Small-angle multiple scattering and spatial resolution in charged particle tomography

A A M Mustafa and Daphne F Jackson
Department of Physics, University of Surrey, Guildford, Surrey, England

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Abstract. The formulae for the RMS scattering angle given by small-angle multiple-scattering theory are discussed and a correction to the standard Rossi formula is derived. Values of the correction factor are calculated for protons, α-particles and heavy ions; these results show that the correction is important for protons and α-particles but negligible for heavy ions with mass number 12–40. The values of the RMS scattering angle and the RMS lateral displacement are calculated for a proton beam passing through water. For protons the correction for energy loss in thick targets is even more important.

1. Introduction

We are studying the feasibility of tomography with charged particles, particularly protons, α-particles and heavy ions up to argon. An important ingredient of this work is an examination of the spatial resolution that can be achieved. The lateral spreading of the beam as it passes through a sample can be related to the RMS angle of deflection arising from multiple small-angle scattering. A simple formula for $\theta_{RMS}$ has been given by Rossi and Greisen (1941) but it has been suggested that this formula needs a correction factor (Highland 1975). Hanson (1978) has plotted such a correction factor for protons and has shown that it is not unimportant for proton tomography.

In this paper we present the key formulae and derive a correction factor which we use to calculate the lateral resolution of protons passing through water.

The theory of small-angle multiple scattering (Scott 1963, Bethe 1953) yields two distribution functions—the function $F(\theta, \phi, t)$ which represents the number of particles scattered into the direction $(\theta, \phi)$ after the beam has passed through a thickness $t$ of material, and the projected-angle function $F_p(\phi, t)$ which represents the projections of $F(\theta, \phi, t)$ on to the $x$-$z$ plane. The initial beam direction is taken to define the $z$ direction. The calculation of these distribution functions requires the probability $W(\theta, t)$ for a single scattering into a solid angle $d\Omega = 2\pi \sin \theta d\theta$ in a thickness $dt$ of material. In section 2 we discuss the treatment of $W(\theta, t)$ and in section 3 we discuss the formulae for $\theta_{RMS}$ extracted from multiple-scattering theory.

2. The single-scattering process

The simplest assumption is that the single-scattering probability can be represented by the Rutherford scattering formula

$$W(\theta, t) = N(t)\sigma_R(\theta)$$

where $\sigma_R(\theta)$ is the differential cross-section for Rutherford scattering, $N(t) = N_A \rho / A$ is...
the number of atoms per unit volume, \( \rho \) is the density, \( A \) is the relative atomic mass and \( N_A \) is Avogadro's number. With \( \beta = v/c, p = \hbar k \) and \( r_e = e^2/m_e c^2 \), we have

\[
W(\theta, t) \, d\Omega = \frac{1}{2} N(t) z^2 Z^2 r_e^2 (m_e c/\beta p)^2 \sin^4(\theta/2) \, d\Omega
\]  
(2)

where \( z \) is the atomic number of the projectile and \( Z \) is the atomic number of the target nucleus. For small angles this becomes

\[
W(\theta, t) \, d\Omega = \frac{4}{3} N(t) z^2 Z^2 \frac{2}{\theta^4} \, d\theta
\]  
(3)

where

\[
\eta = z Z e^2/\hbar c.
\]  
(5)

The effect of screening of the nuclear Coulomb potential by the atomic electrons can be taken into account using a screened potential of the form (Scott 1963)

\[
V(r) = \pm z Z e^2/(r r_0) \]  
(6)

where the screening radius \( r_0 \) is frequently taken to be the Thomas–Fermi radius

\[
r_0 = 0.885 a_0 Z^{-1/3} = 0.885 \hbar /m_e c a Z^{1/3}
\]  
(7)

where \( \alpha = e^2/\hbar c \approx 1/137 \) is the fine-structure constant and \( a_0 \) is the Bohr radius. For the Yukawa potential, which has \( f(r/r_0) = \exp(-r/r_0) \), the first Born approximation yields

\[
W(\theta, t) = \frac{4}{3} N(\eta^2 /k^2 \theta^4) \frac{2\pi}{\theta^4} \, d\theta
\]  
(4)

where \( \theta_0 \) is the Born screening angle:

\[
\theta_0 = \hbar /p r_0 = 1.13\alpha (m_e c/p) Z^{1/3}.
\]  
(9)

In these formulae the nucleus has been treated as a point charge. The finite size of the nucleus is expected to be significant at angles

\[
\theta_N \sim 322 (m_e c/p) A_N^{-1/3}
\]  
(10)

where \( r_N \) is an estimate of the nuclear radius. For \( r_N = 1.2 A^{1/3} \), where \( A_N \) is the mass number of the nucleus, we have

\[
\theta_N \sim 322 (m_e c/p) A_N^{-1/3}.
\]  
(10)

The numerical coefficient derived by Williams (1940) corresponds to \( r_N = 1.38 A^{1/3} \).

