# Confining Interactions in 1+1 and 't Hooft's model of Mesons

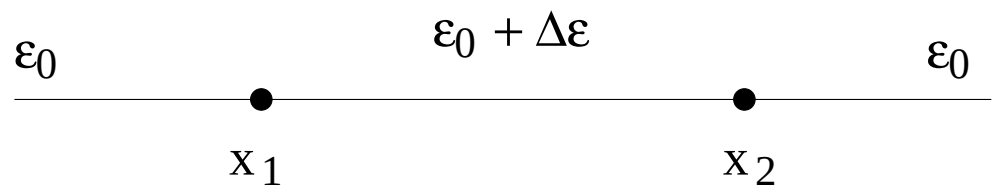
Banks/Fishler 2009

Based on:

- P.Fonseca, AZ, 2001
- P.Fonseca, AZ; 2006
- V.Fateev, S.Lukyanov, AZ, 2009

Confinement is rather common phenomenon in 1+1 models

Its mechanism is relatively simple:




$$V = \Delta\varepsilon \; |x_1 - x_2|$$

Confining potential $\rightarrow$ Tower of "Meson" states (stable & resonances)

May occure due to:

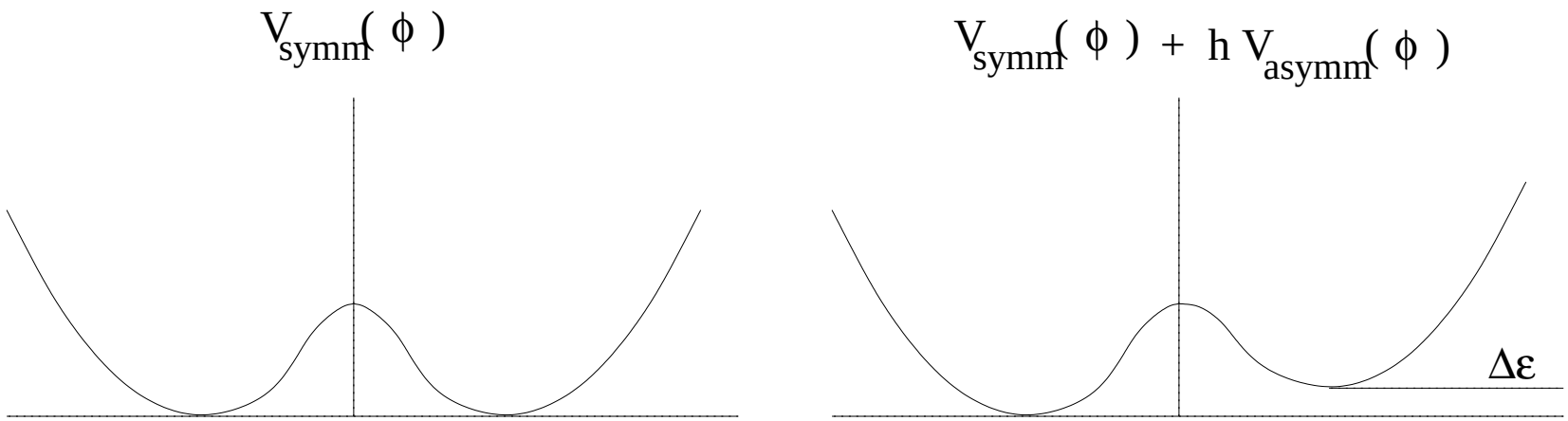
- Adding perturbation which lifts vacuum degeneracy from spontaneously broken symmetry; "Quarks" are domain walls.

- Presence of gauge field (abeelian or non-abelian), $\Delta\varepsilon \sim E$.

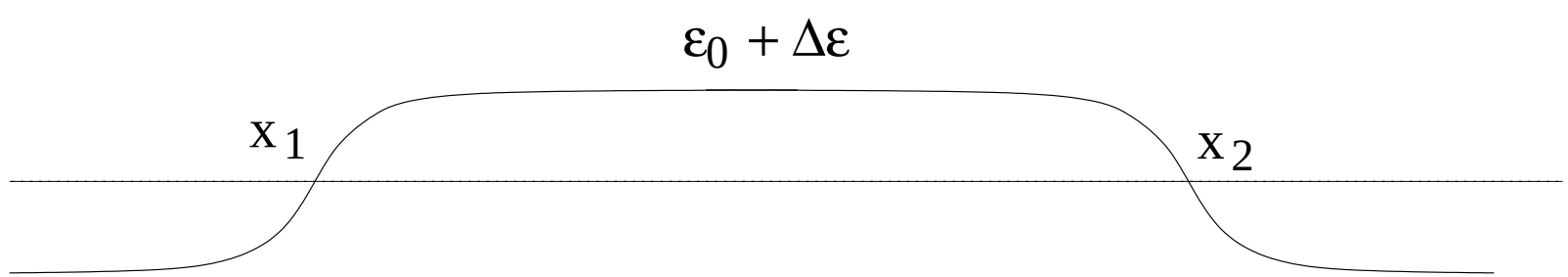
The two may be related through bosonization ($QED_2$)

Typical model:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$




Confining interaction between the kinks ("quarks")



