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Small-angle multiple scattering and spatial resolution in charged particle tomography

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Abstract. The formulae for the RMS scattering angle given by small-angle multiple-scattering theory are discussed and a correction to the standard Rossi formula is derived. Values of the correction factor are calculated for protons, α-particles and heavy ions; these results show that the correction is important for protons and α-particles but negligible for heavy ions with mass number 12–40. The values of the RMS scattering angle and the RMS lateral displacement are calculated for a proton beam passing through water. For protons the correction for energy loss in thick targets is even more important.

1. Introduction

We are studying the feasibility of tomography with charged particles, particularly protons, α-particles and heavy ions up to argon. An important ingredient of this work is an examination of the spatial resolution that can be achieved. The lateral spreading of the beam as it passes through a sample can be related to the RMS angle of deflection arising from multiple small-angle scattering. A simple formula for θ_{RMS} has been given by Rossi and Greisen (1941) but it has been suggested that this formula needs a correction factor (Highland 1975). Hanson (1978) has plotted such a correction factor for protons and has shown that it is not unimportant for proton tomography.

In this paper we present the key formulae and derive a correction factor which we use to calculate the lateral resolution of protons passing through water.

The theory of small-angle multiple scattering (Scott 1963, Bethe 1953) yields two distribution functions—the function $F(\theta, \phi, t)$ which represents the number of particles scattered into the direction (θ, ϕ) after the beam has passed through a thickness t of material, and the projected-angle function $F_p(\phi, t)$ which represents the projections of $F(\theta, \phi, t)$ on to the x–z plane. The initial beam direction is taken to define the z direction. The calculation of these distribution functions requires the probability $W(\theta, t)$ for a single scattering into a solid angle $d\Omega = 2\pi \sin\theta \, d\theta$ in a thickness dt of material. In section 2 we discuss the treatment of $W(\theta, t)$ and in section 3 we discuss the formulae for θ_{RMS} extracted from multiple-scattering theory.

2. The single-scattering process

The simplest assumption is that the single-scattering probability can be represented by the Rutherford scattering formula

$$W(\theta, t) = N(t)\sigma_R(\theta) \tag{1}$$

where $\sigma_R(\theta)$ is the differential cross-section for Rutherford scattering, $N(t) = N_A \rho / A$ is

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the number of atoms per unit volume, ρ is the density, A is the relative atomic mass and N_A is Avogadro's number. With $\beta = v/c$, $p = \hbar k$ and $r_e = e^2/m_e c^2$, we have

$$W(\theta, t)\,\mathrm{d}\Omega = \tfrac{1}{4}N(t)z^2Z^2r_e^2(m_e c/\beta p)^2 \sin^4(\theta/2)\,\mathrm{d}\Omega \tag{2}$$

where z is the atomic number of the projectile and Z is the atomic number of the target nucleus. For small angles this becomes

$$W(\theta, t)\,\mathrm{d}\Omega \simeq 4N(t)z^2Z^2r_e^2(m_e c/\beta p)^2 \frac{2\pi\theta\,\mathrm{d}\theta}{\theta^4} \tag{3}$$

$$\simeq 4N(t)(\eta^2/k^2\theta^4)2\pi\theta\,\mathrm{d}\theta \tag{4}$$

where

$$\eta = zZe^2/\hbar v. \tag{5}$$

The effect of screening of the nuclear Coulomb potential by the atomic electrons can be taken into account using a screened potential of the form (Scott 1963)

$$V(r) = \pm z(Z/r)e^2 f(r/r_0) \tag{6}$$

where the screening radius r_0 is frequently taken to be the Thomas–Fermi radius

$$r_0 = 0.885\, a_0 Z^{-1/3} = 0.885\, \hbar/m_e c\alpha Z^{1/3} \tag{7}$$

where $\alpha = e^2/\hbar c \simeq 1/137$ is the fine-structure constant and a_0 is the Bohr radius. For the Yukawa potential, which has $f(r/r_0) = \exp - r/r_0$, the first Born approximation yields

$$W(\theta, t) = 4N(t)[\eta^2/k^2(\theta^2 + \chi_0^2)^2] \tag{8}$$

where χ_0 is the Born screening angle:

$$\chi_0 = \hbar/pr_0 = 1.13\alpha(m_e c/p)Z^{1/3}. \tag{9}$$

In these formulae the nucleus has been treated as a point charge. The finite size of the nucleus is expected to be significant at angles

$$\theta_N \sim \hbar/pr_N$$

where r_N is an estimate of the nuclear radius. For $r_N = 1.2\,A_N^{1/3}$, where A_N is the mass number of the nucleus, we have

$$\theta_N \sim 322(m_e c/p)A_N^{-1/3}. \tag{10}$$

The numerical coefficient derived by Williams (1940) corresponds to $r_N = 1.38\,A_N^{1/3}$.

