Slewing Extrapolation.

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Abstract

In this memo we describe the extrapolation method used to extend slewing curves for the HiToT calibration. The method is based on random function's approach. The random function framework is set and carried out to the first order of canonical expansion. The statistical error of the extrapolation is shown to be within 1.4 counts at the end of the range.

1 Extrapolation model.

A scheme of estimation of slewing correction for ToT's outside of calibrated range (extrapolation) must model the function beyond the known points by some plausible functional form. In the case of slewing, the functional shape depends on the discriminator threshold level, amplification coefficients, gains of PMTs, wiring and so on. Instead of trying to take all the unknown parameters into account and putting together a physical model of the slewing we have built a statistical one taking the following approach.

We believe that all PMT channels (PMTs themselves and electronic boards) were designed and manufactured to meet common characteristics. Therefore, the study of the channels' responses (calibration) can be viewed as a multiple (about 700 times) measurement of a singe function: T_{start} vs ToT. The fact that the curves obtained for different channels are slightly different can be attributed to the "manufacturing imperfections", and can be treated as such. Here, we mean that the channels differ only due to unavoidable uncontrollable reasons such as spread of characteristics of electronic components and/or actual slewing measurement errors. Thus, we view all slewing curves for all PMTs as particular realizations of the same random function. All slewing curves together form one slewing function family. The validity of this assumption is demonstrated by figure (1) of section 5. For a brief review of the notion of random functions please refer to [1]. The family can be characterized by mean dependence $m(t)$, and correlation function $K(t, t')$ just as mean and dispersion for a random variable. (Here we denote time over threshold as t or t'.) Construction of the statistical model of slewing and extrapolation, within this framework, looks as:

1. Obtain the properties of the random function from the measured data and deduce $m(t)$ and $K(t, t')$. These are the parameters of the model.

2. Using $m(t)$ generate the set of synthetic slewing data beyond the calibrated range for each PMT. This choice provides a best estimate over the class of curves in the least square sense if no additional knowledge is involved.

3. Run a polynomial through all synthetic data points to produce the slewing coefficients for the extended range.

The power of this method is that the extrapolation model is learned on the example of the "fortunate" channels, and the obvious extrapolation test will be automatically satisfied: cut the existing long curve, extrapolate it and compare with the existing data. If, however, long correlations exist (which should be the case because we are dealing with a physical system), there is a hope of improving the extrapolation by including this information into account. Thus, the mean slewing function $m(t)$ serves as a zero order approximation.

2 Higher order corrections.

Inasmuch as a random function is a mathematical concept of great complexity and in most general case it can be regarded as a non-denumerable set of scalar random variables, it is natural to seek the expression of the random function in terms of simpler random concepts: ordinary scalar random variables. We try to express the slewing random function $X(t)$ in the form:

$$X(t) = m(t) + \sum_{\nu} V_{\nu}\phi_{\nu}(t) \tag{1}$$

where $\{\phi_{\nu}(t)\}$ are some, non-random, functions, V_{ν} are random variables. The representation (1) is said to be canonical if $m(t)$ is the mean function and V_{ν}'s are uncorrelated with zero mean:

$$M\left[V_{\nu}\right] = 0 \quad D\left[V_{\nu}\right] = M\left[V_{\nu}^{2}\right] = D_{\nu}$$

$$M\left[V_{\mu}V_{\nu}\right] = 0 \quad \mu \neq \nu \tag{2}$$

If such a representation is found, then for each particular slewing curve $x(t)$ we can calculate the corresponding set of expansion coefficients v_{ν} in the range where calibration data exists and use the series (1) for the extrapolation. These terms will provide higher order corrections to the mean function $m(t)$.

