Confining Interactions in $1+1$ and 't Hooft’s model of Mesons

Banks/Fishler 2009
Based on:

- P. Fonseca, AZ, 2001
- P. Fonseca, AZ; 2006
- V. Fateev, S. Lukyanov, AZ, 2009
Confinement is rather common phenomenon in 1+1 models

Its mechanism is relatively simple:

\[ V = \Delta \varepsilon |x_1 - x_2| \]

Confining potential \( \rightarrow \) Tower of “Meson” states (stable & resonances)

May occur due to:

- Adding perturbation which lifts vacuum degeneracy from spontaneously broken symmetry; “Quarks” are domain walls.

- Presence of gauge field (abelian or non-abelian), \( \Delta \varepsilon \sim E \).

The two may be related through bosonization (QED\(_2\))
Typical model:

\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 - V(\phi) \]

Confining interaction between the kinks ("quarks")
Details may depend on specific model, but basic physics is controlled by one (dimensionless) parameter

\[ \xi = \frac{\Delta \varepsilon}{m_q^2} \]

The “string tension” \( \Delta \varepsilon \) may depend on \( h \) (as well as on other parameters of the model), but at small \( h \)

\[ \Delta \varepsilon \sim h \]

In physical system \( \xi \) is real and \( \geq 0 \), but it is interesting to study analytic properties of physical quantities \( (\varepsilon_0(\xi), M_n(\xi), \text{etc}) \) as the functions of complex \( \xi \).

Basic analytic features are expected to be universal, i.e. shared by all confining interactions in \( 1+1 \).
One is well-known: There is essential singularity at \( \xi = 0 \) (Andreev (1967), Fisher (1964), Langer (1967), Kobzarev, Okun, Voloshin (1975), Coleman (1977))

\[
V_{\text{symm}}(\phi) + h V_{\text{asymm}}(\phi)
\]

Analytic continuation to negative \( h \) turns vacuum into "false vacuum"

\[
V_{\text{symm}}(\phi) + h V_{\text{asymm}}(\phi) \quad \text{vacuum}
\]

\[
V_{\text{symm}}(\phi) - h V_{\text{asymm}}(\phi) \quad \text{"false vacuum"}
\]

"False vacuum" decay:

\[
\Im m \varepsilon_0(\xi) \sim (-\xi) e^{-\frac{\pi}{|\xi|}} \quad \text{at} \quad \xi < 0.
\]
Do we encounter other singularities as we go under the branch cut?

Proposition:

- There are infinitely many singularities under the branch cut, accumulating towards $\xi = 0$.

- The singularities are critical points ($R_c$ diverges), with scaling, critical exponents, and all that.
Generally, physical nature of these singularities is yet to be understood.

Subject of this talk: Evidence for their presence.

Some singularities are well expected: “Quantum spinodal”
Why infinitely many? The masses $M_n(\xi)$ may turn to zero at certain (complex) values of $\xi$.

Simple WKB analysis (valid at small $\xi$): Two quarks with linear confining interaction,

$$H = \omega(p_1) + \omega(p_2) + \Delta\varepsilon \; |x_1 - x_2|$$

Periodic motion $\rightarrow$ Quantization condition:

$$\sinh(2\theta) - 2\theta = \pi \xi \left(n + 1/2\right), \quad n = 0, 1, 2, ...$$

$$M_n = 2m_q \cosh \theta_n.$$
\[ \xi = |\xi| e^{i\phi} \]

\[ M_n = 2m_q \cosh \theta_n, \text{ turns to zero when } \theta_n \text{ hits } i\pi/2 \]

Happens at

\[ \xi_n = \frac{\frac{3\pi i}{e}}{n + 1/2} \]
In fact, WKB breaks down when $\theta$ gets close to $i\pi/2$.

$$\sinh(2\theta) - 2\theta = \pi \xi (n + 1/2) + \xi^2 S_1(\theta) + \xi^3 S_2(\theta) + \ldots$$

where $S_1(\theta), S_2(\theta), \ldots$ diverge at $\theta = i\pi/2$.