Equation (4) can also be written in the form (Molière 1947, 1948)

\[
W(\theta, t) \, d\Omega = 2 \chi_0^2 (\theta d\theta / \theta^4) q(\theta)
\]  
(11)

where \( q(\theta) \) represents the departure from Rutherford scattering due to screening and is given in Molière's formalism by

\[
q(\theta) = \frac{(k \theta)^4}{4 \eta^2} \left[ \int_0^\infty b \, db \, J_0(k \theta b)[\exp(i\Phi(b))-1] \right]^2
\]  
(12)

where \( b \) is the impact parameter,

\[
\Phi(b) = \frac{2\pi}{\eta} \int_0^\infty \frac{f(r/r_0)}{(r^2-b^2)^{1/2}} \, dr
\]  
(13)

and

\[
\Phi((l+1/4)/k) = \delta_l
\]  
(14)
Small-angle multiple scattering and spatial resolution

relates Molière's phase factor $\Phi$ to the phase shift $\delta_i$ found in partial wave treatments of the scattering (Bethe 1953).

The characteristic angle $\chi_c$ is chosen so that the total probability of scattering through an angle greater than $\chi_c$ is unity. For a mixture of scatterers an average scattering probability can be defined as

$$\bar{W}(\theta, t) = \int_0^\chi W(\theta, t') \, dt'$$

and the characteristic angle then becomes

$$\chi^2 = 4\pi \int_0^\chi \sum_i \frac{N_i(t') \eta_i^2(t')}{k^2(t')} \, dt'$$

$$= 4\pi \rho^2 2 \int_0^\chi \sum_i \frac{N_i(t') Z_i^2}{(pv)^2} \, dt'.$$

For a homogeneous sample and zero energy loss this reduces to

$$\chi^2 = 4\pi \rho^2 2 Z \bar{\rho} \bar{v} 2.$$

In Molière's method the screening angle is defined as

$$\log \chi_a = -\frac{1}{2} \lim_{x_m \to \infty} \left( \int_0^x d\theta q(\theta) - \log x_m \right).$$

Using a function $f(r/r_0)$ derived from a fit to the Thomas-Fermi model for the atomic potential and evaluating equations (13), (12) and (18), Molière derived an approximate expression for the screening angle of the form

$$\chi^2 = \frac{\chi_0^2}{(R + 3.76)}$$

where $R = 1.13$ and is a constant for all $Z$. The Thomas-Fermi model is not accurate for light atoms because it is a statistical model.

The possibility of inelastic collisions with electrons must also be taken into account (Fano 1954). For this purpose it is convenient to separate the elastic and inelastic contributions to the cross-sections,

$$\sigma(\theta) = \sigma_{el}(\theta) + \sigma_{inel}(\theta)$$

so that

$$N \sigma_{el}(\theta) \sin \theta \, d\theta = 2Z^2 \chi^2 q(\theta) \sin \theta \, d\theta/4(1 - \cos \theta)^2$$

where $\chi^2$ is as given previously in equations (16) or (17). For scattering of incident electrons the inelastic contribution can be written as

$$N \sigma_{inel}(\theta) \sin \theta \, d\theta = 2Z^2 S(v) \sin \theta \, d\theta/4(1 - \cos \theta)^2$$

where $S(v)$ is the incoherent scattering function and

$$v = 0.333 Z^{-2/3}(pa\rho/h)[2(1 - \cos \theta)]^{1/2}$$

$$= 0.333 Z^{-2/3}(pa\rho/h) \theta.$$
which leads to the results

\[ \chi_c^2 \log \chi_a = 4 \pi \int_0^t \frac{d t'}{k'^2(t')} \sum N_i(t') \eta_i^2(t') \left[ \log \chi_{\text{inel}}^i + \frac{1}{Z_i + 1} \log \left( \frac{\chi_{\text{inel}}^i}{\chi_a^i} \right) \right] \]  

