Pointing with Milagro

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Abstract

This memo reviews various coordinate system definitions needed to correctly reconstruct source locations with Milagro. Depending on what information is returned by the GPS system, we may have systematic pointing errors of up to $0.2°$ due to inconsistent definition of angles. Methods to determine this are presented.

Introduction and definitions

Various coordinate systems are used as a reference for measurements to determine source locations. We present below a review of coordinate systems, a description of what we are doing in MILAGRO, and a discussion of what we must determine to understand whether we are doing this consistently.

The sky

Celestial sphere. — an imaginary sphere with the observer at its center. The stars appear to be located on its surface. It is characterized by the following features.

C - the observer.

CP - line parallel to the axis of rotation of the Earth.

P - North celestial pole.

Celestial Equator - is defined as intersection of a plane perpendicular to CP at point C and the celestial sphere (projection of Earth's equator).

Z - the zenith[1].

$\angle PCZ = 90° - \phi$ - definition of ϕ. ϕ is "latitude" of the observer on the Earth[1].

Horizon - is defined as intersection of a plane perpendicular to CZ at point C and the celestial sphere.

N, S - are North and South of the *Horizon*. N and S are defined as the intersection of a great circle PZ centered at observer C with the *Horizon*. Arc $PZLS$ is called *reference celestial meridian.*

X - is a star.

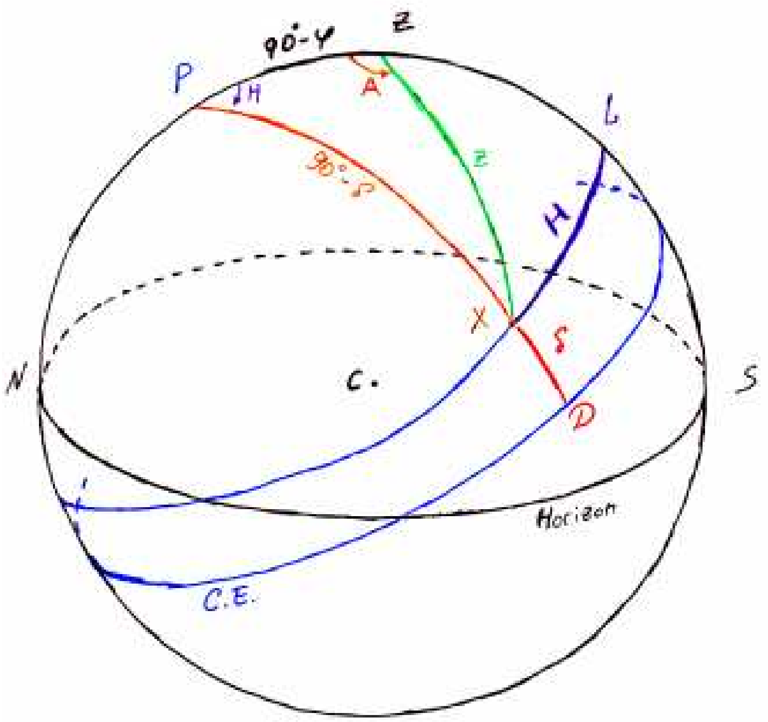
$\angle ZCX = \overset{\smile}{ZX} = z$ - zenith distance of star X.

[1] See latter discussion about its definition in section "The errors we may make".

A - azimuth, is a dihedral angle between reference meridian and ZXC plane.

$\angle LCX = \smile{LX} = H$ - hour angle, is a dihedral angle between reference celestial meridian and that of the star PXD. $\smile{LX}$ is an arc covered by a star due to the Earth's rotation. $\alpha = siderealtime - H$, α - is Right Ascension (R.A.) of a given star. *Sidereal time* is the hour angle of the vernal equinox.

$\angle XCD = \smile{XD} = \delta$ - is Declination of a star (Dec.).



To find relationship between directly measurable quantities (azimuth A and zenith distance z) and celestial coordinates (R.A. α and Dec. δ) we apply the law of cosines for the trihedral angle $CPZX$ (the spherical triangle XPZ) two times. This yields:

$$\cos(90^\circ - \delta) = \cos z \cos(90^\circ - \phi) + \sin z \sin(90^\circ - \phi) \cos A$$

$$\cos z = \cos(90^\circ - \delta) \cos(90^\circ - \phi) + \sin(90^\circ - \delta) \sin(90^\circ - \phi) \cos H$$

or,

$$\sin \delta = \sin \phi \cos z + \cos \phi \sin z \cos A$$

$$\tan H = \frac{\sin z \sin A}{\cos \phi \cos z - \sin z \sin \phi \cos A}$$

These are the formulae used by our Angle2Astro.c routine. This suggests that in order to determine celestial coordinates, one has to supply the correct value of "latitude" ϕ. The currently used value is 35.8784722°.