More elaborated approach is needed.

Two models:

- Ising Field Theory in a magnetic field
- $\mathrm{QCD}_2$ at $N_c = \infty$
Ising Field Theory

\[ V_{\text{symm}}(\phi) \]

Symmetry restoration transition

\[ (\text{Universality class of 2D Ising}) \]

\[ \mathcal{L} = -\bar{\psi} (\gamma \partial) \psi - m_q \bar{\psi} \psi - h \sigma \]

\( \sigma(x) \) - "spin field".

Spontaneous magnetization at \( h = 0 \),

\[ \bar{\sigma} = \langle \sigma \rangle = \left(2^{1/12} e^{-1/8} A^{3/2}\right) m_q^{1/8} \]

At small \( h \)

\[ \Delta \varepsilon = 2h \bar{\sigma} \]
\[ \eta = \frac{1}{\xi^{8/15}} = \frac{m_q}{h^{8/15}} \]

\( YL = "\text{Quantum spinodal}\) \Rightarrow 2D CFT with \( c = -\frac{22}{5} \)
\[ \eta = \frac{1}{\xi^{8/15}} = \frac{m_q}{h^{8/15}} \]
Magnetic field $h \rightarrow$ Confining interaction between the "quarks"

$$\Delta \varepsilon = 2\sigma h + O(h^3)$$

Meson states

$$| M_n, P \rangle = \int \frac{dp}{2\pi} \psi_n(P, p) \ a^\dagger_{P+p} a^\dagger_{P-p} | 0 \rangle + ...$$

Weak coupling (small $h$):

Keeping some multi-quark terms, as needed for Lorentz invariance

$\Rightarrow$ Bethe-Salpeter equation

Rapidity variables:

$$P_+ + p_+ = m_q e^{\beta + \theta}, \quad P_+ - p_+ = m_q e^{\beta - \theta}$$

Lorentz invariance: $\psi_n$ depends only on $\theta$. 
Bethe-Salpeter equation

\[
\left[ m_q^2 - \frac{M_n^2}{4 \cosh^2 \theta} \right] \Psi_n(\theta) = \Delta \varepsilon \int_{-\infty}^{\infty} G(\theta|\theta') \Psi_n(\theta') \frac{d\theta'}{2\pi}
\]

The kernel

\[
G(\theta|\theta') = 2 \frac{\cosh(\theta - \theta')}{\sinh^2(\theta - \theta')} + \frac{1}{4} \frac{\sinh \theta}{\cosh^2 \theta} \frac{\sinh \theta'}{\cosh^2 \theta'}
\]

has second-order pole at \( \theta = \theta' \rightarrow \) Confining interaction.

\[\Rightarrow \text{Tower of eigenvalues } M_n(\eta), \, n = 1, 2, 3, \ldots \]
Real $\eta = m_q/|h|^{8/15}$
Analysis of the BS equation shows infinite set of singularities (square-root) at complex $\eta$:

\[ M_1^{(BS)}(\eta) \sim (\eta - \eta_{YL})^{1/2} \quad \text{in BS approximation} \]

\[ M_1(\eta) \sim (\eta - \eta_{YL})^{5/12} \quad \text{in full theory} \]
In IFT the BS equation is an approximation (uncontrolled at finite $\eta$), as it ignores multi-meson states.

Q: Do the complex singularities exist in full theory?

Physics is similar to QCD$_2$. At $N_c = \infty$ the BS approximation is exact ('t Hooft, 1974)

Q’: Do similar singularities exist in ’t Hooft’s model of mesons?
t’ Hooft’s model: QCD$_2$

\[ \mathcal{L} = \frac{N_c}{4g^2} \text{tr} \left( F^2 \right) - \bar{\psi} (\gamma D + m_q) \psi, \quad D_\nu = \partial_\mu + A_\mu \]

At $N_c = \infty$ the Bethe-Salpeter equation is exact.

\[ \left[ \frac{\alpha}{x} + \frac{\alpha}{1-x} \right] \varphi(x) - \int_0^1 dy \frac{\varphi(y)}{(y-x)^2} = 2\pi^2 \lambda \varphi(x), \]

\[ \alpha = \frac{\pi m_q^2}{g^2} - 1, \quad M^2 = 2\pi g^2 \lambda. \]

Spectral problem for $\lambda \to \lambda_n(\alpha)$.