Details may depend on specific model, but basic physics is controlled by one (dimensionless) parameter

$$\xi = \frac{\Delta\varepsilon}{m_q^2}$$

The "string tension" $\Delta\varepsilon$ may deepend on $h$ (as well as on other parameters of the model), but at small $h$

$$\Delta\varepsilon \sim h$$

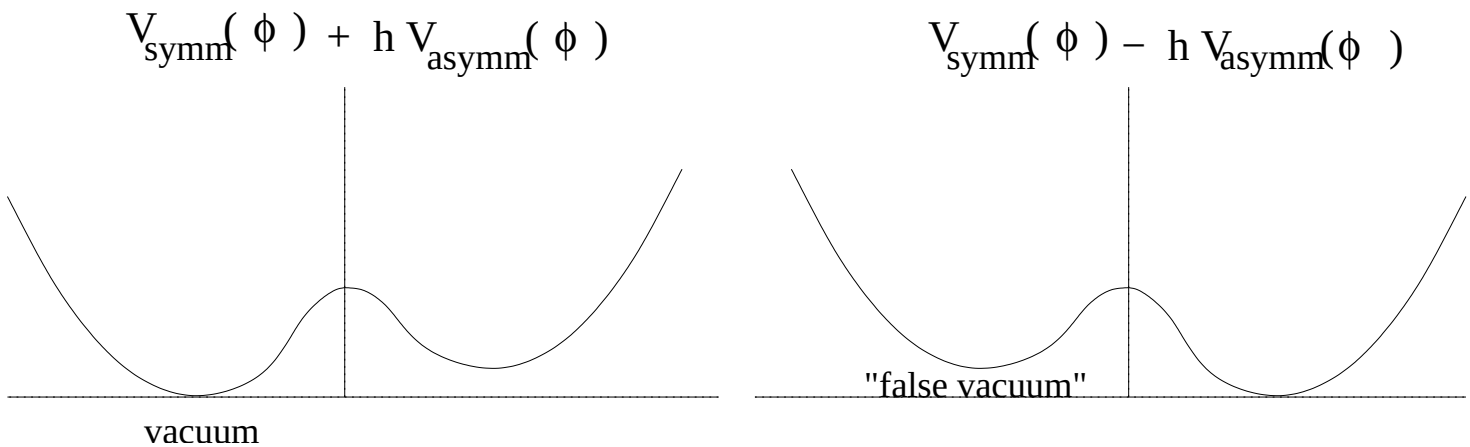
In physical system $\xi$ is real and $\geq 0$, but it is interesting to study analytic properties of physical quantities ($\varepsilon_0(\xi)$, $M_n(\xi)$, etc) as the functions of complex $\xi$.

Basic analytic features are expected to be universal, i.e. shared by all confining interactions in 1+1.

One is well-known: There is essential singularity at $\xi = 0$ (Andreev (1967), Fisher (1964), Langer (1967), Kobzarev, Okun, Voloshin (1975), Coleman (1977))

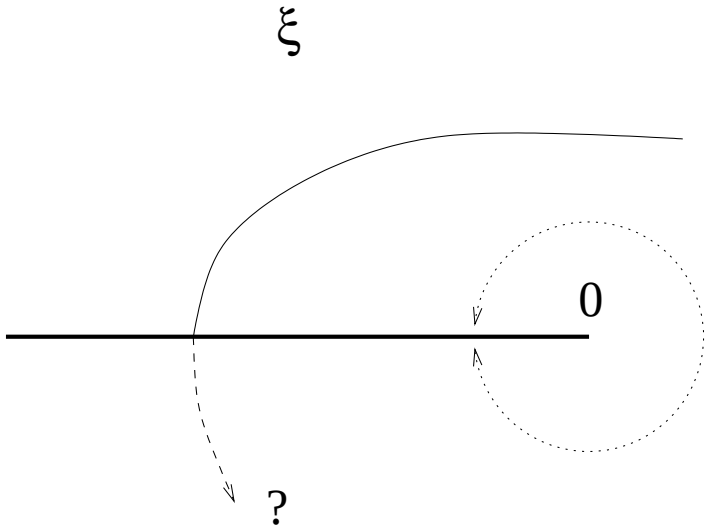
$$V_{\text{symm}}(\phi) + h\, V_{\text{asymm}}(\phi)$$

Analytic continuation to negative $h$ turns vacuum into "false vacuum"




"False vacuum" decay:

$$\Im m\, \varepsilon_0(\xi) \sim (-\xi)\, e^{-\frac{\pi}{|\xi|}} \qquad \text{at} \quad \xi < 0\,.$$



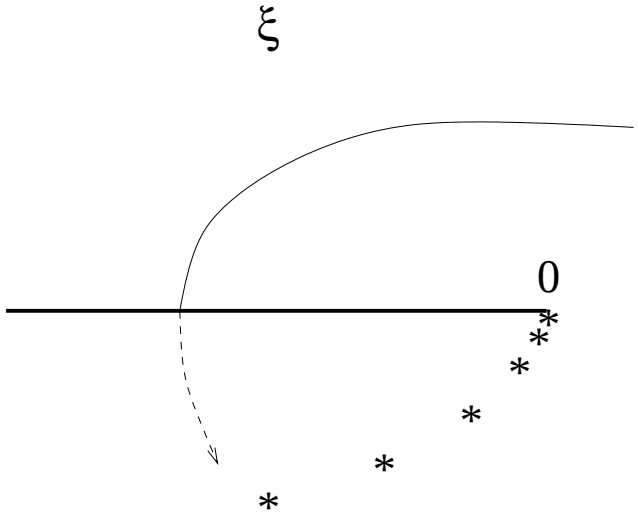

$$\mathrm{disc}(\ \varepsilon_0\ )\ \sim\ \mathrm{i}\,(-\xi\ )\,\exp(\pi/\xi\ )$$

Do we encounter other singularities as we go <u>under</u> the branch cut?

Proposition:

- There are infinitely many singularities under the branch cut, accumulating towards $\xi = 0$.