The Earth

Geoid — a surface which coincides with the unperturbed surface of water in the ocean and is extended under the dry land (which is above sea level) in such a way that plumb lines cross its surface at square angle at each point. (**Remark:** by definition it includes all effects of impurities and rotation of the Earth and does not include external effects such as tidal forces.)

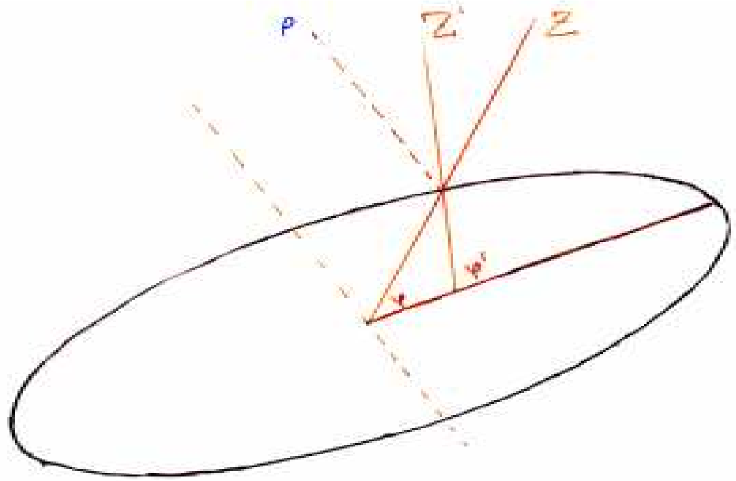
General Earth Ellipsoid (the ellipsoid) — the ellipsoid centered at the Earth's center of mass with minor axis lying on the axis of rotation of the Earth. Parameters of the ellipsoid are defined from the least square fit to the geoid. Parameters of the General Earth Ellipsoid are[2]:

a	- major semi-axis,	$a = 6378245.00000\ (m)$
b	- minor semi-axis,	$b = 6356863.01877\ (m)$
e	- eccentricity,	$e^2 = \frac{a^2-b^2}{a^2} = 0.006693421623$

It is the **ellipsoid's surface** with respect to which all coordinates (*latitude* and *longitude*) are defined. Since the Earth's surface differs from the ellipsoid, measurements done on the Earth must be "reduced" to the ellipsoid before they can be used to calculate angles, distances, areas, etc.

In order to describe position of a point on the ellipsoid one can use a number of different coordinate systems: Cartesian systems, different kinds of spherical coordinate systems, etc. Here, we discuss only the three following choices.

1. Geocentric coordinate system.
   - *Latitude* — is defined as an angle ϕ between a radius-vector from the center of the ellipsoid to the point on the surface and the equatorial plane.
   - *Longitude* — is dihedral angle between the reference meridian plane and the meridian plane of the point in question.



[2]P.S. Zakatov, Course on Higher Geodesy. (Russian) Nedra Publisher, Moscow, 1976, p. 14

2. Geodesic coordinate system.

   - *Latitude* — is defined as an angle ϕ' between a normal line to the surface of the ellipsoid at the given point and the equatorial plane.
   - *Longitude* — is the same as in the geocentric coordinate system.

3. Astronomical coordinate system.

   - *Latitude* — is defined as an angle between the plumb line at the given point and the equatorial plane.
   - *Longitude* — is the dihedral angle between the reference meridian plane and the astronomical meridian plane of the point. The astronomical meridian plane is defined by the plumb line and the line parallel to the axis of rotation of the Earth which intersect each other at the observation point.

Relationship between different coordinate systems

It is clear that geocentric latitude ϕ is not the same as geodesic ϕ' one, but there exist a simple geometrical relationship between them. To find it we use the equation of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

A tangent at point (x, y) on the ellipse has a slope of $\frac{dy}{dx}$. The normal to the ellipse at the same point (x, y) therefore has slope equal to $(-\frac{dx}{dy})$:

