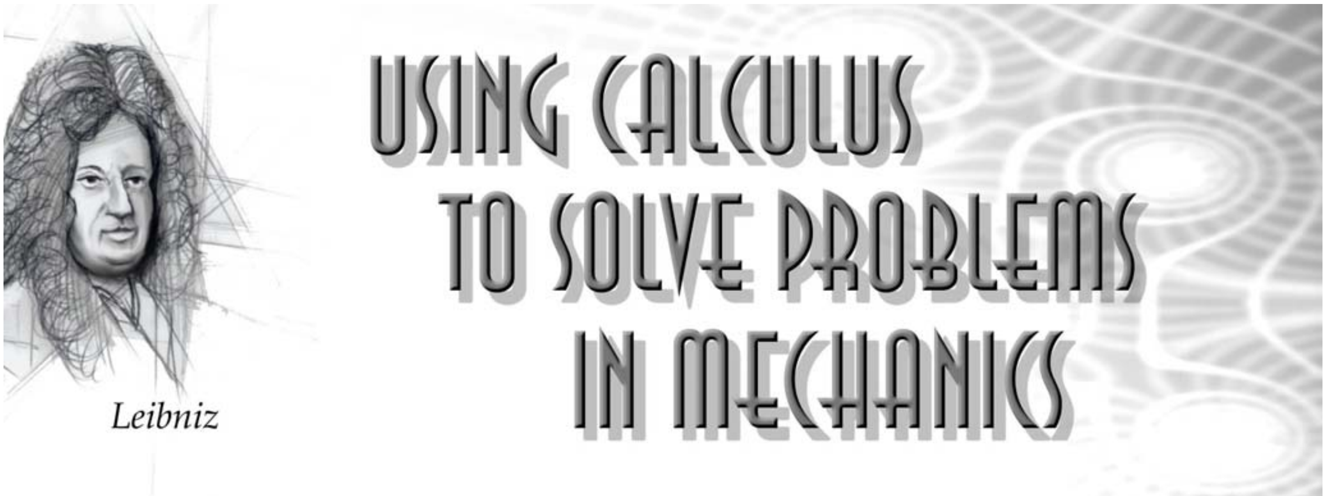


The Big Idea:

In most realistic situations forces and accelerations are not fixed quantities but vary with time or displacement. In these situations algebraic formulas cannot do better than approximate the situation, but the tools of calculus can give exact solutions. The *derivative* gives the instantaneous rate of change of displacement (velocity) and of the instantaneous rate of change of velocity (acceleration). The *integral* gives an infinite sum of the product of a force that varies with displacement times displacement (work), or similarly if the force varies with time (impulse).

The Key Concepts:

- **Acceleration** is the derivative of velocity with respect to time. The slope of the tangent to the line of a graph of velocity vs. time is the acceleration.
- **Velocity** is the derivative of displacement with respect to time. The slope of the tangent to the line of a graph of displacement vs. time is the velocity
- **Work** is the integral of force as a function of displacement times displacement. The area under the curve of a graph of force vs. displacement is the work.
- **Impulse** is the time integral of force as a function of time. The area under the curve of a graph of force vs. time is the impulse.
- **Other Derivatives** include rotational velocity—angle with respect to time; angular acceleration—rotational velocity with respect to time
- **Other Integrals** include moment of inertia, where mass varies with radius and rotational work, where torque varies with angle
- **Harmonic Motion** can be written as a *differential equation.*

The Key Equations:

$\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{x}/dt^2$; Acceleration is the time derivative of velocity.

$\mathbf{v} = d\mathbf{x}/dt = \int a\, dt$; Vvelocity is the time derivative of displacement.

$x = \int v\, dt$; the third of the"Big Three" equations for kinematics

$W = \int \mathbf{F}(\mathbf{x}) \cdot d\mathbf{x}$; Work is the integral of force times displacement.

$P = dW/dt$; Power is the time derivative of work.

$\mathbf{J} = \int \mathbf{F}(t)\, dt = \Delta\mathbf{p}$ **;** Impulse is the integral of force times time.

$\mathbf{r_c} = 1/M \int \mathbf{r}(m)\, dm$; The vector position of the center of mass can be found by integration. $M = \Sigma\, m_i$ where $\mathbf{r}(m)$ is the radius a s a function of mass, for non-uniform bodies.

$\boldsymbol{\omega} = d\theta/dt$; Angular velocity is a derivative too.

$\alpha = d\omega/dt$; Angular acceleration is a derivative.

$W = \int \tau(\theta)\, d\theta$; Work in rotational motion integrates torque and angle.

$\boldsymbol{\tau} = d\mathbf{L}/dt$; Torque is the derivative of angular momentum..

$m\, d^2x/dt^2 = -k\, x(t)$; The differential equation of a spring in simple harmonic motion.

$d^2\theta/dt^2 = -g/l\, \theta(t)$; The differential equation of a pendulum, if θ is small such that $\sin\theta \approx \theta$.

Problem Set: Mechanics with Calculus

1. A particle moves in a straight line with its position, x, given by the following equation: $x(t) = t^4 - 4t^3 + 2t^2 + 3t + 6$.

   a. Find its position after 1 second
   b. Find its velocity after 2 seconds.
   c. Find its acceleration after 3 seconds.
   d. What is the rate of change of the acceleration at 1 second.
   e. Graph the rate of change of acceleration vs. time.