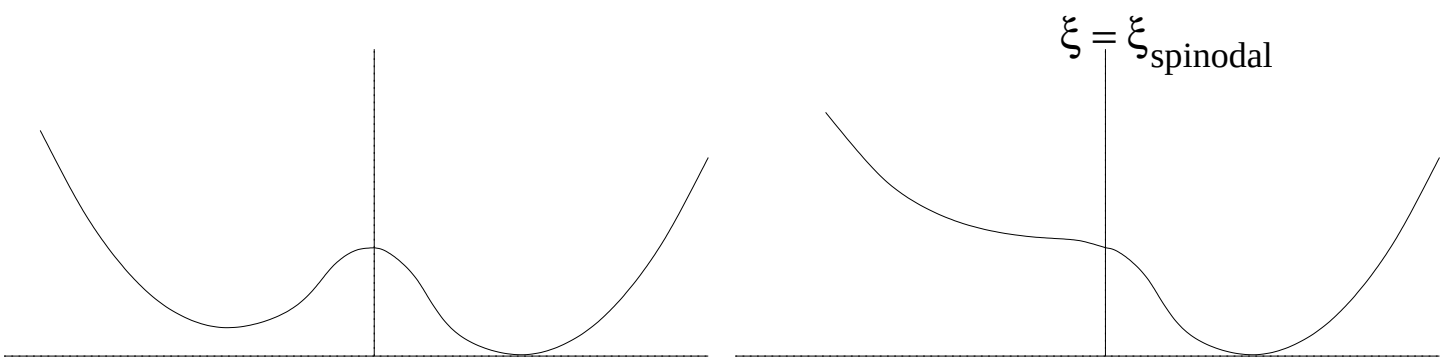
- The singularities are <u>critical points</u> ($R_c$ diverges), with scaling, critical exponents, and all that.




Generally, physical nature of these singularities is yet to be understood.

Subject of this talk: Evidence for their presence.

Some singularities are well expected: "Quantum spinodal"



Why infinitely many? The masses $M_n(\xi)$ may turn to zero at certain (complex) values of $\xi$.

Simple WKB analysis (valid at small $\xi$): Two quarks with linear confining interaction,

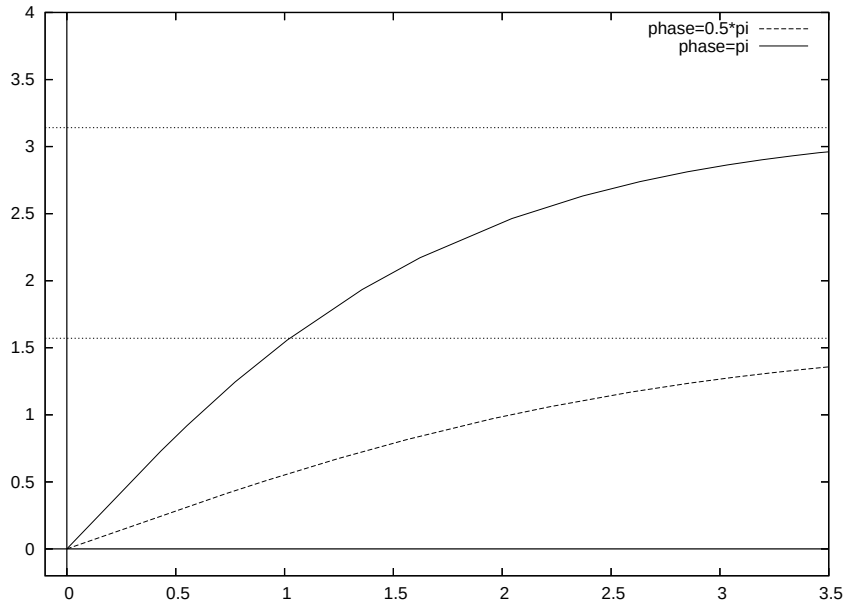
$$H = \omega(p_1) + \omega(p_2) + \Delta\varepsilon \; |x_1 - x_2|$$

Periodic motion $\rightarrow$ Quantization condition:

$$\sinh(2\theta) - 2\theta = \pi\,\xi \; (n + 1/2)\,, \quad n = 0, 1, 2, ...$$

$$M_n = 2m_q \cosh\theta_n\,.$$

$$\xi = |\xi|\, e^{i\phi}$$




$M_n = 2m_q \cosh\theta_n$, turns to zero when $\theta_n$ hits $i\pi/2$

Happens at

$$\xi_n = \frac{e^{\frac{3\pi i}{2}}}{n + 1/2}$$

In fact, WKB breaks down when $\theta$ gets close to $i\pi/2$.

$$\sinh(2\theta) - 2\theta = \pi\,\xi\ (n + 1/2) + \xi^2\ S_1(\theta) + \xi^3\ S_2(\theta) + ...$$

where $S_1(\theta), S_2(\theta), ...$ diverge at $\theta = i\pi/2$.

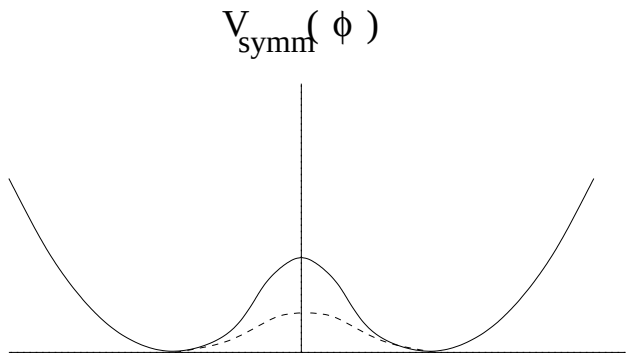
More elaborated approach is needed.

Two models:

- Ising Field Theory in a magnetic field

- $\mathrm{QCD}_2$ at $N_c = \infty$

# Ising Field Theory

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Symmetry restoration transition

(Universality class of 2D Ising)

$$\mathcal{L} = -\bar{\psi}\,(\gamma\partial)\,\psi - m_q\,\bar{\psi}\psi - h\,\sigma$$

$\sigma(x)$- "spin field".