3 Construction of Canonical expansion.

Not all sets $\{\phi_{\nu}(t)\}$ can be used as a basis of canonical representation for random function family $X(t)$. Indeed, if we represent the centered random function $X^{\circ}(t)$ in the form

$$X^{\circ}(t) = X(t) - m(t) = \sum_{\nu} V_{\nu}\phi_{\nu}(t)$$

multiply both sides by V_{μ} and average then, in virtue of (2), we obtain

$$M\left[X^{\circ}(t)V_{\mu}\right] = D_{\mu}\phi_{\mu}(t)$$

Hence, the basis functions should be such that

$$\phi_{\mu}(t) = \frac{1}{D_{\mu}} M\left[X^{\circ}(t)V_{\mu}\right] \tag{3}$$

We will use that to construct the canonical basis. It is also worth noting that random variables V_{μ} are correlated with $X^{\circ}(t)$, otherwise $\phi_{\mu}(t) \equiv 0$. The simplest form of random variables correlated with $X^{\circ}(t)$ and having zero expected values is a linear combination of values of function $X^\circ(t)$ corresponding to different values of argument t:

$$V_\mu = \sum_k a_{\mu k} X^\circ(t_k) \tag{4}$$

Let us check the correlations of the pairs:

$$M\left[V_\mu V_\nu\right] = \sum_{k,l} a_{\mu k} a_{\nu l} M\left[X^\circ(t_k) X^\circ(t_l)\right])$$

For the representation to be canonical $M\left[V_\mu V_\nu\right]$ should be zero for $\mu \neq \nu$ (2), which can always be done choosing appropriate $a_{\mu k}$ because the condition

$$\sum_{k,l} a_{\mu k} a_{\nu l} K(t_k, t_l)) = 0 \quad \mu \neq \nu \tag{5}$$

can be satisfied in an infinite number of ways.[1] (If $\mu, \nu = 1, 2, .., n$ and $k, l = 1, 2, ..m$ then there are $n \cdot m$ coefficients $a_{\mu k}$ with $\frac{n(n-1)}{2}$ equations on them. The number of sampling points m can be as large as needed.) Thus, coming back to the construction of the canonical basis (3), making use of (4) and assuming that the coefficients $a_{\mu k}$ are found (5) we obtain:

$$\phi_\mu(t) = \frac{1}{D_\mu} M\left[X^\circ(t) \sum_k a_{\mu k} X^\circ(t_k)\right] = \frac{1}{D_\mu} \sum_k a_{\mu k} M\left[X^\circ(t) X^\circ(t_k)\right]$$

$$\phi_\mu(t) = \frac{1}{D_\mu} \sum_k a_{\mu k} K(t, t_k) \tag{6}$$

As we can see, the correlation function $K(t, t')$ provides a way of constructing the canonical basis.[2] For example choosing $a_{1,1} = 1$ and all other $a_{1,k} = 0$ we obtain the first term of the expansion, the first correction to the mean slewing curve:

$$V_1 = X^\circ(t_1) \quad D_1 = K(t_1, t_1)$$

$$\phi_1(t) = \frac{1}{D_1} K(t, t_1)$$

[1]Here, by definition, the correlation function is: $K(t, t') = M\left[X^\circ(t) X^\circ(t')\right]$.

[2]Note that the basis $\phi_\mu(t)$ of random function $X(t)$ is not a basis in a function space in a usual sense. This is a "statistical" basis adjusted to the given random function.

The expansion coefficient V_1 is simply the deviation of the particular slewing curve form the mean dependence at the end of calibrated range denoted as t_1: $v_1 = x(t_1) - m(t_1)$. Extrapolation, then, is obtained

$$x(t) = m(t) + \frac{x(t_1) - m(t_1)}{K(t_1, t_1)} K(t, t_1) \qquad (7)$$

4 Extrapolation error.

Let us first prove that the introduced set of basis functions (3) for any given number of terms in the truncated canonical representation series (1) and for the given selection of random coefficients V_μ gives the best approximation to the random function $X(t)$ in the sense that among all possible series having the same number of terms this set minimizes the mathematical expectation of the square of the residual term for any t. The square of the residual is used because the expectation of the residual is zero. Let $\{\psi_\mu(t)\}$ denote a different basis, then

$$X^\circ(t) = \sum_{\mu=1}^{n} V_\mu \psi_\mu(t) + R_n(t)$$

The expected value of the square of the residual term, then:

$$M\left[(R_n(t))^2\right] = M\left[\left\{X^\circ(t) - \sum_{\mu=1}^{n} V_\mu \psi_\mu(t)\right\}\left\{X^\circ(t) - \sum_{\nu=1}^{n} V_\nu \psi_\nu(t)\right\}\right]$$

or, after some rearrangements and using the expression (3) for ϕ_μ's

$$M\left[(R_n(t))^2\right] = K(t,t) + \sum_{\mu=1}^{n} D_\mu (\psi_\mu(t) - \phi_\mu(t))^2 - \sum_{\mu=1}^{n} D_\mu (\phi_\mu(t))^2$$

