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OPTIMUM TRACK FITTING IN THE PRESENCE OF MULTIPLE SCATTERING

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A method for track fitting is proposed which attempts to be as close as possible to the real track along the full path length. This is done by the introduction of scattering planes in which the particle is allowed to change its direction. A fit over the full track length includes the probability of direction change by scattering. Using matrix notation a fairly simple formalism for error estimation has been developed. Results of this method are compared to those of more widely used procedures for "typical" examples of high energy spectrometers.

1. Introduction

A standard problem in high energy physics experiments is the determination of track parameters (momentum, position and angle) as well as the estimation of their errors from measurements along the path of the particle. A first theoretical investigation of this problem, which is illustrated in fig. 1, has been done by Gluckstern [1] in 1963. It is still today considered as the standard reference. The particle in general passes a magnetic field region, so that the momentum can be determined by the curvature. Detectors measuring the particle position (as well as other material along the path of the particle) have the unwanted property of changing the particle direction due to multiple scattering. The situation is indicated in fig. 1.

For the determination of the track parameters ad hoc procedures are frequently invented in which multiple scattering is either neglected, or taken into account by degrading the weight of the measuring points by adding to the measurement precision a momentum dependent contribution from multiple scattering.

As will be demonstrated later on, both of these procedures do not completely exploit the information available in the apparatus. An additional problem in the second procedure lies in the fact that there is no way of degrading the measurement precision which at the same time gives best results for all track parameters. For the vertex position one would like to give largest weight to the measurements close to the vertex, while for momentum one would give highest weight to the high field region.

The method proposed in this paper avoids these problems by including the scattering explicitly into the fit. After completion of this work I have come along an old unpublished review of track fitting methods [2] in which the introduction of kinks into the track fit has already been proposed. In a more recent paper [3]

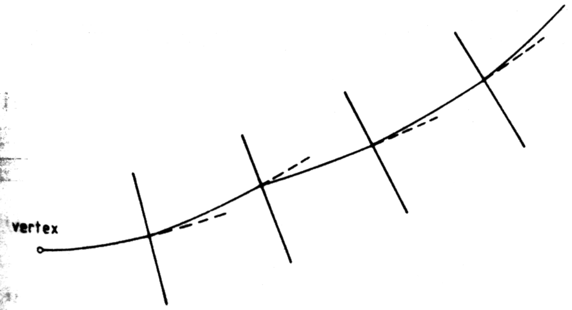

Fig. 1. Particle trajectory in a magnetic field with scattering in the detector planes of a spectrometer.

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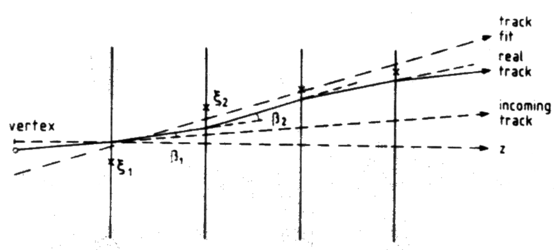

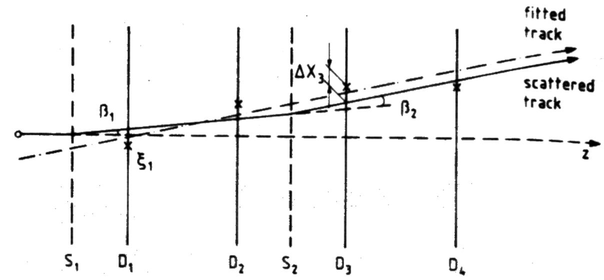

a single projected scattering angle was included in the track fit. In order to give a simple introduction in the method and the mathematical formalism used and for comparison of the results a short presentation of more standard methods (sections 2 and 3) precedes the description of the new proposed procedure in section 4. The study of these non-optimum procedures is not a prerequisite for an understanding of the new proposed procedure and it is therefore possible to proceed directly to section 4.

2. Straight line fit in field free region

A particle of momentum P is assumed to enter nearly orthogonal to a set of parallel measuring planes (detectors). These detectors, in addition to measuring the crossing point of the track, will change the particle direction by multiple scattering. The situation is shown in fig. 2 where the path of the particle is drawn as a kinked continuous line, the measured coordinates are indicated by crosses while the result of a straight line fit is shown as the dash–dotted line.

In order to allow for the more general case of additional scatterers and for easier mathematical treatment we will assume the existence of separate measuring and scattering planes which do not necessarily coincide in position (fig. 3). Here we have chosen for simplification of the presentation of error analysis a particle incident along the z-axis.

The measured track position ξ_i in a plane will be given by the crossing point of the real (kinked) track altered by the measuring error Δx_i in each detector:

$$\xi_i = \sum_{J=1}^{N} \beta_J (z_i - Z_J) \Theta(z_i - Z_J) + \Delta x_i. \tag{1}$$

Here and in the following we use small letters for measuring planes and capitals for scattering planes. β_J is the kink angle in scattering plane J placed at position Z_J. The symbol

$$\Theta(z) = \begin{cases} 1 & \text{for } z \geq 0, \\ 0 & \text{for } z < 0, \end{cases} \tag{2}$$

has been introduced for restriction to scatters ahead of the measuring plane.