$$\tan\phi = \frac{y}{x}$$

$$\tan\phi' = -\frac{dx}{dy} = \frac{a^2}{b^2}\frac{y}{x}$$

so that

$$\tan\phi' = \frac{a^2}{b^2}\tan\phi$$

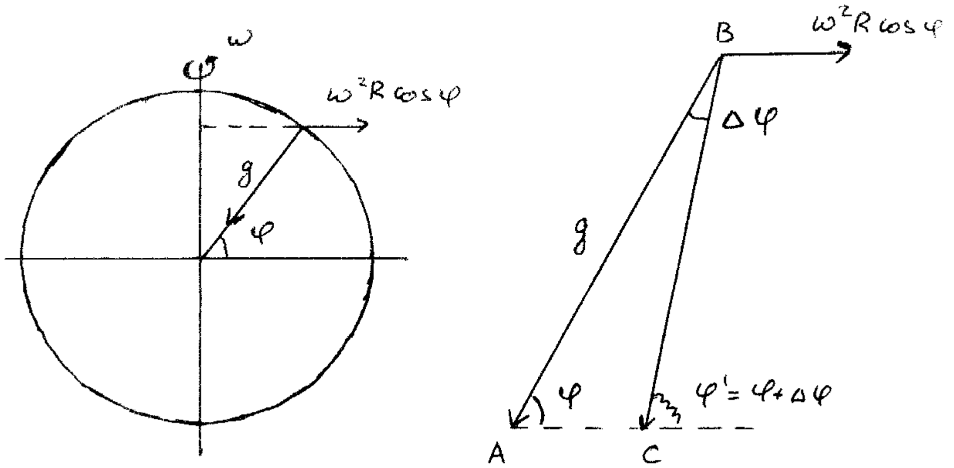
$$\sin(\phi' - \phi) = e^2 \sin\phi' \cos\phi \approx \frac{1}{2}e^2 \sin 2\phi$$

Putting in the latitude of Milagro $\phi \sim 36^\circ$ we obtain an estimate of geodesic and geocentric latitude difference which comes from different definitions of the zenith:

$$\Delta\phi = (\phi' - \phi) \sim 0.18^\circ$$

Note again, that this includes the centrifugal force effect (see definition of the geoid).

It is easy to make an estimate of $\Delta\phi$ due to centrifugal effect itself for a spherical earth of radius R.



Applying the law of sines for ΔABC we obtain:

$$\frac{\sin(\Delta\phi)}{AC} = \frac{\sin(\phi')}{AB}$$

$$\frac{\sin(\Delta\phi)}{\omega^2 R\cos\phi} = \frac{\sin(\phi+\Delta\phi)}{g}$$

$$\frac{g}{\omega^2 R}\frac{\sin(\Delta\phi)}{\cos\phi} = \sin\phi\cos(\Delta\phi) + \cos\phi\sin(\Delta\phi)$$

$$\tan(\Delta\phi) = \frac{\sin\phi\cos\phi}{\frac{g}{\omega^2 R} - \cos^2\phi}$$

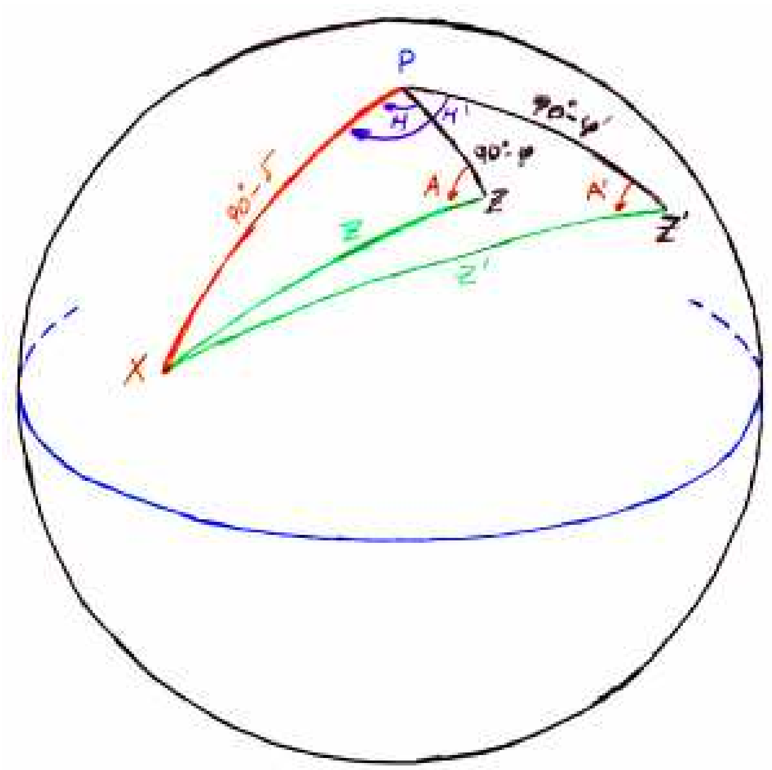
$$\Delta\phi \sim 0.09^\circ$$

This means that about half of the difference in geocentric and geodesic latitudes comes from centrifugal effect, the rest is oblateness of the Earth itself.