It is easy to see now that for given n the expected value of the square of the residual term is minimized when $\psi_\mu(t) \equiv \phi_\mu(t)$. Thus, using the introduced set $\{\phi_\mu(t)\}$ gives the best estimate with the residual:

$$RMS = \sqrt{M\left[(R_n(t))^2\right]} = \sqrt{K(t,t) - \sum_{\mu=1}^{n} D_\mu (\phi_\mu(t))^2}$$

For the example considered with $n = 1$ the RMS of the extrapolation error is:

$$RMS = \sqrt{K(t,t) - \frac{1}{D_1}\left(K(t,t_1)\right)^2} \quad (8)$$

5 Application of the method and results.

In order to apply the method of random functions we first note that we are interested in the shape of the slewing curve only, not in the timing pedestal. The latter is being measured directly at the accessible light levels. The shape of the slewing beyond the calibrated range is needed for shower data processing because real events may be more energetic than the laser ones. Thus, the family of the derivatives of the slewing realizations was studied.

The original slewing data T_{start} vs ToT was sliced in bins of 25 counts of ToT for all tubes. The differences of T_{start}'s for adjacent ToT bins were recorded. These represent the numerical derivative of the slewing curve at a particular ToT. The examples of such T_{start} steps for all PMTs at particular ToT are histogramed on figure (1). The average of each distribution provides a value of $m(t)$ at that ToT — the zeroth approximation for extrapolation. The integrated average curve is shown on figure (2). Only first order correction was used in the form of equation (7) where $K(t_1,t)$ represents correlation between distributions of figure (1) for the adjacent ToT's. Only ToT bins with more than 50 entries were used, which correspond to $HiToT < 675$ counts. Beyond that, linear extrapolation is used with the average slope of $0.0381\frac{T_{start}\ count}{ToT\ count}$. After the synthetic data points were generated with the help of equation (7) a separate polynomial was fit in the extrapolated region and it is the one used online for the HiToT extrapolation. The original polynomial fitted to the calibration data was left intact. LoToT was not extrapolated because its use is not expected at such high light levels.

The extrapolation error is estimated using equation (8) as $0.09\sqrt{extrapolation\ range}$. The typical extrapolation range from shower data is 100-200 counts (see figure (3)), leading to the upper bound on the extrapolation error of 1.4 counts within the frame of the model.