Fitting the n measurements ξ_i with a straight line

$$x = a + bz, \tag{3}$$

by minimizing

$$\chi^2 = \chi^2(a, b) = \sum_{i=1}^{n} w_i (\xi_i - a - bz_i)^2 \tag{4}$$

with respect to the two parameters a and b, with w_i representing the weights given to the individual measurements (normally taken to be the precision $1/\sigma_i^2$ of the measurement) one obtains the linear equations

$$a \sum_{i=1}^{n} w_i + b \sum_{i=1}^{n} w_i z_i = \sum_{i=1}^{n} w_i \xi_i, \tag{5}$$

$$a \sum_{i=1}^{n} w_i z_i + b \sum_{i=1}^{n} w_i z_i^2 = \sum_{i=1}^{n} w_i z_i \xi_i,$$

which may be written in matrix notation

$$\mathbf{A} \boldsymbol{q} = \boldsymbol{B}, \tag{6}$$

with the 2×2 matrix $\mathbf{A}$: $A_{1,1} = \sum_{i=1}^{n} w_i$; $A_{1,2} = A_{2,1} = \sum_{i=1}^{n} w_i z_i$; $A_{2,2} = \sum_{i=1}^{n} w_i z_i^2$, the column vector $\boldsymbol{q}$ representing the track parameters $q_1 = a$; $q_2 = b$, and the right hand side of the equations $B_1 = \sum_{i=1}^{n} w_i \xi_i$; $B_2 = \sum_{i=1}^{n} w_i z_i \xi_i$. Writing the relation between the right hand side of the equations and the coordinates again in matrix form,

$$\boldsymbol{B} = \mathbf{D} \boldsymbol{\Xi}, \tag{7}$$

with $D_{1,j} = w_j$; $D_{2,j} = w_j z_j$ and $\boldsymbol{\Xi}$ the column vector representing the n measurements ξ_i one obtains a linear relation between measurements and track parameters

$$\mathbf{A} \boldsymbol{q} = \mathbf{D} \boldsymbol{\Xi}. \tag{8}$$

It can be solved by inversion:

$$\boldsymbol{q} = \mathbf{A}^{-1} \boldsymbol{B} = \mathbf{A}^{-1} \mathbf{D} \boldsymbol{\Xi}. \tag{9}$$

This equation, to be used for fitting the track to the measured coordinates, may now also serve for the estimation of the track parameter errors if one goes back to eq. (1) in which we expressed for a particular event the measured coordinates by the individual kink angles β_J and errors Δx_i in the position measurement. These equations may again be written in matrix form:

$$\boldsymbol{\Xi} = \boldsymbol{r} + \mathbf{G} \boldsymbol{s} \tag{10}$$

with $\boldsymbol{r}$ the n measuring errors ($r_i = \Delta x_i$), $\boldsymbol{s}$ the N scattering angles ($s_I = \beta_I$) and the $n \times N$ matrix $G_{j,K} = (z_j - Z_K)\Theta(z_j - Z_K)$.

One then obtains from eq. (9)

$$\boldsymbol{q} = \mathbf{A}^{-1} \mathbf{D}(\boldsymbol{r} + \mathbf{G}\boldsymbol{s}) = \mathbf{A}^{-1} \mathbf{D} \boldsymbol{r} + \mathbf{A}^{-1} \mathbf{D} \mathbf{G} \boldsymbol{s}, \tag{11}$$

with $\boldsymbol{r}$ containing the measuring errors and $\boldsymbol{s}$ the scattering angles.

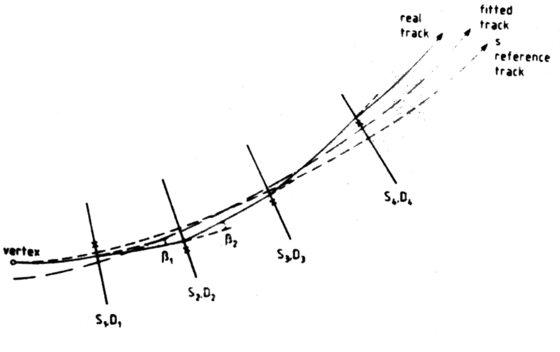

Fig. 4. Track fit in the presence of a magnetic field. The fit has the shape expected if no scattering occurred. Detector and scattering planes are assumed to coincide.

Using the law of propagation of errors, the error matrix $\mathbf{E}_q$ of the track parameters is easily calculated from the diagonal matrices for position measurement $\mathbf{E}_r$ and for multiple scattering $(1/P^2)\mathbf{E}_s$ taking into account the absence of correlation between multiple scattering and position measurement errors:

$$E_{r_{i,j}} = \langle (\Delta x_i)^2 \rangle \delta_{i,j}, \quad E_{s_{I,J}} = \langle (p\Delta\beta)^2 \rangle \delta_{I,J}, \tag{12}$$

$$\mathbf{E}_q = \mathbf{A}^{-1}\mathbf{D}\mathbf{E}_r(\mathbf{A}^{-1}\mathbf{D})^{\mathrm{T}} + \frac{1}{P^2}\mathbf{A}^{-1}\mathbf{D}\mathbf{G}\mathbf{E}_s(\mathbf{A}^{-1}\mathbf{D}\mathbf{G})^{\mathrm{T}} = \mathbf{E}_q^r + \frac{1}{P^2}\mathbf{E}_q^s. \tag{13}$$

Here we have taken a factor $1/P^2$ out of the scattering error matrix in order to make it dependent on geometry only. Therefore $\mathbf{E}_q^r$ and $\mathbf{E}_q^s$ are independent of particle momentum as long as no momentum dependent weights are entered into the matrix $\mathbf{A}$. The errors on the track parameters are then given by the quadratic sum of a constant and a term proportional to $1/P$:

$$\langle (\Delta a)^2 \rangle = E_{q1,1}^r + \frac{1}{P^2}E_{q1,1}^s, \quad \langle (\Delta b)^2 \rangle = E_{q2,2}^r + \frac{1}{P^2}E_{q2,2}^s, \tag{14}$$

$$\langle \Delta a \Delta b \rangle = E_{q1,2}^r + \frac{1}{P^2}E_{q1,2}^s.$$

3. Fit of a trajectory in the presence of a magnetic field

The treatment in two dimensions of section 2 will now be generalized to include bending in a magnetic field perpendicular to the plane of the trajectory. We will assume that particles have fairly high momentum, so as to keep bending angles small. Furthermore the measuring and scattering planes are assumed to be roughly perpendicular to the track of a (unscattered) particle of momentum P_0. The path length s along this reference track and the perpendicular distance x are chosen as coordinates. A real track will be characterized by the transverse distance a and slope difference b at the origin ($s = 0$), as well as the deviation in the inverse momentum $p = \delta(1/P) = -\delta P/P_0^2$ from the reference track.