(25)

\[ \log(\chi_{\text{inel}}^i/\chi_a^i) = \frac{1}{2}(u_{\text{inel}} - u_a) \]  

(26)

\[ u_a = \log(0.1148 Z^{-2/3}(R + 3.76111)) \]  

(27)

\[ u_{\text{inel}} = \lim_{U \to \infty} \left( \int_U^\infty du S(\exp \frac{1}{2}u) + 1 - U \right). \]  

(28)

Estimates of \( u_{\text{inel}} \) in the Thomas-Fermi model yield \(-5.8\) for all \( Z \) while exact calculations for hydrogen yield \(-3.6\) (Fano 1954).

For heavy incident particles these formulae for inelastic scattering cannot be used because of the limitation on the recoil energy \( Q \) that can be imparted to the recoiling electron. Equation (22) must be replaced by (Fano 1954)

\[ N_{\sigma_{\text{inel}}}(Q) \, dQ = Z \chi_c^2 S(v) \left( \frac{p^2}{2m_e^2} \right) \frac{dQ}{Q^2} \left( 1 - \frac{Q\beta^2}{Q_{\text{max}}} \right). \]  

(29)

This yields

\[ \chi_c^2 \log \chi_a = 4 \pi \int_0^t \frac{d t'}{k'^2(t')} \sum N_i(t') \eta_i^2(t') \left( \log \chi_{\text{inel}}^i - \frac{1}{2Z_i} D \right) \]  

(30)

where

\[ D = \int_{Q_{\text{min}}}^{Q_{\text{max}}} \frac{dQ}{Q} \left( S(v) - \frac{Q}{Q_{\text{max}}} \right) \left( 1 - \frac{Q\beta^2}{Q_{\text{max}}} \right) \]  

(31)

\[ = \log(1129 Z^{-4/3} \beta^2 \gamma^2) - u_{\text{inel}} - \frac{1}{2} \beta^2 \]  

(32)

where \( \gamma = (1 - \beta^2)^{-1/2} \). For a homogeneous sample with no energy loss, \( \chi_c^2 \) is again given by equation (17) and \( \log \chi_a \) is given by

\[ \log \chi_a = \sum_i N_i Z_i^2 \left( \log \chi_{\text{inel}}^i - \frac{1}{2Z_i} D \right). \]  

(33)

3. Multiple scattering

Rossi and Greisen (1941) give the RMS scattering angle for multiple scattering when the energy loss can be neglected as

\[ \theta_{\text{RMS}} = \left( \frac{E_i z}{p \beta c} \right) \rho t / X_0 \]  

(34)

where \( E_i = (4 \pi / \alpha)^{1/2} m_e c^2 \) and \( X_0 \) is the radiation length (in units of mass per length\(^2\)). The definition of \( X_0 \) used by Rossi is

\[ X_0 = (1/4 \alpha)(A/N_A) \left[ 1/Z^2 r_0^2 \log(183 Z^{-1/3}) \right] \]  

(35)

and hence

\[ \theta_{\text{RMS}}^{\text{Rossi}} = 4 \pi e^4 (z Z / p \beta c)^{2}(N_A / A) \rho t 4 \log(183 Z^{-1/3}). \]  

(36)

A more accurate expression for the radiation length has been given by Tsai (1974) in the form

\[ X_0 \propto A / [Z^2 (L_{\text{rad}} - f) + Z L_{\text{rad}}]. \]  

(37)
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where \( L_{\text{rad}} \) is obtained from the atomic form factor, \( L'_{\text{rad}} \) is obtained from the incoherent scattering function and \( f \) is the Coulomb correction (Davies et al 1954)

\[
f = (\alpha Z)^2 [(1 + (\alpha Z)^2)^{-1} + 0.20206 - 0.0369(\alpha Z)^2 + 0.0083(\alpha Z)^4 \ldots].
\]

(38)

From Molière’s fit to the Thomas–Fermi potential, Tsai obtained

\[
L_{\text{rad}} = \log(184.15 Z^{-1/3}) \quad (39)
\]

\[
L'_{\text{rad}} = \log(1194 Z^{-2/3}). \quad (40)
\]