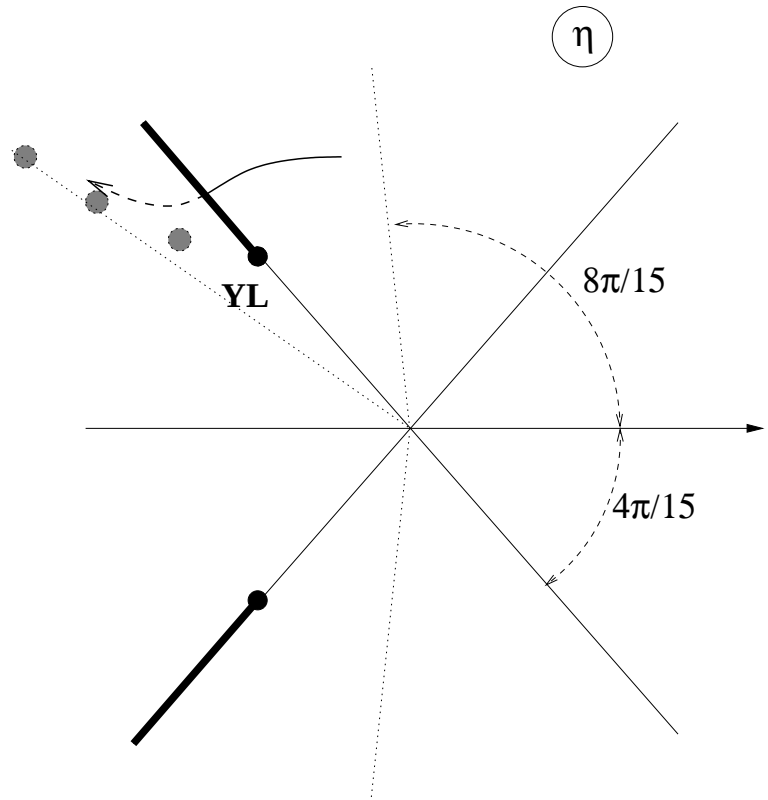
Spontaneous magnetization at $h = 0$,

$$\bar{\sigma} = \langle\,\sigma\,\rangle = \left(2^{1/12}\,e^{-1/8}\,A^{3/2}\right)\,m_q^{1/8}$$

At small $h$

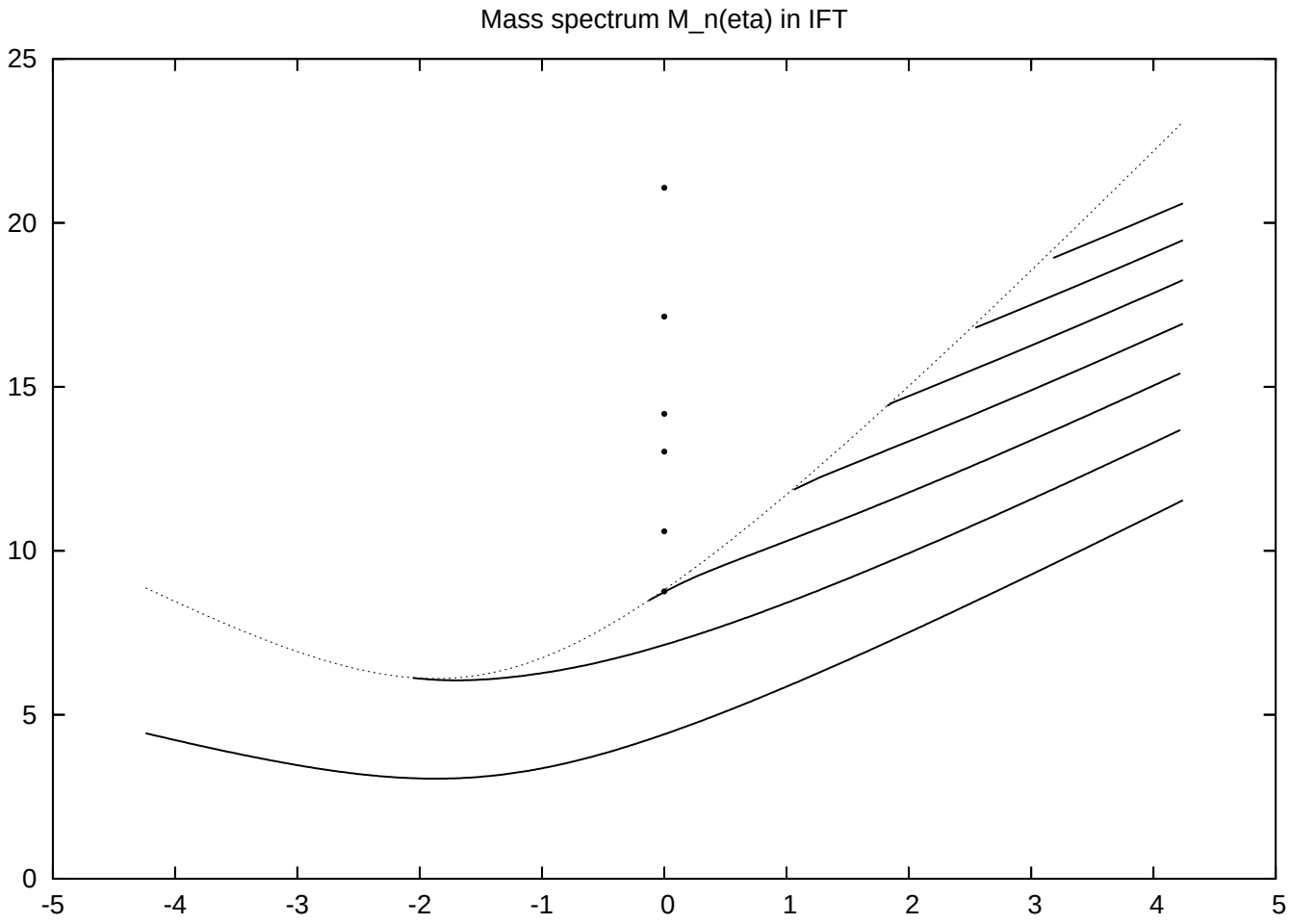
$$\Delta\varepsilon = 2h\,\bar{\sigma}$$