This way of parametrizing tracks may not be very suitable in many real applications as it does not make use of symmetries present in the detector and therefore would require a high number of reference tracks. It serves, however, very well for the purpose of calculating the expected errors on the track parameters.

An arrangement with coinciding measuring and scattering planes together with the unscattered (dashed) reference track and the real (continuous) kinked track is shown in fig. 4. Without scattering and for small deviations of a real track from the reference track (fig. 5) the transverse distance to the reference track will be given by

$$x(s) = a + bs + 0.3p\int_{l=0}^{s}(s-l)B\,\mathrm{d}l, \tag{15}$$

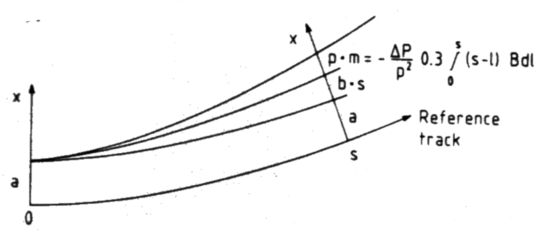

Fig. 5. Relation between coordinates and track parameters.

if we use units of meter, Tesla and GeV/c. Adding multiple scattering and measuring errors and calling the field integrals up to the measuring planes

$$m_i = 0.3\int_{l=0}^{s_i} (s_i - l) B \, dl, \tag{16}$$

one finds for the measured coordinates

$$\xi_i = a + bs_i + pm_i + \sum_{J=1}^{N} \beta_J (s_i - S_J)\Theta(s_i - S_J) + \Delta x_i. \tag{17}$$

Fitting the measurements with a track as expected in the magnetic field but without scattering one obtains by minimizing

$$\chi^2 = \chi^2(p, a, b) = \sum_{i=1}^{n} w_i(\xi_i - x(s_i))^2, \tag{18}$$

a system of linear equations for the track parameters $p = \delta(1/P)$, a and b:

$$\begin{aligned} p\sum_{i=1}^{n} w_i m_i^2 + a\sum_{i=1}^{n} w_i m_i + b\sum_{i=1}^{n} w_i s_i m_i &= \sum_{i=1}^{n} w_i m_i \xi_i, \\ p\sum_{i=1}^{n} w_i m_i + a\sum_{i=1}^{n} w_i + b\sum_{i=1}^{n} w_i s_i &= \sum_{i=1}^{n} w_i \xi_i, \\ p\sum_{i=1}^{n} w_i m_i s_i + a\sum_{i=1}^{n} w_i s_i + b\sum_{i=1}^{n} w_i s_i^2 &= \sum_{i=1}^{n} w_i s_i \xi_i, \end{aligned} \tag{19}$$

which may be written in matrix form as

$$\mathbf{A} \boldsymbol{q} = \boldsymbol{B} = \mathbf{D}\Xi \tag{20}$$

with the now 3×3 matrix $\mathbf{A}$: $A_{0.0} = \Sigma_{i=1}^n w_i m_i^2$; $A_{0.1} = A_{1.0} = \Sigma_{i=1}^n w_i m_i$; $A_{0.2} = A_{2.0} = \Sigma_{i=1}^n w_i s_i m_i$; $A_{1.1} = \Sigma_{i=1}^n w_i$; $A_{1.2} = A_{2.1} = \Sigma_{i=1}^n w_i s_i$; $A_{2.2} = \Sigma_{i=1}^n w_i s_i^2$; the column vector $\boldsymbol{q}$ representing the three track parameters $q_0 = p = -\delta P/P^2$; $q_1 = a$; $q_2 = b$; and the right hand side of the equations $\boldsymbol{B}$ with $B_0 = \Sigma_{i=1}^n w_i m_i \xi_i$; $B_1 = \Sigma_{i=1}^n w_i \xi_i$; $B_2 = \Sigma_{i=1}^n w_i s_i \xi_i$. Ξ as before represents the n measurements ξ_i and the now $3 \times n$ matrix $\mathbf{D}$ is given by $D_{0,j} = w_j m_j$, $D_{1,j} = w_j$; $D_{2,j} = w_j s_j$.

Track parameters are obtained by inversion of eq. (20):

$$\boldsymbol{q} = \mathbf{A}^{-1}\boldsymbol{B} = \mathbf{A}^{-1}\mathbf{D}\Xi. \tag{21}$$

Note that matrix $\mathbf{A}$ contains only quantities describing the setup while $\boldsymbol{B}$ depends on the measured coordinates of the track.

Calculation of the now 3×3 error matrix for the track parameters is analogous to the procedure applied in the previous section and we obtain:

$$\mathbf{E}_q = \mathbf{A}^{-1}\mathbf{D}\mathbf{E}_r\mathbf{D}^{\mathrm{T}}(\mathbf{A}^{-1})^{\mathrm{T}} + \frac{1}{P^2}\mathbf{A}^{-1}\mathbf{D}\mathbf{G}\mathbf{E}_s\mathbf{G}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}(\mathbf{A}^{-1})^{\mathrm{T}} = \mathbf{E}_q^r + \frac{1}{P_2}\mathbf{E}_q^s, \tag{22}$$

with the $n \times N$ matrix $\mathbf{G}$: $G_{j,K} = (s_j - S_K)\Theta(s_j - S_K)$ and the error matrices $\mathbf{E}_r$, $\mathbf{E}_s$ for measuring errors and multiple scattering given in eqs. (12).