In Molière’s model of multiple scattering the RMS angle of the Gaussian part of the distribution is given by

\[
\theta_{\text{RMS}} = \chi_c B^{1/2}
\]

(41)

where (Scott 1963)

\[
B = 1.153 + 2.583 \log (\chi_c/\chi_a)^2.
\]

(42)

Hence, using equation (17), we have

\[
\theta_{\text{RMS}}^2 (\text{Molière}) = 4\pi e^4 (zZ/\beta_c)^2 (N_a/A) \rho B.
\]

(43)

We now follow K M Hanson (1978 and 1979, private communication) and define a correction to the Rossi RMS angle of the form

\[
\theta_{\text{RMS}}(C) = \theta_{\text{RMS}}(\text{Rossi}) (1 + \epsilon)
\]

(44)

and setting \( \theta_{\text{RMS}}(C) = \theta_{\text{RMS}}(M) \) we have

\[
\epsilon = [0.25 Z^2 B / [Z^2 (L_{\text{rad}} - f) + ZL'_{\text{rad}}]]^{1/2} - 1.
\]

(45)

and

\[
\theta_{\text{RMS}}(C) = \left( 4\pi e^4 N_a z^2 \frac{\rho t}{A (\rho \beta_c)^2} [Z^2 (L_{\text{rad}} - f) + ZL'_{\text{rad}}]^{1/2} (1 + \epsilon) \right)^{1/2}.
\]

(46)

Highland (1975) has investigated the angle \( \theta_{1/e} \), which is the angle at which the measured distribution falls to \( 1/e \) of its value at \( \theta = 0 \), and is given by (Scott 1963)

\[
\theta_{1/e} = \chi_c (1.007 B - 1.33)^{1/2}.
\]

(47)

This angle is preferred for examination of experimental data because it is least affected by the unmeasured tail of the distribution, and Highland (1975) suggests that it should be calculated from the formula

\[
\theta_{1/e} = (17.5/p\beta c)(\rho t/X_0)^{1/2}(1 + \epsilon')
\]

(48)

\[
\epsilon' = a' \log (\rho t/b' X_0)
\]

(49)

where \( a' \) and \( b' \) are constants, independent of \( p \) and \( z \).

Hungerford et al (1972) have derived a correction to \( \theta_{\text{RMS}} \) by applying a conversion factor from the laboratory to the centre-of-mass frame of reference. However, since all integrals in multiple-scattering theory are of the form

\[
\int \sin \theta \, d\theta \int W(\theta, t) \, dt
\]

and \( W(\theta, t) \sin \theta \, d\theta \) is invariant with respect to the frame of reference it is not easy to see the justification for their procedure.
4. Calculations

Our ultimate aim is to calculate the lateral displacement of a beam of charged particles at various depths. This is given by

\[ y_{\text{RMS}} = 6^{-1/2} \theta_{\text{RMSf}}. \]  

(50)

We first carried out a complete calculation in the Molière (M) model, disregarding energy loss, i.e., calculating \( X^2 \) from equation (17), \( \chi_{\text{el}} \) from equation (19) with \( R = 1.13 \) for all \( Z \), \( \log X_a \) from equations (32) and (33) with \( u_{\text{inel}} = -5.8 \) for all \( Z \), \( B \) from equation (42) and finally \( \theta_{\text{RMSf}}(M) \) from equation (43). Then we calculated \( \epsilon \) from equation (45) using \( L_{\text{rad}} \) and \( L_{\text{rad}} \) given by equations (39) and (40). The value of \( X_0 \) was calculated from equation (37).

K M Hanson (1979, private communication) has recalculated \( X^2 \) from equation (18) using the atomic form factors tabulated by Hubbell et al. (1975) and has deduced values of \( R \) for each \( Z \). He has also recalculated \( L_{\text{rad}}, L_{\text{rad}}, X_0 \) and \( u_{\text{inel}} \) using the atomic form factors and incoherent scattering factors tabulated by Hubbell et al. (1975). These

| Table 1. (a) Values of \( \epsilon \) for incident protons calculated with the Molière model for copper \((Z = 29, \quad A = 63.54)\) with \( R = 1.13, \quad X_0 = 12.86. \) (b) Values of \( \epsilon \) calculated with Hanson’s parameters \( R = 1.287, \quad X_0 = 13.02, \quad u_{\text{inel}} = -5.068. \)(The units for \( X \) and \( X_0 \) are \( \text{g cm}^{-2} = 10 \text{ kg m}^{-2}. \))

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tabulations are based on the best available atomic wavefunctions and the parameters derived from them should represent an improvement on the values derived by Molière and others from the Thomas–Fermi model, particularly for light elements. We have used Hanson’s values of $R_0$, $E_{rad}$, $L_{rad}$ and $\mu_{inel}$ in a second set of calculations using the same equations as listed above.