2. A sky-diver of mass, m, opens her parachute and finds that the air resistance, F_a, is given by the formula $F_a = bv$, where b is a constant and v is the velocity.

   a. Set up, but do not solve a differential equation for velocity as a function of time.
   b. Set up but do not solve a differential equation for distance as a function of time.
   c. Find the terminal velocity in terms of m, b, and g.
   d. If in a different situation the formula for air resistance were $F_a = bv + cv^2$, where c is another constant find the terminal velocity in terms of the above plus c.

3. Students are pulling a 2 kg friction block along a rough, but level, surface. In one case it is determined that the position of the block as a function of time is given by : $x(t) = .3t^3 - .1t^2 + .2t$.

   a. Find the speed of the block at t = 2 sec.
   b. Find an expression for acceleration as a function of time.
   c. Find an expression for force as a function of time.
   d. Find the initial kinetic energy of the block
   e. Find the change in kinetic energy of the block from t = 0 to t = 2 sec.
   f. Another lab group determines that the force as a function of distance is given by: $F(x) = x^2 + 2x + 2$ and the block is pulled at an angle of 15 degrees to the horizontal. Find the change in kinetic energy from x = 0 to x = 2 meters.
   g. For the above group find a differential equation for power.

4. An 800 kg sports car traveling at 20 m/s crashes into a SUV in a completely inelastic collision. The position of the wreck for the first 3 seconds is given by: $x(t) = 8t + t^{-1} + 2t^{-2}$, where t = 0 is the time of collision.

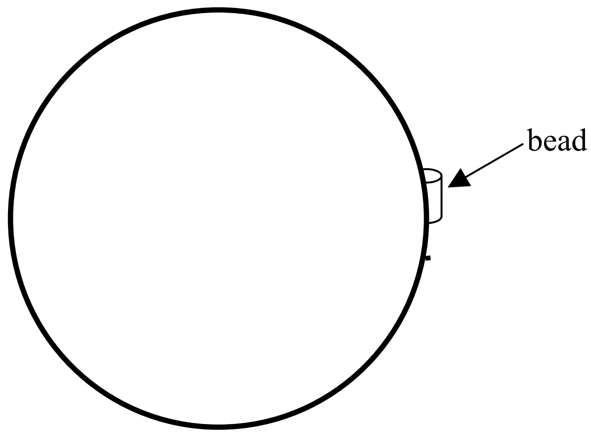
   a. Give an expression for the velocity of the wreck as a function of time.
   b. Find an expression for the acceleration of the wreck as a function of time.
   c. Find the mass of the SUV.
   d. Find an expression for the force as a function of time.
   e. Find the impulse from t = 0 to t = 3 sec.

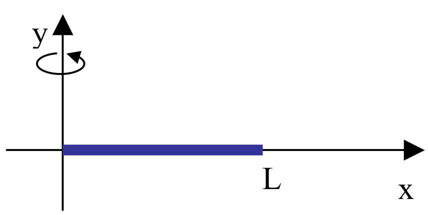
5. The vector position of a particle is given by
   $\mathbf{r} = 3\ \sin(2\pi t)\ \mathbf{i} + 2\ \cos(2\pi t)\ \mathbf{j}$
   where t is in seconds.

a. Plot the path of the particle in the x-y plane.
b. Find the velocity vector.
c. Find the acceleration vector and show that its direction is along r; that is, it is radial.
d. Find the times for which the speed is a maximum or minimum

6. Consider a bead of mass *m* that is free to move on a thin, circular wire of radius *r*. The bead is given an initial speed v_0 and there is a coefficient of sliding friction μ_k. The experiment is performed in a spacecraft drifting in space (i.e. no gravity to worry about)

    a. Show that the speed of the bead at any subsequent time *t* is given by $v(t) = v_0/[1+(\mu_k/r)v_0t]$.
    b. Plot *v* vs. *t* for $v_0 = 10$ m/s, r = 5 m, and $\mu_k = 0.5$. Label both axes with at least 5 numbers.





7. The above rod of length *L* is rotating about one end. It has a linear density given by

$$\lambda = \lambda_0\left[1+\frac{x}{L}\right] \text{ where } \lambda_0 = \frac{M}{L}.$$

    a. Find I_y.
    b. Find the moment of inertia about an axis perpendicular to the rod and through its CM, letting x_0 be the coordinate of its CM.
    c. Where is the CM?

8. The position of a certain system with mass of 10 kg exhibits simple harmonic motion, where

$$x(t) = 20\cos\left(15.2t + \frac{\pi}{4}\right)$$ and is in units of meters.

a. What is the total Energy of the system (let the Potential Energy be zero at the equilibrium position)?

b. At t = 0, what is the Potential Energy?

9. A device when compressed has a restoring force given by: $F(x) = k_1x + k_2x^2$. When $x = 0$, $F = 0$.

a. Find an expression for the potential energy as a function of x.

b. When the device is released it goes through damped harmonic motion. The resisting friction force is given by $-k_3v$, where v is the velocity. Write but do not solve a differential equation describing the motion.