Again we see that the errors on the track parameters are given as the quadratic sum of a momentum independent term and a term proportional to $1/P$ as long as the weights of the measurements are chosen momentum independent.

4. Optimum fitting method

4.1. Optimum fitting without magnetic field

In the previous two sections we have assumed that measured coordinates are fitted by a trajectory of a shape which was expected if no scattering occurred. Such a trajectory will not be a good approximation to the real track over its full length. The region of good approximation (e.g. close to the origin) can be chosen by giving higher weights in the fit to measurements close to this region. As a further improvement it seems natural to try to approximate the kinked trajectory in the fit (fig. 6), so as to obtain optimum precision over the full length of the track. For clarity we will derive the formalism for the most simple case of a field free region in two dimensions. As before we will assume that scattering material not necessarily coincides with measuring planes. This should simplify application to real situations as, e.g., high energy spectrometers, as it allows limitation in the number of kinks in the trajectory to reasonable low values even if the number of measuring planes is very high. The formalism can thus be applied to also include scattering material in front of the first measuring device thus providing correct track errors for the vertex fit.

The measured coordinates ξ_i are now fitted by a kinked trajectory whose position x_i in the measuring plane is parametrized by

$$x_i = a + bz_i + \sum_{J=1}^{N} \alpha_J (z_i - Z_J)\Theta(z_i - Z_J) = a + bz_i + \sum_{J=1}^{N} \alpha_J \zeta_{z_i,J}, \tag{23}$$

where a and b are position and slope of the incoming track, α_J the kink in scattering plane J and $\Theta(z_i - Z_J)$ defined in eq. (2) provides restriction to upstream kinks only. The short hand notation

$$\zeta_{z,J} = (z - Z_J)\Theta(z - Z_J) \tag{24}$$

has been introduced for simplification of the following formulae.

The χ^2 to be minimized by variation of the kinked track parameters a, b and the kink angles α_K in the N scattering planes contains now also terms describing the probability of scattering:

$$\chi^2 = \chi^2(a, b, \alpha_1, \dots \alpha_N) = \sum_{i=1}^{n} \frac{(\xi_i - x_i)^2}{\sigma_{x_i}^2} + \sum_{J=1}^{N} \frac{(\beta_J - \alpha_J)^2}{\sigma_{s_J}^2}, \tag{25}$$

where σ_{x_i} is the rms measuring error of plane i and σ_{s_J} is the rms of the scattering angle β_J in scattering plane J (dependent on the momentum of the particle).

In writing down this equation, we pretend that the scattering angle in plane J is being measured independent of the coordinates of the measuring planes, leading to a value β_J for which we later insert the value zero. This at first glance awkward procedure is necessary for proper evaluation of the track parameters errors.

It may be worth stressing in this context that the number of degrees of freedom of the fit does not change by the introduction of kinks in the fit since each additional parameter is compensated by the corresponding zero "measurement" of this angle.

Taking the derivatives with respect to the $N+2$ track parameters a, b, and α_K in eq. (25) we obtain the system of equations:

$$\begin{aligned}
&a\sum_{i=1}^{n} w_{x_i} + b\sum_{i=1}^{n} w_{x_i} z_i + \sum_{J=1}^{N} \alpha_J \sum_{i=1}^{n} w_{x_i}\zeta_{z_i,J} = \sum_{i=1}^{n} w_{x_i}\xi_i,\\
&a\sum_{i=1}^{n} w_{x_i} z_i + b\sum_{i=1}^{n} w_{x_i} z_i^2 + \sum_{J=1}^{N} \alpha_J \sum_{i=1}^{n} w_{x_i} z_i\zeta_{z_i,J} = \sum_{i=1}^{n} w_{x_i} z_i\xi_i,\\
&a\sum_{i=1}^{n} w_{x_i}\zeta_{z_i,K} + b\sum_{i=1}^{n} w_{x_i} z_i\zeta_{z_i,K} + \sum_{J=1}^{N} \alpha_J \sum_{i=1}^{n} w_{x_i}\zeta_{z_i,J}\zeta_{z_i,K} + w_{s_K}\alpha_K = \sum_{i=1}^{n} w_{x_i}\zeta_{z_i,K}\xi_i + w_{s_K}\beta_K,\\
&K = 1,\dots, N,
\end{aligned} \tag{26}$$

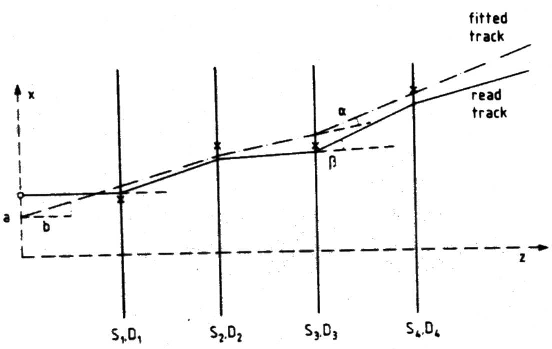

Fig. 6. Optimum fitting method in the field free case: Approximation of the real linked track with kink angles β by a kinked trajectory with kink angles α. For simplification of the drawing detector and scattering planes are assumed to coincide. The scattering angles β are assumed to be measured with the result zero.