Analytic properties of $\lambda_n(\alpha)$ at complex $\alpha$? Singular points?
Singularity at $\alpha = -1$. Chiral limit $m_q \to 0$:

$$M_\pi^2 \sim m_q g \to \lambda_0(\alpha) \sim \sqrt{\alpha + 1}$$

Critical point. At finite $N_c$

$$N_c \text{ WZW } [G_{\text{flavor}}]$$

[Gepner, 1988; Affleck, 1989]

Other (complex) singularities?

Preliminary study [V. Fateev, S. Lukyanov, AZ, 2009]
Rapidity form: 

\[ x = \frac{1}{2} (1 + \tanh \theta) \]

\[
\left[ 2\alpha - \frac{\pi^2 \lambda}{\cosh^2 \theta} \right] \Psi(\theta) = \int_{-\infty}^{\infty} G(\theta - \theta') \Psi(\theta) \, d\theta
\]

\[ G(\theta - \theta') = \frac{1}{\sinh^2(\theta - \theta')} \]

has second-order pole at \( \theta = \theta' \).

Yet more convenient form

\[ \Psi(\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} e^{i\nu\theta} \Phi(\nu) \]

\[
\left[ \alpha + \frac{\pi \nu}{2} \coth \frac{\pi \nu}{2} \right] \Phi(\nu) = \frac{\pi \lambda}{2} \int_{-\infty}^{\infty} d\nu' \, S(\nu - \nu') \Phi(\nu')
\]

\[ S(\nu) = \frac{\pi \nu}{2 \sinh \frac{\pi \nu}{2}} \]
• $\Phi(\nu)$ is meromorphic function of $\nu$, with poles at

$$\nu_l + 2iN, \quad -\nu_l - 2iN,$$

with $N = 0, 1, 2, \ldots$, and $\nu_l$ - roots of

$$\alpha + \frac{\pi \nu}{2} \coth \frac{\pi \nu}{2} = 0$$

with $\Im \nu_l \geq 0$ at real $\alpha > -1$
At complex $\alpha$ the pole $-\nu_0$ can wander into the upper half-plane, and at special $\alpha_k$ it hits another pole there.
This gives rise to singularities (square-root branching points of \( \lambda_n(\alpha) \)) at

\[
\alpha_k = -\frac{1}{2} \left[ 1 + \cosh(\pi \nu_k) \right]
\]

\[
\sinh(\pi \nu_k) - \pi \nu_k = 0 , \quad \Re \nu_k \leq 0
\]
\[
\eta = \sqrt{\alpha + 1}
\]

Singular points in the plane of \(\sqrt{\alpha + 1}\)

**Proposition:** \(\eta_k\) are critical points:

\[
\lambda_{2k}(\alpha) \sim \sqrt{\eta - \eta_k}
\]
The operator
\[
\hat{S} : \Phi(\nu) \rightarrow \int_{-\infty}^{\infty} d\nu' \ S(\nu - \nu')\Phi(\nu')
\]
is inverse to a finite-difference operator \( \Rightarrow \) Finite difference equation

\[
Q(\nu + 2i) + Q(\nu - 2i) - 2Q(\nu) = U(\nu) Q(\nu)
\]
for

\[
Q(\nu) = \left[ \alpha \sinh \frac{\pi \nu}{2} + \frac{\pi \nu}{2} \cosh \frac{\pi \nu}{2} \right] \Phi(\nu)
\]
with

\[
U(\nu) = 2\pi^2 \lambda \left[ \alpha + \frac{\pi \nu}{2} \coth \frac{\pi \nu}{2} \right]^{-1}
\]

Baxter’s TQ equation (with \( T(\nu) = 2 + U(\nu) \))