In general, there is no relation between astronomical latitude as defined by a plumb line and the geodesic one, but according to indications in literature the difference is of the order of $15''$ in magnitude.

The errors we may make

In general, it does not matter which coordinate system we choose to use (which line we take as a reference line and call it the zenith) as long as we do it consistently. Since Milagro is a water based telescope, it is natural to choose a plumb (perpendicular to water surface) line as a reference direction and call it the zenith. It is also convenient to build the PMT plane parallel to the water surface. Therefore, our reference line constitutes an astronomical coordinate system . It is, however, not clear (although must be possible to find out) in which coordinate system GPS outputs the values of latitude and longitude. To calculate R.A. and Dec., as it was mentioned earlier, the routine needs to know the angle $\angle PCZ$ which, for consistency, should be given in the same astronomical coordinate system.



Given all this, we investigate a possibility of systematic error due to improper definition of ϕ. Recall that

$$\sin\delta = \sin\phi\cos z + \cos\phi\sin z\cos A$$

After differentiating it with respect to ϕ we obtain:

$$\frac{\Delta\delta}{\Delta\phi} = \cos H$$

This means that at each given moment of time H, the declination of a star as determined by our routine might be shifted by $\cos H\Delta\phi$ from its true value. This effect smears the source over a region which depends on the range of H and also results in an average shift of

$$< \Delta\delta >=< \cos H > \Delta\phi$$

If a star is always above the horizon, then $< \Delta\delta >= 0$, otherwise $< \cos H > > 0$ and there is a non-zero shift of the same sign as $\Delta\phi$.

We can also have a systematic error in *hour angle* H determination if our reference celestial meridian is not consistent with the *sidereal time* definition. This will produce a systematic error in H, constant for all stars and times of observation. We would expect this effect to be present due to "sticking out" of plumb line from a plane of symmetry of the Earth containing point of observation. The estimate of this kind of error is rather small, about $15''$.

Conclusion

We hope that after this memo it becomes evident that correct value of latitude ϕ which defines our local coordinate system is necessary to avoid systematic error in Dec. δ. A consistency check of the "local" sidereal time and the one we use must be performed to avoid possible additive systematic error in R.A. α. We see two ways to attack the problem.

1. The easiest way is to try to gain the information required from our GPS system. It involves documentation search and contacts with knowledgeable people which should be possible to accomplish.

2. The second possibility is to perform measurements by ourselves using a theodolite or optical telescope set up according to a local plumb line near the site. This approach involves knowledge of astronomo-geodesic methods of making accurate measurements and skills in operating the equipment. Here we sketch the ideas which can be employed.

   - By observing a star which is always above the horizon one can measure its zenith distances in upper and lower culminations z_{upper} and z_{lower}. Then

     $$z_{upper} = \phi - \delta$$

     $$z_{lower} = 180^\circ - \phi - \delta$$

     therefore

     $$\phi = 90^\circ - \frac{z_{lower} - z_{upper}}{2}$$

     This method does not require knowledge of celestial coordinates of the star observed, which means that any star can be used. But, actual observation must be made 12 hours apart which makes it difficult in practice. It is possible to modify this method for 2 stars, one in upper, the other in lower transit with known declination difference:

     $$z_{upper} = \phi - \delta_1$$

     $$z_{lower} = 180^\circ - \phi - \delta_2$$

     $$\phi = 90^\circ - \frac{z_{lower} - z_{upper}}{2} - \frac{\delta_2 - \delta_1}{2}$$

   - If we recall that

     $$\sin\delta = \sin\phi\cos z + \cos\phi\sin z\cos A$$

     then, after differentiation with respect to time we obtain:

$$\tan\phi = \cot z \cos A - \frac{\dot{A}}{\dot{z}} \sin A$$

or

$$\tan\phi = \cot z \cos A - \frac{dA}{dz} \sin A$$

This method does not impose long-time-of-observation constraints but relies on the ability to measure rate of change of azimuth as a function of zenith distance. Any star at all can be used. This seems to be possible to accomplish.

- A sidereal time synchronization check can be performed only by comparing measured R.A. of a known star with the one listed in Astronomical Almanac. This can also be done using optical instruments.

Perhaps other methods and ideas can be devised to perform these test measurements. The above suggestions are provided in order to indicate that this problem is important and soluble.

References

1. R.M. Green, Textbook on Spherical Astronomy. Cambridge University Press, 1977.
2. P.S. Zakatov, Course on Higher Geodesy. (Russian) Nedra Publisher, Moscow, 1976.