \mathcal{O}$

$= \int D\varphi Z_{\text{CFT}}[\varphi] e^{i \int d^d x J \varphi}$

UV fine-tuning: $m^2\varphi^2...$adds $\mathcal{O\mathcal{O}}$ operator. Eg: $\mathcal{O} = \bar{\psi}\psi$, $\mathcal{O\mathcal{O}} = \bar{\psi}\psi\bar{\psi}\psi$
AdS/CFT cont’d:

As with QM example, 2 different solutions $\Rightarrow$ 2 different CFTs

\[ \varphi = \varphi_0 z^\Delta^+ : \quad Z_{\text{grav.}} \bigg|_{\varphi \to \varphi_0 z^\Delta^+} = Z_{\text{CFT}}[\varphi_0] \]

\[ \varphi = J z^\Delta^- : \quad Z_{\text{grav.}} \bigg|_{\varphi \to J z^\Delta^-} = Z_{\text{CFT}}[J] \]

UV fine-tuning: $m^2 \varphi^2$...adds OO operator. Eg: $O = \bar{\psi} \psi$, $OO = \bar{\psi} \psi \bar{\psi} \psi$

\[ \Rightarrow \text{analog of } \delta(r) \text{ in QM example tuned to unstable UV fixed pt.} \]
A relativistic example: defect Yang-Mills theory
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Charged relativistic fermions on a d-dimensional defect + 4D conformal gauge theory (eg, N=4 SYM)

\[ S = \int d^{d+1}x \, i \bar{\psi} \gamma^{\mu} D_{\mu} \psi - \frac{1}{4g^2} \int d^4x \, F^{a}_{\mu\nu} F^{a,\mu\nu} \]
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\(g\) doesn’t run
g doesn’t run by construction

Expect a phase transition as a function of g:

\[ \langle \bar{\psi} \psi \rangle = \begin{cases} 
0 & g < g_* \\
\Lambda_{d}^{d} & g > g_* 
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Add a contact interaction to the theory (as in QM & AdS/CFT examples!) and study its running:

\[ \Delta S = \int d^{d+1}x \left( -\frac{c}{2} (\bar{\psi} \gamma_{\mu} T_a \psi)^2 \right) \]
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Phase transition is in perturbative regime for d=1+\varepsilon (spatial dimensions of “defect”): compute β-function
The phase transition occurs at

So we find that there is a phase transition occurring at

The dynamically generated mass gap is

The RG equation can be written in a way very similar to the RG equation for the

The solution is

when

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This substitution also works for mass gap at $g > g^*$ and conforms with BKT scaling.

$\beta(c)$:

$1/\varepsilon$ pole for $d = (1 + \varepsilon)$
\[ \beta(c) = \frac{-g^2}{2\pi} - \epsilon c - \frac{N_c}{2\pi} c^2 \]
\[ = \frac{1}{2\pi} \left( \frac{\pi^2 \epsilon^2}{N_c} - g^2 \right) - \frac{N_c}{2\pi} \left( c - \frac{\epsilon \pi}{N_c} \right)^2 \]
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- Find BKT transition at \( g^2 = g_*^2 = (\epsilon \pi)^2/N_c \)
  \( \Lambda_{\text{IR}} \sim \Lambda_{\text{UV}} \exp[-\pi/\sqrt{(g^2-g_*^2)}] \)

- Schwinger-Dyson gap eq (rainbow approx) gives qualitatively same results
Back to QCD at LARGE $N_c$ and $N_f$:

Asymptotic freedom $\Rightarrow$ conformal $\Rightarrow$ trivial

$\langle \bar{\psi}\psi \rangle \neq 0$

Transition at $x=x_c$?

Gauge coupling: $\alpha_*$

$0 \cdots x_c \cdots 11/2 \cdots x$

Banks-Zaks fixed point
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Miransky 1985

Appelquist, Terning, Wijewardhana 1996
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Schwinger-Dyson (rainbow approximation):

Found: BKT scaling for $\langle \bar{\psi} \psi \rangle$...not rigorous, but qualitatively correct?
Conjecture: loss of conformality for QCD at $x_c$ is of BKT type, due to fixed point merger.
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Near Banks-Zaks (IR) fixed point:

$\Delta^+ + \Delta^- = 4$?
**Conjecture:** loss of conformality for QCD at $x_c$ is of BKT type, due to fixed point merger.

Near Banks-Zaks (IR) fixed point:

\[ \Delta^+ + \Delta^- = 4? \]

\[ \Delta^+ \psi \bar{\psi} = 3 - \# g^2 N_c \]

(almost free quarks)
Conjecture: loss of conformality for QCD at $x_c$ is of BKT type, due to fixed point merger.

Near Banks-Zaks (IR) fixed point:

QCD:
$$\Delta^+_{\psi\bar{\psi}} = 3 - \# g^2 N_c$$
(almost free quarks)

Partner theory QCD*:
$$\Delta^-_{\psi\bar{\psi}} = d-\Delta^+_{\psi\bar{\psi}} = 1 + \# g^2 N_c$$
(almost free scalar?)
WANTED

Conformal theory
defined at nontrivial
UV fixed point
to merge with QCD
at $x = x_c$

LAST SEEN WITH WEAKLY
COUPLED SCALAR
Consider:

- SU($N_c$) gauge theory
- $N_f$ massless Dirac fermions $\psi$
- $M_f^2$ scalars $\varphi$, tuned to be massless
- coupling $\bar{\psi}\varphi\psi$
- Model has SU($M_f$)$\times$SU($M_f$) chiral symmetry, $\varphi = (\bullet, \square)$
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Conformal fixed point?

Find analog of Banks-Zaks pt. for:

$$\text{iff } M_f \leq \frac{5}{2\sqrt{11}} N_f \approx .75 N_f$$
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Find analog of Banks-Zaks pt. for:

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\text{iff } M_f \leq \frac{5}{2\sqrt{11}} N_f \approx .75 N_f
\]

..but QCD* needs full flavor symmetry. Possibly only at stronger coupling?
QCD* ?

Free fermions

\[ \Delta \bar{\psi}\psi \]

UV fixed point starts at strong-ish coupling?

\[ \Delta^+ \]

\[ \Delta^- \]

\[ QCD \]

\[ QCD^* \]

\[ x_{BZ} = 11/2 \]
QCD*? 

Free fermions

UV fixed point starts at strong-ish coupling?

Or possibly \((\Delta^+ + \Delta^-) \neq d\) in QCD?

Eg: like effect of Casimir energy in AdS/CFT
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III. Both relativistic & non-relativistic examples
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V. Implications for QCD with many flavors? Is there a pair of conformal QCD theories? What is QCD*?
Finding QCD* should be on field theory / lattice QCD “to-do” list.