where we have replaced the inverse squares of position measurement error and multiple scattering angles with the symbols

$$w_{x_i}=1/\sigma_{x_i}^2, \quad w_{s_J}=1/\sigma_{s_J}^2. \tag{27}$$

This system of equations can again be written in matrix form:

$$\mathbf{A}\boldsymbol{q}=\boldsymbol{B}, \tag{28}$$

with the $N+2$ track parameters $q_1=a$; $q_2=b$; $q_{J+2}=\alpha_J$, the symmetric matrix $\mathbf{A}$:

$$A_{1,1}=\sum_{i=1}^{n} w_{x_i}, \quad A_{1,2}=A_{2,1}=\sum_{i=1}^{n} w_{x_i}z_i, \quad A_{2,2}=\sum_{i=1}^{n} w_{x_i}z_i^2,$$
$$A_{1,J+2}=A_{J+2,1}=\sum_{i=1}^{n} w_{x_i}\zeta_{z_i,J}, \quad A_{2,J+2}=\sum_{i=1}^{n} w_{x_i}z_i\zeta_{z_i,J}, \tag{29}$$
$$A_{J+2,K+2}=A_{K+2,J+2}=\sum_{i=1}^{n} w_{x_i}\zeta_{z_i,J}\zeta_{z_i,K}+w_{s_J}\delta_{J,K}$$

and the right hand column $\boldsymbol{B}$:

$$B_1=\sum_{i=1}^{n} w_{x_i}\xi_i, \quad B_2=\sum_{i=1}^{n} w_{x_i}z_i\xi_i, \quad B_{K+2}=\sum_{i=1}^{n} w_{x_i}\zeta_{z_i,K}\xi_i+w_{s_K}\beta_K. \tag{30}$$

Notice that here and in the following capital indices are running over the N scattering planes and small indices over the n measuring planes.

Introducing the $(N+2)\times n$ dimensional matrix $\mathbf{D}$

$$D_{1,i}=w_{x_i}, \quad D_{2,i}=w_{x_i}z_i, \quad D_{J+2,i}=w_{x_i}\zeta_{z_i,J}, \tag{31}$$

and the $(N+2)\times N$ dimensional matrix $\mathbf{F}$

$$F_{1,I}=F_{2,I}=0, \quad F_{J+2,I}=w_{s_J}\delta_{J,I}, \tag{32}$$

we can express in matrix notation the right hand side of the equations $\boldsymbol{B}$ by the measurements Ξ $(\Xi_i=\xi_i)$ and the scattering angle "measurements" $\boldsymbol{S}(S_I=\beta_I)$:

$$\boldsymbol{B}=\mathbf{D}\Xi+\mathbf{F}\boldsymbol{S}, \tag{33}$$

and obtain the linear relations between measurements and track parameters

$$\mathbf{A}\boldsymbol{q}=\mathbf{D}\Xi+\mathbf{F}\boldsymbol{S} \tag{34}$$

The system of equations (34) can be solved by inversion of the matrix $\mathbf{A}$:

$$\boldsymbol{q}=\mathbf{A}^{-1}\mathbf{D}\Xi+\mathbf{A}^{-1}\mathbf{F}\boldsymbol{S}. \tag{35}$$

Remember that $\beta_I=0$ and that therefore the second term on the right hand side vanishes. It has only been kept for formal reasons, so as to be able to do proper error analysis.

Contrary to section 2, where we considered multiple scattering to contribute to the error on the measurement of the track coordinates, we are now trying to fit the real kinked path of the trajectory, pretending that the kink angles are being measured. Consequently we simply take for the error matrix of the coordinate measurements the diagonal matrix representing the measurement precision of the detectors

$$E_{\xi_{i,j}} = \left\langle (\Delta x_i)^2 \right\rangle \delta_{i,j}, \tag{36}$$

and for the error matrix of the angle "measurement" the multiple scattering angles themselves

$$E_{\beta_{I,J}} = \left\langle \Delta(\beta_I)^2 \right\rangle \delta_{I,J}. \tag{37}$$

The error matrix for the $N+2$ track parameters is then simply obtained as

$$\mathbf{E}_q = \mathbf{A}^{-1}\mathbf{D}\mathbf{E}_\xi \mathbf{D}^{\mathrm{T}}(\mathbf{A}^{-1})^{\mathrm{T}} + \mathbf{A}^{-1}\mathbf{F}\mathbf{E}_\beta \mathbf{F}^{\mathrm{T}}(\mathbf{A}^{-1})^{\mathrm{T}}. \tag{38}$$

This $(N+2)\times(N+2)$ dimensional error matrix may now serve for a determination of the errors of position and slope at any location along the track. Since the parameters describing the track are position and slope of the incoming track and the N kink angles in all scattering planes, we obtain for the track slope in region K (between scattering planes K and $K+1$)

$$b_K = b + \alpha_1 + \alpha_2 + \cdots \alpha_K = \sum_{J=2}^{K+2} q_J, \quad b(z) = q_2 + \sum_{I=1}^{N} q_{I+2}\Theta(z - Z_I), \tag{39}$$

$$\left\langle \Delta(b_K)^2 \right\rangle = \sum_{I=2}^{K+2}\sum_{J=2}^{K+2} E_{q_{I,J}},$$

$$\langle (\Delta b(z))^2 \rangle = E_{q_{2,2}} + 2\sum_{I=1}^{N} E_{q2,I+2}\Theta(z - Z_I) + \sum_{I=1}^{N}\sum_{J=1}^{N} E_{qI+2,J+2}\Theta(z - Z_I)\Theta(z - Z_J). \tag{40}$$

Above and in the following we use the short hand notations (2) and (24) in order to obtain expressions valid for arbitrary z.