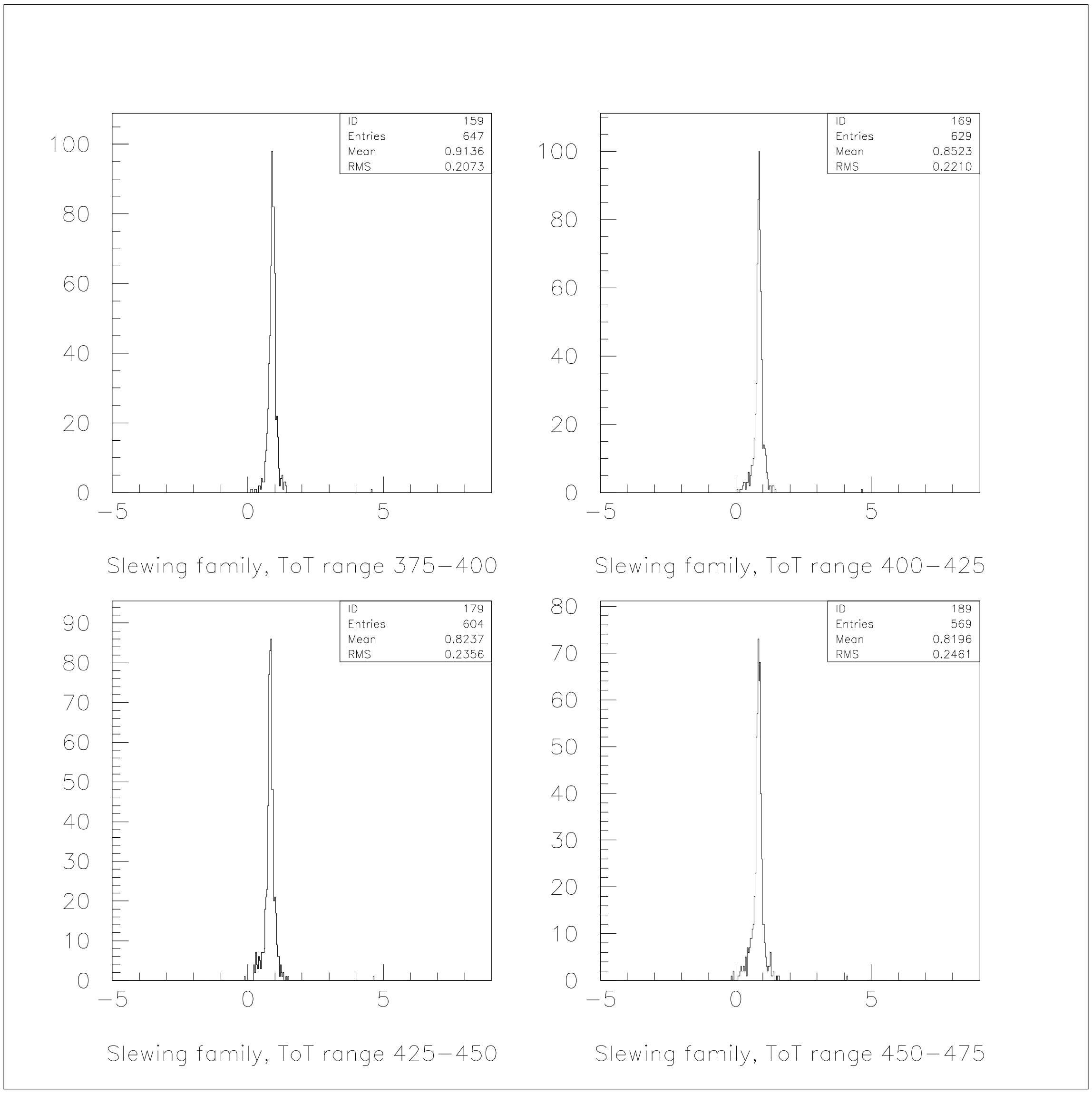

Figure 1: Examples of distributions of T_{start} steps between adjacent ToT bins. Means of the histograms are used as zeroth approximation, correlations between the histograms are used for the first oder correction.

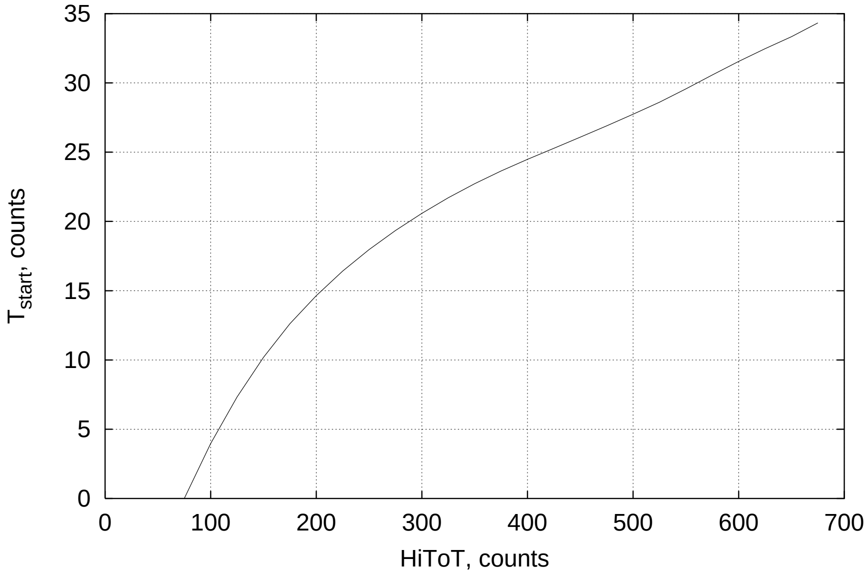

Figure 2: Average curve used for HiToT slewing extrapolation.