$$\eta = \frac{1}{\xi^{8/15}} = \frac{m_q}{h^{8/15}}$$




YL = "Quantum spinodal" $\Rightarrow$ 2D CFT with c $= -22/5$

# Mass spectrum of IFT (numerical)




$$\eta = \frac{1}{\xi^{8/15}} = \frac{m_q}{h^{8/15}}$$

Magnetic field $h \to$ Confining interaction between the "quarks"

$$\Delta\varepsilon = 2\bar{\sigma}\, h + O(h^3)$$

Meson states

$$\mid M_n, P \rangle = \int \frac{dp}{2\pi}\, \Psi_n(P,p)\ \mathbf{a}^\dagger_{P+p}\mathbf{a}^\dagger_{P-p} \mid 0 \rangle + ...$$

Weak coupling (small $h$):

Keeping some multi-quark terms, as needed for Lorentz invariance
$\Rightarrow$ Bethe-Salpeter equation

Rapidity variables:

$$P_+ + p_+ = m_q\, e^{\beta+\theta}\,, \qquad P_+ - p_+ = m_q\, e^{\beta-\theta}$$

Lorentz invariance: $\Psi_n$ depends only on $\theta$.

Bethe-Salpeter equation

$$\left[m_q^2 - \frac{M_n^2}{4\,\cosh^2\theta}\right]\Psi_n(\theta) = \Delta\varepsilon \int_{-\infty}^{\infty} G(\theta|\theta')\,\Psi_n(\theta')\,\frac{d\theta'}{2\pi}$$

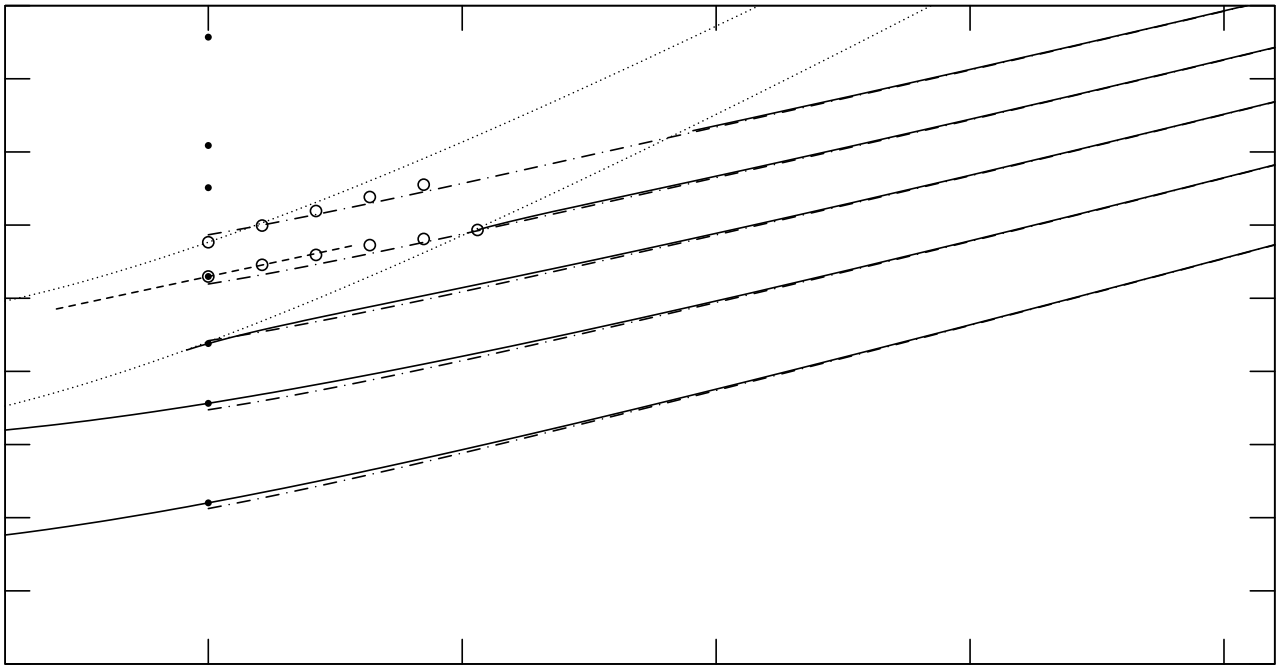
The kernel

$$G(\theta|\theta') = 2\,\frac{\cosh(\theta-\theta')}{\sinh^2(\theta-\theta')} + \frac{1}{4}\,\frac{\sinh\theta}{\cosh^2\theta}\,\frac{\sinh\theta'}{\cosh^2\theta'}$$

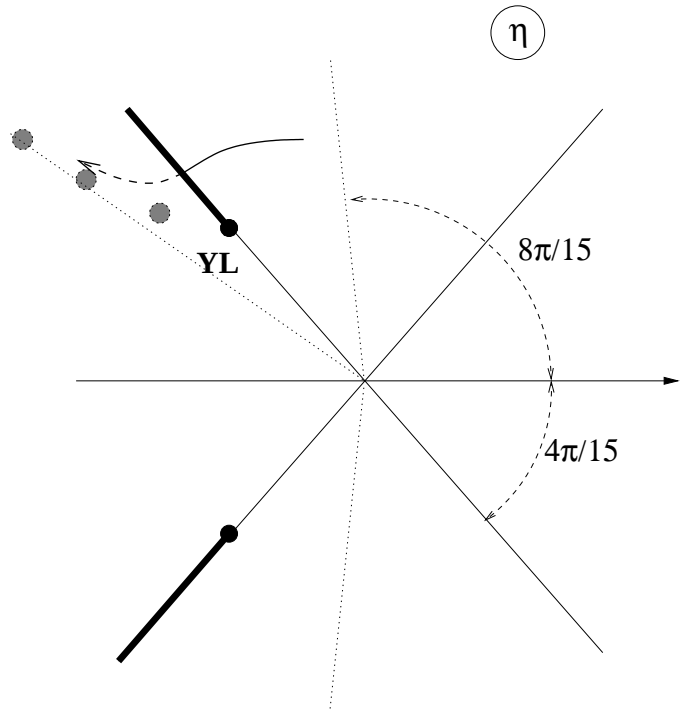
has second-order pole at $\theta = \theta' \rightarrow$ Confining interaction.