From the expression for the position of the track in the region between scattering planes K and $K+1$

$$x = a + bz + \alpha_1(z - Z_1) + \alpha_2(z - Z_2) + \cdots \alpha_k(z - Z_K), \quad x = q_1 + q_2 z + \sum_{J=1}^{K} q_{J+2}(z - Z_J),$$

$$x(z) = q_1 + q_2 z + \sum_{J=1}^{N} q_{J+2}\zeta_{z,J}, \tag{41}$$

one finds the position error:

$$\begin{aligned}\langle (\Delta x)^2 \rangle = {} & \left\langle (\Delta q_1)^2 \right\rangle + 2z\langle \Delta q_1 \Delta q_2 \rangle + z^2 \langle (\Delta q_2)^2 \rangle \\ & + 2\sum_{J=1}^{K} (z - Z_J)\langle \Delta q_1 \Delta q_{J+2} \rangle + 2\sum_{J=1}^{K} z(z - Z_J)\langle \Delta q_2 \Delta q_{J+2} \rangle \\ & + \sum_{I=1}^{K}\sum_{J=1}^{K} (z - Z_I)(z - Z_J)\langle \Delta q_{I+2} \Delta q_{J+2} \rangle,\end{aligned}$$

$$\begin{aligned}\langle (\Delta x(z))^2 \rangle = {} & E_{q1,1} + 2zE_{q1,2} + z^2 E_{q2,2} + 2\sum_{I=1}^{N} E_{q1,I+2}\zeta_{z,I} + 2z\sum_{I=1}^{N} E_{q2,I+2}\zeta_{z,I} \\ & + \sum_{I=1}^{N}\sum_{J=1}^{N} E_{qI+2,J+2}\zeta_{z,I}\zeta_{z,J}.\end{aligned} \tag{42}$$

We may in a similar way derive the error correlation between position and slope and obtain:

$$\langle \Delta x(z)\Delta b(z)\rangle = E_{q1,2} + \sum_{I=1}^{N} E_{q1,I+2}\Theta(z-Z_I) + zE_{q2,2} + \sum_{I=1}^{N} E_{q2,I+2}\left[z\Theta(z-Z_I) + \zeta_{z,I}\right] + \sum_{I=1}^{N}\sum_{J=1}^{N} E_{qI+2,J+2}\Theta(z-Z_I)\zeta_{z,J}. \tag{43}$$

4.2. Optimum fitting in a magnetic field

The method described above can fairly easily be generalized to include situations where the particle passes magnetic field regions. This is done analogously to the procedure described in section 3 if one adds a momentum term (fig. 5) to eq. (23) describing the (kinked) track:

$$x_i = pm_i + a + bs_i + \sum_{J=1}^{N} \alpha_J \zeta_{s_i,J}. \tag{44}$$

Remember that $p = \delta(1/P)$ is the deviation of the inverse momentum from that of a reference track (P_0) and m_i are the magnetic field integrals given by eq. (16). $\zeta_{s_i,J}$ is now defined in terms of the path length along the reference track:

$$\zeta_{s,J} = (s - S_J)\Theta(s - S_J). \tag{45}$$

In addition to position a, slope b at the origin and the kink angles α_J the track is now characterized by the deviation of the inverse momentum p from the reference track. We will take the index zero for p so as to keep the similarity to previous equations in this section as close as possible. The expression for χ^2 (eq. (25)) remains unchanged but one now has to take also the derivative to p leading to a system of $N+3$ equations which may again be written in matrix form:

$$\boldsymbol{q} = \mathbf{A}^{-1}\boldsymbol{B}, \tag{46}$$

with

$$q_0 = p,\quad A_{0,0} = \sum_{i=1}^{n} w_{x_i} m_i^2,\quad A_{0,1} = A_{1,0} = \sum_{i=1}^{n} w_{x_i} m_i,\quad A_{0,2} = A_{2,0} = \sum_{i=1}^{n} w_{x_i} m_i s_i,$$

$$A_{0,J+2} = A_{J+2,0} = \sum_{i=1}^{n} w_{x_i} m_i \zeta_{s_i,J}\quad B_0 = \sum_{i=1}^{n} w_{x_i} m_i \zeta_i,$$

and the other terms identical to those in eqs. (27)–(30) with the obvious replacement of z by s. Extending similarly the dimensions of matrices **D** and **F** (eqs. (31) and (32)) to $(N+3)\times n$ and $(N+3)\times N$ with

$$D_{0,i} = w_{x_i} m_i,\quad F_{0,I} = 0,$$

one retains eq. (38) for the error matrix of the $N+3$ track parameters.

Calculation of transverse position for arbitrary location s along the track and of the root mean square errors follows from eq. (44) in complete analogy to the procedure applied in the field free case. One obtains for the region between scattering planes K and $K+1$

$$x(s) = pm(s) + a + bs + \sum_{J=1}^{K} \alpha_J(s - S_J) = q_0 m(s) + q_1 + q_2 s + \sum_{J=1}^{N} q_{J+2}\zeta_{s,J}, \tag{47}$$

$$\langle \Delta x^2(s)\rangle = E_{q0,0}m^2(s) + 2E_{q0,1}m(s) + 2E_{q0,2}sm(s) + 2m(s)\sum_{J=1}^{N} E_{q0,J+2}\zeta_{s,J} + E_{q1,1} + 2sE_{q1,2} + 2\sum_{I=1}^{N} E_{q1,I+2}\zeta_{s,I} + s^2E_{q2,2} + 2s\sum_{I=1}^{N} E_{q2,I+2}\zeta_{s,I} + \sum_{I=1}^{N}\sum_{J=1}^{N} E_{qI+2,J+2}\zeta_{s,I}\zeta_{s,J}, \tag{48}$$

where one notices the similarity with eq. (42) of all terms not containing the magnetic field integral $m(s)$ defined in eq. (16).