A References

1. Lazar Fleysher, Roman Fleysher. An Invitation to Random Functions: a hypothetic lecture on a hypothetic Milagro seminar. May 7, 1999. Available at http://pages.nyu.edu/~rqf6512/MILAGRO/random.ps

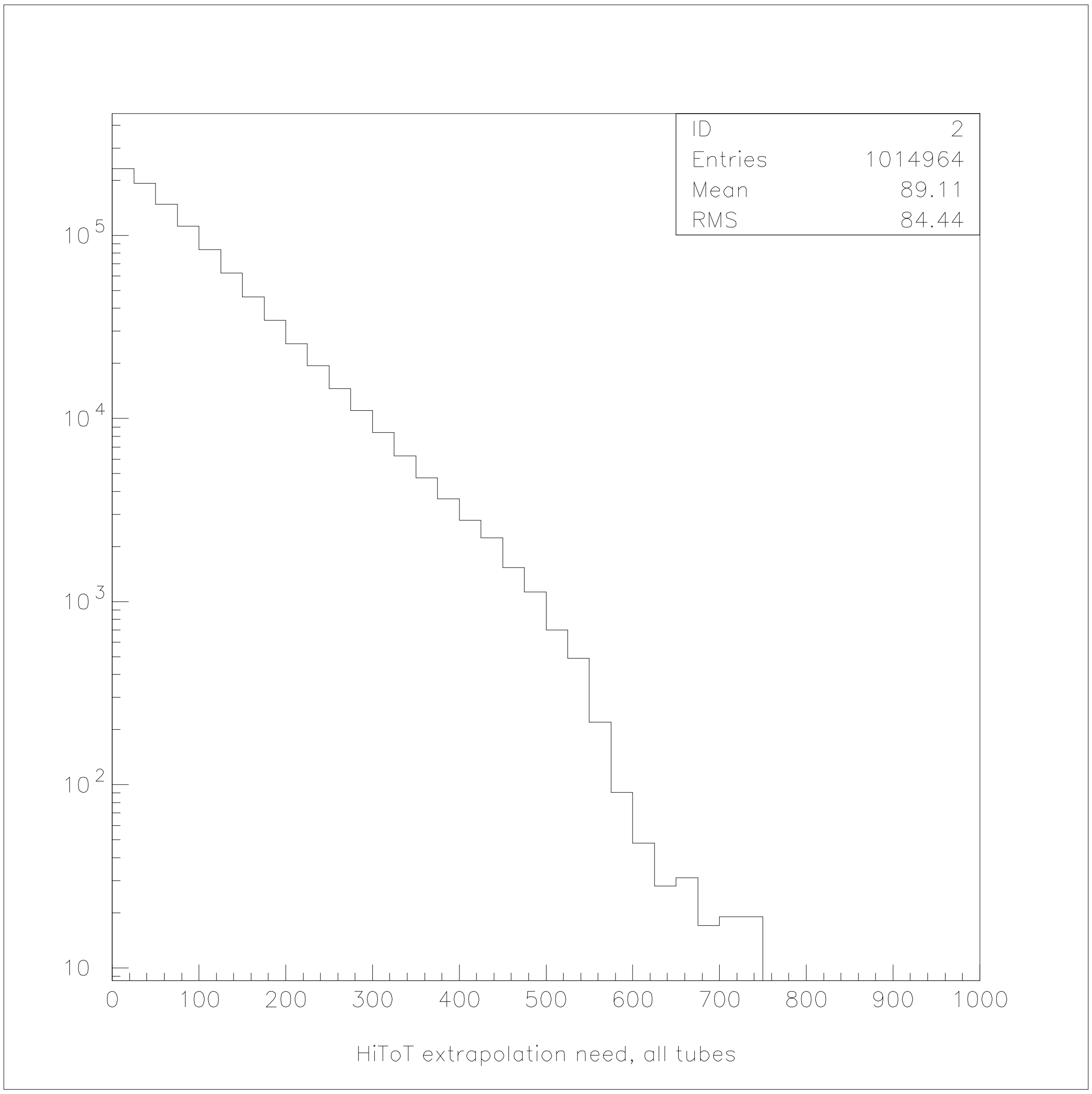

Figure 3: Histogram of the needed TOT extrapolation from the shower data. Note the exponential shape of the distribution.