Analytic results for \( \lambda_n(\alpha) \)
1. Systematic large-$n$ expansions of $\lambda_n(\alpha)$:

\[
\begin{align*}
n &= 2\lambda - \frac{2\alpha}{\pi^2} \log(2\lambda) - C_0(\alpha) + \frac{\alpha^2}{\pi^4\lambda} + \frac{C_2(\alpha)}{\lambda^2} + \\
& \quad \frac{1}{\lambda^3} \left[ C_3(\alpha) - \frac{(-)^n(1+\alpha)}{\pi^6} \left( \log \left(2\pi \lambda e^{\gamma_E}\right) + C'_3(\alpha) \right) \right] + \ldots
\end{align*}
\]

where

\[
\begin{align*}
C_0(\alpha) &= \frac{3}{4} + \frac{2\alpha}{\pi^2} \log \left(4\pi e^{\gamma_E}\right) - \\
& \quad \frac{\alpha^2}{2\pi^2} \int_{-\infty}^{\infty} \frac{dt}{t} \frac{\sinh(t) \left( \sinh(2t) - 2t \right)}{\cosh^2(t) \left( \alpha \sinh(t) + t \cosh(t) \right)},
\end{align*}
\]

\[
\begin{align*}
C_2(\alpha) &= \frac{1}{2\pi^6} \left[ \alpha^3 + (-1)^n \pi^2 (1 + \alpha) \right],
\end{align*}
\]

\[
\begin{align*}
C_3(\alpha) &= \frac{1}{12\pi^8} \left[ 5\alpha^4 + \pi^2 (1 + \alpha)^2 \right],
\end{align*}
\]

\[
\begin{align*}
C'_3(\alpha) &= -\frac{1 + 3\alpha}{3} + \frac{\alpha}{8} \int_{-\infty}^{\infty} dt \frac{\sinh(2t) - 2t}{t \sinh(t) \left( \alpha \sinh(t) + t \cosh(t) \right)},
\end{align*}
\]

[’t Hooft, 1974; Brauer, Spence, Weis, 1979; Fateev, Lukyanov, AZ, 2009]
2. Exact sum rules:

\[ G^{(s)}_{+}(\alpha) = \sum_{m=0}^{\infty} \frac{1}{\lambda_{2m}^{s}(\alpha)} , \quad G^{(s)}_{-}(\alpha) = \sum_{m=0}^{\infty} \frac{1}{\lambda_{2m+1}^{s}(\alpha)} \]

E.g.

\[ G^{(1)}_{\pm}(\alpha) = \log(8\pi) - 2 \pm 1 - \frac{\alpha}{4} \int_{-\infty}^{\infty} \frac{dt}{t} \frac{\sinh(t) (\sinh(2t) \pm 2t)}{\cosh^{2}(t) (\alpha \sinh(t) + t \cosh(t))} . \]

\[ G^{(s)}_{+}(\alpha) \sim (\alpha - \alpha_{k})^{-s/2} , \quad G^{(s)}_{-}(\alpha) \sim (\alpha - \alpha_{k})^{1/2} \]

\( \alpha_{k} \) are critical points: \( M_{2k}^{2}(\alpha_{k}) \sim \sqrt{\alpha - \alpha_{k}} \).
Speculation: $N_c < \infty$, 

$$M_{2k}^2(\alpha) \sim (\alpha - \alpha_k)^{\beta_k}$$

with critical exponents 

$$\beta_k = \frac{1}{2} + \frac{b_k}{N_c} + ...$$

$\alpha_k$ are likely to become non-trivial (non-unitary) CFT.

Q: What kind of criticality $\alpha_k$ correspond to?

Requires study of finite $N_c$ QCD$_2$. 
Summary:

- $N_c = \infty$ QCD$_2$ has infinitely many critical points at complex
  $\alpha = m_q^2/g^2 - 1$

- This phenomenon seems to be common for confining theories in 1+1 (e.g. IFT in a magnetic field).

Main Question: What these critical points try to tell us about basic mechanism of confinement?