Similarly one may also calculate the slope and its mean square deviation from the true value between scattering planes K and $K+1$ and obtains:

$$\alpha(s) = pM(s) + b + \sum_{J=1}^{K} \alpha_J = q_0 M(s) + q_2 + \sum_{J=1}^{N} q_{J+2}\Theta(s - S_J), \tag{49}$$

$$\langle \Delta\alpha^2(s)\rangle = M^2(s)E_{q0,0} + 2M(s)E_{q0,2} + 2M(s)\sum_{J=1}^{N} E_{q0,J+2}\Theta(s - S_J) + E_{q2,2} + 2\sum_{J=1}^{N} E_{q2,J+2}\Theta(s - S_J) + \sum_{I=1}^{N}\sum_{J=1}^{N} E_{qI+2,J+2}\Theta(s - S_I)\Theta(s - S_J), \tag{50}$$

where $M(s)$ is the field integral up to position s along the reference track:

$$M(s) = 0.3\int_{l=0}^{s} B\,\mathrm{d}l. \tag{51}$$

The error correlation term between position and slope is given by:

$$\langle \Delta x(s)\Delta\alpha(s)\rangle = m(s)M(s)E_{q0,0} + M(s)E_{q0,1} + (m(s) + sM(s))E_{q0,2} + \sum_{J=1}^{N} E_{q0,J+2}(m(s)\Theta(s - S_J) + M(s)\zeta_{s,J}) + E_{q1,2} + \sum_{J=1}^{N} E_{q1,J+2}\Theta(s - S_J) + sE_{q2,2} + \sum_{j=1}^{N} E_{q2,J+2}[s\Theta(s - S_J) + \zeta_{s,J}] + \sum_{i=1}^{N}\sum_{J=1}^{N} E_{qI+2,J+2}\zeta_{s,I}\Theta(s - S_J). \tag{52}$$

5. Comparison of methods

In order to demonstrate the superiority of the new proposed fitting procedure it will be applied as well as two other more conventional methods to a "typical" setup consisting of 4 high resolution silicon detectors uniformly distributed over 12 cm track length followed by 13 gas detectors again at equal spacing over 120 cm track length. Properties of this setup are summarized in table 1. The field integrals are given for a 1 T transverse magnetic field, but the situation without magnetic field will also be treated. Comparison of the fitting procedures will be done by considering the measurement precision of the transverse track position (impact parameter) and the direction at the interaction point ($z = 0$) located 40 mm upstream of the first silicon detector ($z = 0$). In the presence of the magnetic field the measuring precision for the momentum will be considered in addition.

Furthermore the deviation of the fit from the real (scattered) track will be considered along the full length of the track. This is e.g. of interest for linking of the track to other parts of the spectrometer (e.g. calorimeters) downstream of the tracking detector. As will be seen in the following figures the proposed new fitting method gives best results in all cases.

Analytical results on the track parameter precision have been compared with a Monte Carlo simulation in order to check for internal consistency of the formalism and perfect agreement has been obtained for all three methods.

The three methods already described in detail in this paper will be shortly characterized again for the special arrangement under consideration:

Table 1
"Typical" detector arrangement used in comparing the track fitting methods

s [mm]	l/l_{rad} [%]	σ_{meas} [μm]	m_i [T m^2]	Detector type
40	0.4	5	0.24	silicon
80	0.4	5	0.96	silicon
120	0.4	5	2.16	silicon
160	0.4	5	3.84	silicon
200	0.1	200	6.0	gas
300	0.1	200	13.5	gas
400	0.1	200	24.0	gas
500	0.1	200	37.5	gas
600	0.1	200	54.0	gas
700	0.1	200	73.5	gas
800	0.1	200	96.0	gas
900	0.1	200	121.5	gas
1000	0.1	200	150.0	gas
1100	0.1	200	181.5	gas
1200	0.1	200	216.0	gas
1300	0.1	200	253.5	gas
1400	0.1	200	294.0	gas

(I) Single circle fit (straight line fit for $B = 0$) with weights given to the individual measurements according to the measurement precision of the detectors: $w = 1/\sigma^2_{meas}$.

(II) As the previous method but with momentum dependent weights obtained by adding an error from multiple scattering to the measuring precision: ($w = 1/(\sigma^2_{meas} + \sigma^2_{scatt})$). Multiple scattering in and behind the second measuring plane has been taken into account for calculation of σ_{scatt}.

(O) Optimum fitting method. Fitted tracks represented by segments of circles (or straight lines for $B = 0$) linked at the scattering planes by small kink angles.

Considering first the field free case the momentum dependences of impact parameter and angle measurement errors are shown in figs. 7 and 8. For easier graphical presentation the abscissa has been chosen linear in the square root of momentum. Below 10 GeV momentum the proposed optimum fitting method gives substantially better results than the two "standard" methods. At 4 GeV, e.g., we find from fig. 7 a decrease of the impact parameter measurement error by a factor 0.58 with respect to method I and by 0.78 with respect to method II in which more weight is given to the measurements in the front part of the track.

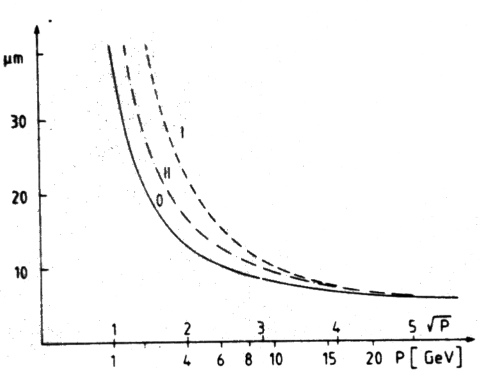

Fig. 7. Transverse track position measuring error at the vertex (impact parameter error) as a function of the particle momentum in the situation without magnetic field for the three fitting procedures described in the text.

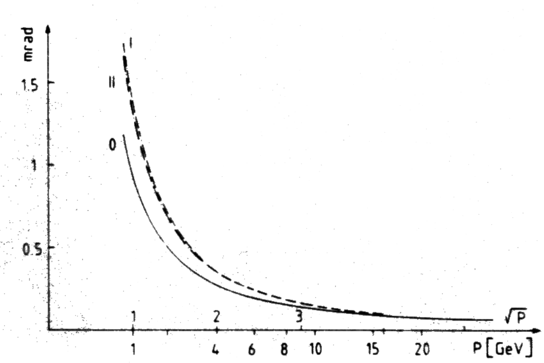

Fig. 8. Direction measuring error at the vertex as a function of the particle momentum in the situation without magnetic field.