The results obtained for $\epsilon$ for incident protons are given in tables 1–3 for copper, calcium and water. It can be seen that the magnitude and variation of $\epsilon$ is such that $\theta_{\text{RMS}}$ should always be calculated from a formula that takes this correction into account. The discrepancies between the values of $\epsilon$ calculated from Molière’s model and those calculated with Hanson’s parameters are small and arise mainly from differences in $X_0$. We have investigated the effect of replacing equation (19) by the approximate expression

$$\chi_0^2 = X_0 R (1 + 3.33 \eta^2)$$

(51)

in the calculation with Hanson’s parameters. The effect is negligible for $\beta = 0.6$, while

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| 0.005       | -0.3030       | -0.3315       | -0.3630       | -0.3878       | -0.4087       |
| 0.020       | -0.2426       | -0.2687       | -0.2974       | -0.3197       | -0.3386       |
| 0.050       | -0.2051       | -0.2300       | -0.2572       | -0.2783       | -0.2960       |
| 0.100       | -0.1779       | -0.2020       | -0.2282       | -0.2485       | -0.2654       |
| 0.200       | -0.1516       | -0.1749       | -0.2002       | -0.2197       | -0.2361       |
| 0.500       | -0.1180       | -0.1404       | -0.1647       | -0.1834       | -0.1990       |
| 1.000       | -0.0934       | -0.1152       | -0.1387       | -0.1569       | -0.1720       |
| 2.000       | -0.0695       | -0.0907       | -0.1136       | -0.1312       | -0.1458       |
| 5.000       | -0.0388       | -0.0593       | -0.0814       | -0.0984       | -0.1125       |
| 10.000      | -0.0162       | -0.0362       | -0.0578       | -0.0743       | -0.0880       |
| 20.000      | 0.0059        | -0.0136       | -0.0347       | -0.0509       | -0.0642       |
| 50.000      | 0.0344        | 0.0154        | -0.0051       | -0.0207       | -0.0337       |
Table 3. (a) Values of ε for incident protons calculated with the Molière model for water (mean Z = 7.30, A = 18.02) with R = 1.13, X₀ = 36.70. (b) Values of ε calculated with Hanson's parameters R = 1.221, X₀ = 36.70, u_{min} = -4.64. (The units for X and X₀ are g cm⁻² = 10 kg m⁻².)

(a)  

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for small β and large values of X = pt the maximum change is ~9%. Since B ~ 15 the difference between θ_{RMS} and θ_{1/ε} is small in all cases.

Typical behaviour of ε as a function of ρ/pt is shown in figure 1. It is clear that the simple formula (49) for ε could not reproduce this behaviour over the whole range of β and for different absorbing materials.

Figure 2 shows the behaviour of ε for different ions calculated at β = 0.65 as a function of X = pt while figure 3 shows the behaviour of ε as a function of β for a fixed depth t = 200 mm in water. It is interesting to note that the correction to the Rossi formula (34) for θ_{RMS} is negligible for heavy ions with mass numbers in the range 12–40. For the same target material ε varies with β/z.

The RMS lateral displacement y_{RMS} of a proton beam as it passes through water has been calculated using equation (50) with θ_{RMS} given by equation (46). Results are given in table 4 and show that inclusion of the correction ε gives an apparent improvement in the lateral resolution of ~10%.
Figure 1. Behaviour of $\epsilon$ for protons passing through copper as a function of $X/X_0$ where $X = \rho t$ and $X_0$ is the radiation length. The values of $\epsilon$ correspond to those tabulated in table 1(b). The values of $\beta$ are given on the curves.

From these results, we conclude that it is necessary to include the correction $\epsilon$, i.e. to correct the Rossi formula for $\theta_{RMS}$, for protons and $\alpha$-particles but not for heavy ions. The Molière model is sufficiently accurate for the calculation of $\epsilon$ and $\theta_{RMS}$.