$\Rightarrow$ Tower of eigenvalues $M_n(\eta)$, $n = 1, 2, 3, ...$

Real $\eta = m_q/|h|^{8/15}$



Analysis of the BS equation shows infinite set of singularities (square-root) at complex $\eta$:




E.g.

$$M_1^{(\mathrm{BS})}(\eta) \sim (\eta - \eta_{\mathrm{YL}})^{1/2} \qquad \text{in BS approximation}$$
$$M_1(\eta) \sim (\eta - \eta_{\mathrm{YL}})^{5/12} \qquad \text{in full theory}$$

In IFT the BS equation is an approximation (uncontrolled at finite $\eta$), as it ignores multi-meson states.

Q: Do the complex singularities exist in full theory?

Physics is similar to $QCD_2$. At $N_c = \infty$ the BS approximation is exact ('t Hooft, 1974)

Q': Do similar singularities exist in 't Hooft's model of mesons?

t' Hooft's model: $\mathrm{QCD}_2$

$$\mathcal{L} = \frac{N_c}{4g^2}\,\mathrm{tr}\left(F^2\right) - \bar{\psi}\left(\gamma D + m_q\right)\psi\,, \qquad D_\nu = \partial_\mu + A_\mu$$

At $N_c = \infty$ the Bethe-Salpeter equation is exact.

$$\left[\frac{\alpha}{x} + \frac{\alpha}{1-x}\right]\varphi(x) - \int_0^1 \mathrm{d}y\ \frac{\varphi(y)}{(y-x)^2} = 2\pi^2\,\lambda\ \ \varphi(x)\ ,$$

$$\alpha = \frac{\pi m_q^2}{g^2} - 1\,, \qquad M^2 = 2\pi g^2\ \lambda\,.$$

Spectral problem for $\lambda \to \lambda_n(\alpha)$.

Analytic properties of $\lambda_n(\alpha)$ at complex $\alpha$? Singular points?

Singularity at $\alpha = -1$. Chiral limit $m_q \to 0$:

$$M_\pi^2 \sim m_q\, g \quad \to \quad \lambda_0(\alpha) \sim \sqrt{\alpha + 1}$$

Critical point. At finite $N_c$

$$N_c \text{ WZW } [G_{\text{flavor}}]$$

[Gepner, 1988; Affleck, 1989]

Other (complex) singularities?

Preliminary study [V.Fateev, S.Lukyanov, AZ, 2009]

Rapidity form: $x = \frac{1}{2}\,(1 + \tanh\theta)$

$$\left[2\alpha - \frac{\pi^2\,\lambda}{\cosh^2\theta}\right]\Psi(\theta) = \int_{-\infty}^{\infty} G(\theta - \theta')\,\Psi(\theta)\,d\theta$$

$$G(\theta - \theta') = \frac{1}{\sinh^2(\theta - \theta')}$$

has second-order pole at $\theta = \theta'$.

Yet more convenient form

$$\Psi(\theta) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi}\; e^{i\nu\theta}\;\Phi(\nu)$$

$$\left[\alpha + \frac{\pi\nu}{2}\,\coth\frac{\pi\nu}{2}\right]\Phi(\nu) = \frac{\pi\lambda}{2}\int_{-\infty}^{\infty} d\nu'\;S(\nu - \nu')\,\Phi(\nu')$$

$$S(\nu) = \frac{\pi\nu}{2\sinh\frac{\pi\nu}{2}}$$

- $\Phi(\nu)$ is meromorphic function of $\nu$, with poles at

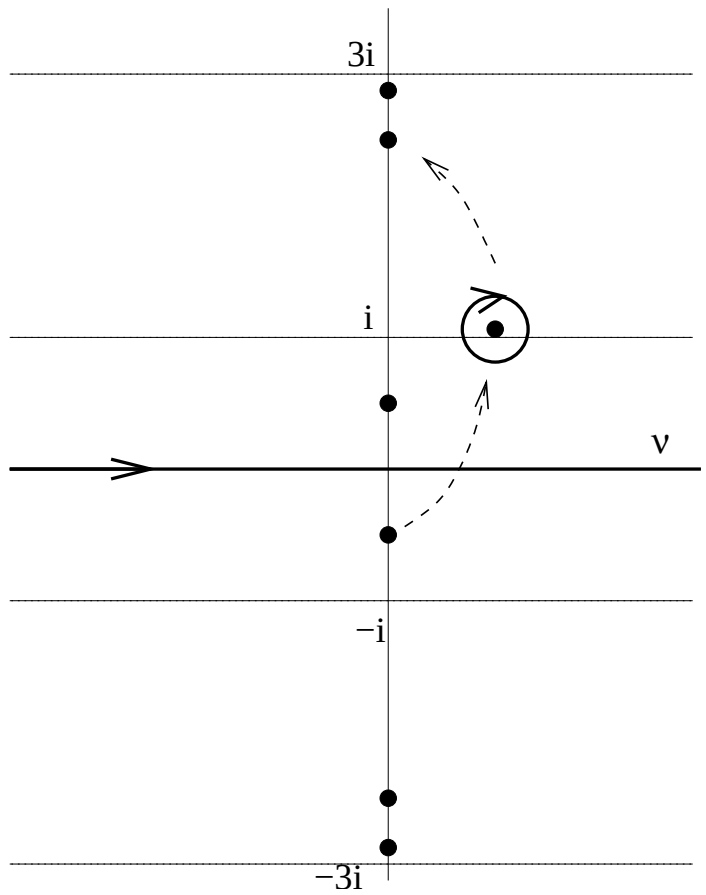
$$\nu_l + 2iN\,, \qquad -\nu_l - 2iN\,,$$