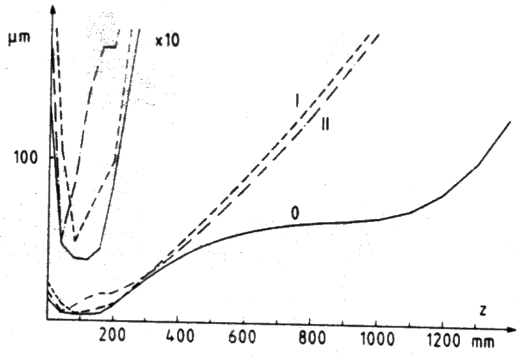

Fig. 9. Transverse deviation of the fitted track from the real track as a function of the position along the track for a 4 GeV particle in the situation without magnetic field.

The deviation Δx of the fitted track from the real track as a function of z is shown in fig. 9 for a 4 GeV particle. The front part of the track has been blown up by a factor 10. One recognizes the superiority of the new fitting method over the full length of the track.

The results of fits including the momentum in the presence of a magnetic field of 1 T are shown figs. 10–13. As expected one obtains at high momentum somewhat worse results compared with the case $B = 0$ for impact parameter and direction while the optimum method gives almost identical measurement

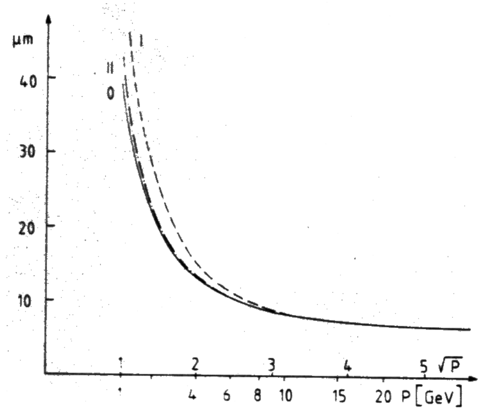

Fig. 10. Transverse track position measuring error at the vertex (impact parameter error) as a function of the particle momentum in the presence of a 1 T magnetic field.

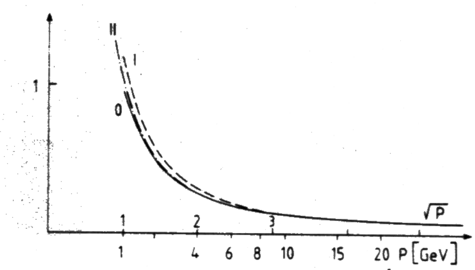

Fig. 11. Direction measuring error at the vertex as a function of the particle momentum in the presence of a 1 T magnetic field.

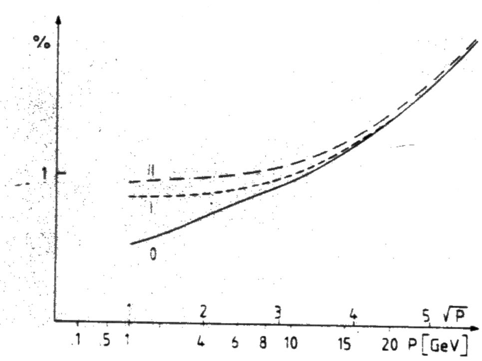

Fig. 12. Momentum measuring error as a function of the particle momentum in the presence of a 1 T magnetic field.

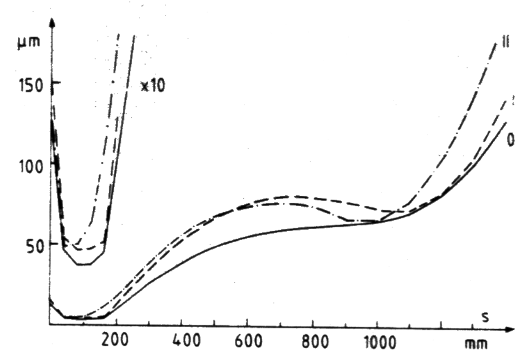

Fig. 13. Transverse deviation Δx of the fitted track from the real track for a 4 GeV particle in the presence of a 1 T magnetic field.

precision at low momentum. The other two methods deviate less from the optimum than in the field free case probably due to absorption of part of the multiple scattering into the fitted curvature. This in effect results in giving higher weight to the measurements in the front part when extrapolating to the vertex.

Consideration of the momentum measurement precision (fig. 12) proves the point that the weighting procedure designed for improvement of the impact parameter measurement (method II) leads to worse results in momentum measurement than the naive method I. Here again the optimum fitting method is shown to be superior at all momenta.

6. Summary

A method for track fitting has been proposed which allows for particle direction change in discrete scattering planes and thereby approximating the track by a kinked trajectory over its full length. The formalism for performing the fit and for estimating the errors of the track parameters has been developed. It makes convenient use of the matrix formalism and leads to a full error correlation matrix of the track parameters (momentum, position, slope and N scattering angles). The method has been demonstrated to lead to more precise results than some "standard" procedures also described in this paper. A further advantage of this method is the correctness of the calculation of the fit quality (χ^2), being independent of momentum of the particle. Therefore it is possible to use correct probability criteria for acceptance of track candidates during pattern recognition.

The formalism has been presented for two dimensions only and does not include energy loss from passage through material. Furthermore some approximations (eq. (15)) used for tracking inside a magnetic field may be too coarse for real applications as far as the determination of the track parameters is concerned although it may suffice for an estimation of the errors.

Application in real situations may therefore require modifications and additions which, however, do not change the basic ideas presented in this paper. Generalization to three dimensions seems straightforward and energy loss might be included in an iterative fashion if necessary. This seems, however, only necessary in exotic cases, such as, e.g., tracking of muons through a calorimeter or very low energy particles. Finally it may be useful to stress that even for non-optimum fitting methods a correct estimation of errors of the track parameters is extremely important for decisions on probability basis, such as, e.g., in vertex topology analysis.

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