For thick targets energy loss should be taken into account by using equations (16) and (30) instead of equations (17) and (33). Equations (16) and (30) can be rewritten as

$$\chi^2 = \int_0^t \text{d}r' \chi^2(r')$$

$$\chi^2 \log \chi^2 = \int_0^t \text{d}r' \chi^2(r') \log \chi^2(r').$$

Figure 2. Behaviour of $\epsilon$ as a function of $X = \rho t$ for different ions with $\beta = 0.65$. (The units of $X$ are g cm$^{-2} = 10$ kg m$^{-2}$.)
These equations can be converted to integrals over the kinetic energy $T$ of the form (Berger and Seltzer 1964)

$$
\chi^2 = \int_T^{T_0} dT' \chi^2(T') \left( -\frac{1}{\rho} \frac{dE(T')}{dx} \right)^{-1} 
$$

$$
\chi^2 \log \chi^2 = \int_T^{T_0} dT' \chi^2(T') \log \chi^2(T') \left( -\frac{1}{\rho} \frac{dE(T')}{dx} \right)^{-1}
$$

where $dE/dx$ is the stopping power, $T_0$ is the initial kinetic energy and $T$ is the kinetic energy at thickness $t$. These formulae are reliable when the final energy is not near to zero. This condition is satisfied in all our examples.

Results for a proton beam at various energies passing through various thicknesses in water are given in table 5. Both $\chi_e$ and $\chi_a$ increase quite substantially compared with the results for no energy loss, leading to an increase in $B$ of only a few per cent and to a decrease in $\epsilon$ of 10–30% depending on the energy and thickness. The large increase in $\chi_e$ is reflected in the large increases in the values of $\theta_{RMS}$ and $\gamma$ given in table 5 compared

<table>
<thead>
<tr>
<th>Proton energy (MeV)</th>
<th>Depth, $t$ (mm)</th>
<th>$\beta$</th>
<th>$\epsilon$</th>
<th>$\theta_{RMS}$ (rad)</th>
<th>$\gamma_{RMS}$ (mm)</th>
<th>Uncorrected $\theta_{RMS}$ (rad)</th>
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Small-angle multiple scattering and spatial resolution

Table 5. Results for a proton beam passing through water when energy loss is taken into account and using the Molière model.

<table>
<thead>
<tr>
<th>Proton energy (MeV)</th>
<th>Depth, t (mm)</th>
<th>ε</th>
<th>θ_{RMS} (rad)</th>
<th>Y_{RMS} (mm)</th>
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<tr>
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<td>300</td>
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</table>

with those given in table 4. Thus for protons passing through thick targets it is essential to take energy loss into account rather accurately. It is also evident from tables 4 and 5 that spatial resolution in proton tomography will be poor for thick targets, as already noted by Hanson (1978).

The uncertainties in the values given in tables 4 and 5 are small. We estimate that, in the energy region of interest, uncertainties due to errors in measured values of stopping power of a single element and to departures from the Bragg rule should be less than 1%. This will have a negligible effect on our calculations. Uncertainties of \( \sim 10\% \) in the mean residual energy \( T \) lead to a change in \( Y_{RMS} \) of \( \sim 0.5\% \). Hence, provided that \( T \) does not fall below about 10 MeV, the results given by such calculations should be very reliable.

Acknowledgments

We are greatly indebted to Dr K M Hanson for sending us his parameters derived from improved atomic models and details of his calculations. This work is associated with a NATO grant.

Resumé

Diffusions multiples selon des petits angles et résolution spatiale dans la tomographie avec des particules chargées.

Les formules, données par la théorie de la diffusion multiple selon des petits angles, pour l’écart type de l’angle de diffusion sont discutées et il en est déduit une correction de la formule standard de Rossi. Les valeurs du facteur correctif sont calculées pour des protons, des particules alpha et des ions lourds: ces résultats montrent que cette correction est importante pour les protons et les particules alpha, mais négligeable pour les ions lourds de nombre de masse compris entre 12 et 40. Nous avons calculé les valeurs de l’écart-type de l’angle de diffusion et de l’écart-type du déplacement latéral pour un faisceau de protons traversant de l’eau. Pour les protons, la correction de la perte d’énergie à travers des cibles épaisse sont encore plus importante.

Zusammenfassung

Vielfachstreuung bei kleinen Winkeln und räumliche Auflösung bei der Tomographie mit geladenen Teilchen.


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