with $N = 0, 1, 2, ...$, and $\nu_l$ - roots of

$$\alpha + \frac{\pi\nu}{2}\coth\frac{\pi\nu}{2} = 0$$

with $\Im m\,\nu_l \geq 0$ at real $\alpha > -1$

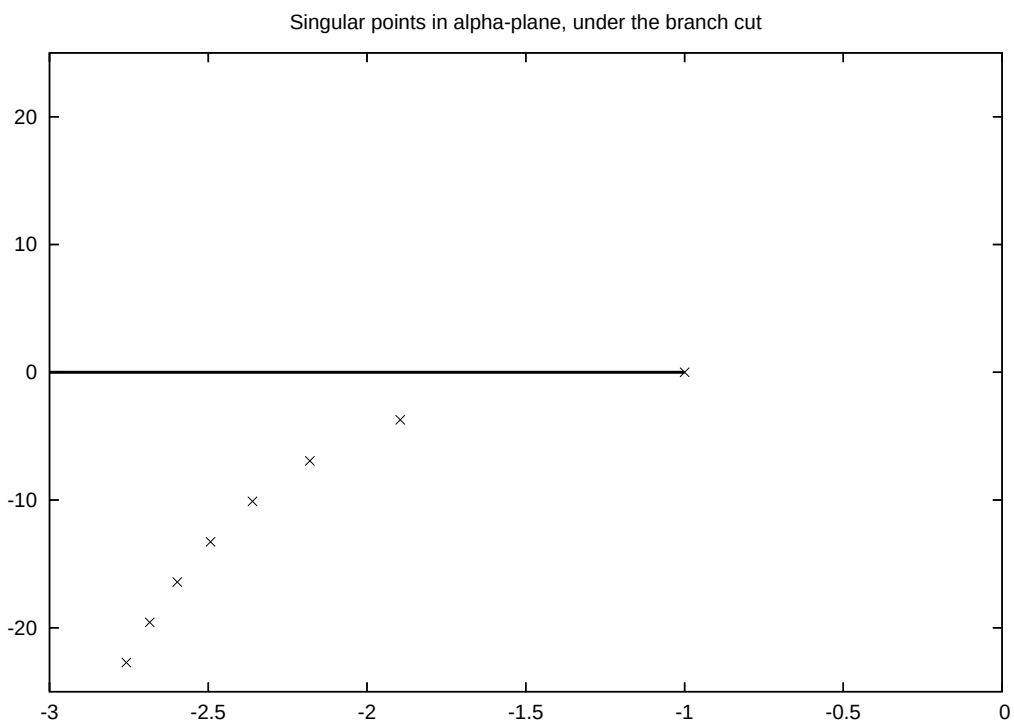
• At complex $\alpha$ the pole $-\nu_0$ can wander into the upper half-plane, and at special $\alpha_k$ it hits another pole there.



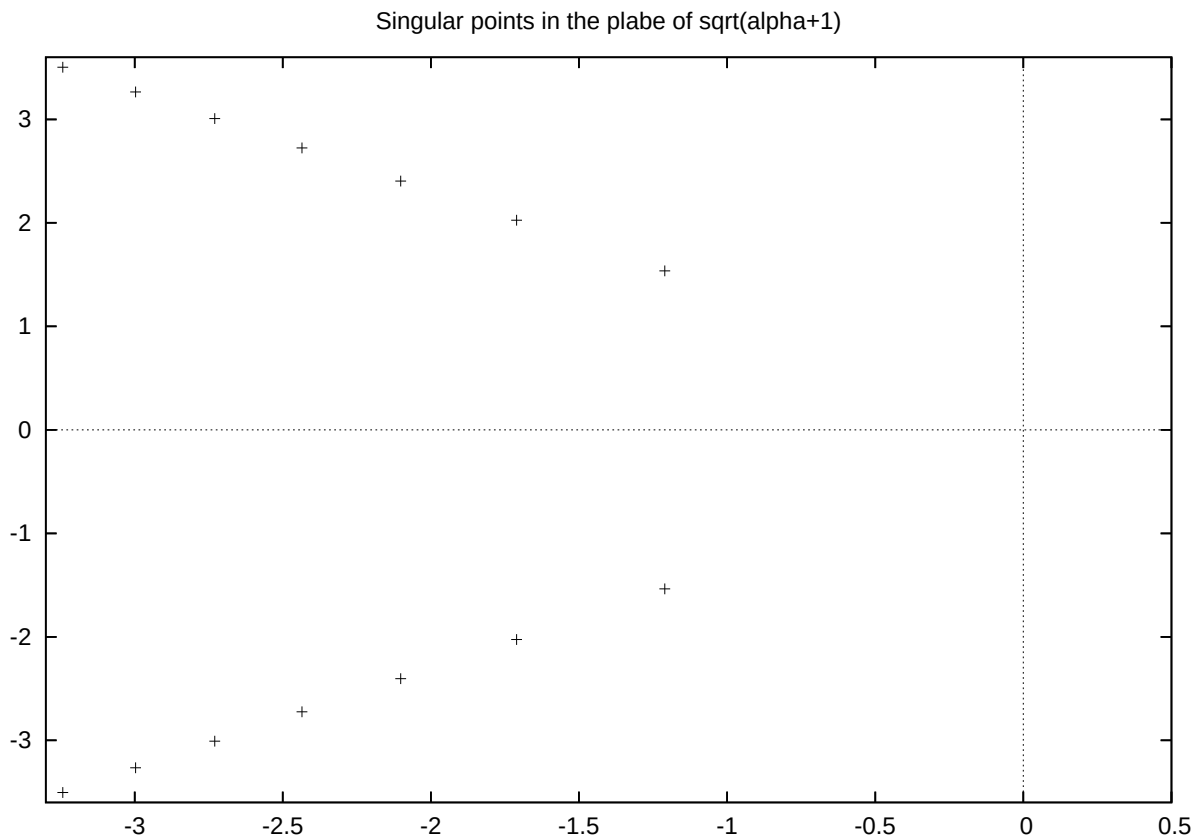
• This gives rise to singularities (square-root branching points of $\lambda_n(\alpha)$) at

$$\alpha_k = -\frac{1}{2}\,[1 + \cosh(\pi\nu_k)]$$

$$\sinh(\pi\nu_k) - \pi\nu_k = 0\ , \qquad \Re e\,\nu_k \leq 0$$



$$\eta = \sqrt{\alpha + 1}$$




Proposition: $\eta_k$ are critical points:

$$\lambda_{2k}(\alpha) \ \sim \ \sqrt{\eta - \eta_k}$$

The operator

$$\widehat{S} : \Phi(\nu) \to \int_{-\infty}^{\infty} d\nu' \; S(\nu - \nu')\Phi(\nu')$$

is inverse to a finite-difference operator $\Rightarrow$ Finite difference equation

$$Q(\nu + 2i) + Q(\nu - 2i) - 2\,Q(\nu) = U(\nu)\,Q(\nu)$$

for

$$Q(\nu) = \left[\alpha \sinh \frac{\pi\nu}{2} + \frac{\pi\nu}{2} \cosh \frac{\pi\nu}{2}\right] \Phi(\nu)$$

with

$$U(\nu) = 2\pi^2 \lambda \left[\alpha + \frac{\pi\nu}{2} \coth \frac{\pi\nu}{2}\right]^{-1}$$

Baxter's TQ equation (with $T(\nu) = 2 + U(\nu)$)

Analytic results for $\lambda_n(\alpha)$

1. Systematic large-$n$ expansions of $\lambda_n(\alpha)$:

$$n = 2\lambda - \frac{2\alpha}{\pi^2}\,\log(2\lambda) - C_0(\alpha) + \frac{\alpha^2}{\pi^4\lambda} + \frac{C_2(\alpha)}{\lambda^2} + \frac{1}{\lambda^3}\left[C_3(\alpha) - \frac{(-)^n\,(1+\alpha)}{\pi^6}\left(\log\left(2\pi\lambda e^{\gamma_E}\right) + C_3'(\alpha)\right)\right] + \ldots$$

where

$$C_0(\alpha) = \frac{3}{4} + \frac{2\alpha}{\pi^2}\,\log\left(4\pi e^{\gamma_E}\right) - \frac{\alpha^2}{2\pi^2}\int_{-\infty}^{\infty}\frac{dt}{t}\,\frac{\sinh(t)\,(\sinh(2t) - 2t)}{\cosh^2(t)\left(\alpha\,\sinh(t) + t\,\cosh(t)\right)}\,,$$

$$C_2(\alpha) = \frac{1}{2\pi^6}\left[\alpha^3 + (-1)^n\,\pi^2\,(1+\alpha)\right]\,,$$

$$C_3(\alpha) = \frac{1}{12\pi^8}\left[5\alpha^4 + \pi^2\,(1+\alpha)^2\right]\,,$$

$$C_3'(\alpha) = -\frac{1+3\alpha}{3} + \frac{\alpha}{8}\int_{-\infty}^{\infty} dt\;\frac{\sinh(2t) - 2t}{t\,\sinh(t)\left(\alpha\,\sinh(t) + t\,\cosh(t)\right)}\,.$$

['t Hooft, 1974; Brauer, Spence, Weis, 1979; Fateev, Lukyanov, AZ, 2009]

2. Exact sum rules:

$$G_+^{(s)}(\alpha) = \sum_{m=0}^{\infty} \frac{1}{\lambda_{2m}^s(\alpha)} \ , \quad G_-^{(s)}(\alpha) = \sum_{m=0}^{\infty} \frac{1}{\lambda_{2m+1}^s(\alpha)}$$

E.g.

$$G_\pm^{(1)}(\alpha) = \log(8\pi) - 2 \pm 1 - \frac{\alpha}{4} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t} \frac{\sinh(t)\ (\sinh(2t) \pm 2t)}{\cosh^2(t)\,(\alpha \sinh(t) + t\,\cosh(t)\,)} \ .$$

$$G_+^{(s)}(\alpha) \sim (\alpha - \alpha_k)^{-s/2} \ , \qquad G_-^{(s)}(\alpha) \sim (\alpha - \alpha_k)^{1/2}$$

$\alpha_k$ are critical points: $M_{2k}^2(\alpha_k) \sim \sqrt{\alpha - \alpha_k}$.

Speculation: $N_c < \infty$,

$$M_{2k}^2(\alpha) \sim (\alpha - \alpha_k)^{\beta_k}$$

with critical exponents

$$\beta_k = \frac{1}{2} + \frac{b_k}{N_c} + ...$$

$\alpha_k$ are likely to become non-trivial (non-unitary) CFT.

Q: What kind of criticality $\alpha_k$ correspond to?

Requires study of finite $N_c$ $\mathrm{QCD}_2$.

Summary:

- $N_c = \infty$ $\mathrm{QCD}_2$ has infinitely many critical points at complex $\alpha = m_q^2/g^2 - 1$

- This phenomenon seems to be common for confining theories in 1+1 (e.g. IFT in a magnetic field).

Main Question: What these critical points try to tell us about basic mechanism of confinement?