Equation (4) can also be written in the form (Molière 1947, 1948)

$$W(\theta, t)\,\mathrm{d}\Omega = 2\chi_c^2(\theta\mathrm{d}\theta/\theta^4)q(\theta) \tag{11}$$

where $q(\theta)$ represents the departure from Rutherford scattering due to screening and is given in Molière's formalism by

$$q(\theta) = \frac{(k\theta)^4}{4\eta^2}\left|\int_0^\infty b\,\mathrm{d}b\,J_0(k\theta b)[\exp(\mathrm{i}\Phi(b)) - 1]\right|^2 \tag{12}$$

where b is the impact parameter,

$$\Phi(b) = \frac{2\eta}{\hbar v}\int_b^\infty \frac{f(r/r_0)}{(r^2 - b^2)^{1/2}}\,\mathrm{d}r \tag{13}$$

and

$$\Phi((l+\tfrac{1}{2})/k) = \delta_l \tag{14}$$

relates Molière's phase factor Φ to the phase shift δ_l found in partial wave treatments of the scattering (Bethe 1953).

The characteristic angle χ_c is chosen so that the total probability of scattering through an angle greater than χ_c is unity. For a mixture of scatterers an average scattering probability can be defined as

$$t\bar{W}(\theta, t) = \int_0^t W(\theta, t')\,\mathrm{d}t' \tag{15}$$

and the characteristic angle then becomes

$$\begin{aligned}\chi_c^2 &= 4\pi \int_0^t \mathrm{d}t' \sum_i \frac{N_i(t')\eta_i^2(t')}{k^2(t')} \\ &= 4\pi e^4 z^2 \int_0^t \mathrm{d}t' \sum_i \frac{N_i(t')Z_i^2}{(pv)^2}.\end{aligned} \tag{16}$$

For a homogeneous sample and zero energy loss this reduces to

$$\begin{aligned}\chi_c^2 &= 4\pi e^4 z^2 Z^2 Nt/p^2v^2 \\ &= 4\pi e^4 z^2 Z^2 N_A \rho t/Ap^2v^2.\end{aligned} \tag{17}$$

In Molière's method the screening angle is defined as

$$\log \chi_a = -\tfrac{1}{2} - \lim_{\chi_m \to \infty} \left(\int_0^{\chi_m} \mathrm{d}\theta \frac{q(\theta)}{\theta} - \log \chi_m \right). \tag{18}$$

Using a function $f(r/r_0)$ derived from a fit to the Thomas-Fermi model for the atomic potential and evaluating equations (13), (12) and (18), Molière derived an approximate expression for the screening angle of the form

$$\chi_a^2 = \chi_0^2(R + 3.76\eta^2) \tag{19}$$

where $R = 1.13$ and is a constant for all Z. The Thomas-Fermi model is not accurate for light atoms because it is a statistical model.

The possibility of inelastic collisions with electrons must also be taken into account (Fano 1954). For this purpose it is convenient to separate the elastic and inelastic contributions to the cross-sections,

$$\sigma(\theta) = \sigma_{el}(\theta) + \sigma_{inel}(\theta) \tag{20}$$

so that

$$Nt\sigma_{el}(\theta) \sin\theta\, \mathrm{d}\theta = 2Z^2\chi_c^2 q_{el}(\theta) \sin\theta\, \mathrm{d}\theta/4(1-\cos\theta)^2 \tag{21}$$

where χ_c^2 is as given previously in equations (16) or (17). For scattering of incident electrons the inelastic contribution can be written as

$$Nt\sigma_{inel}(\theta) \sin\theta\, \mathrm{d}\theta = 2Z\chi_c^2 S(v) \sin\theta\, \mathrm{d}\theta/4(1-\cos\theta)^2 \tag{22}$$

where $S(v)$ is the incoherent scattering function and

$$\begin{aligned}v &= 0.333\, Z^{-2/3}(pa_0/\hbar)[2(1-\cos\theta)]^{1/2} \\ &\simeq 0.333\, Z^{-2/3}(pa_0/\hbar)\theta.\end{aligned} \tag{23}$$

Thus $q(\theta)$ in equation (11) is to be replaced by

$$q(\theta) = q_{el}(\theta) + (1/Z)S(v) \tag{24}$$

which leads to the results

$$\chi_c^2 \log \chi_\alpha = 4\pi \int_0^t \frac{dt'}{k^2(t')} \sum_i N_i(t') \eta_i^2(t') \left[\log \chi_{\alpha i}^{el} + \frac{1}{Z_i + 1} \log \left(\frac{\chi_\alpha^{inel}}{\chi_\alpha^{el}} \right)_i \right] \quad (25)$$

$$\log(\chi_\alpha^{inel}/\chi_\alpha^{el}) = \tfrac{1}{2}(u_{inel} - u_a) \quad (26)$$

$$u_a = \log(0.1148\, Z^{-2/3}(R + 3.76\eta^2)) \quad (27)$$

$$u_{inel} = \lim_{U \to \infty} \left(\int_{-\infty}^{U} du\, S(\exp \tfrac{1}{2}u) + 1 - U \right). \quad (28)$$

Estimates of u_{inel} in the Thomas–Fermi model yield -5.8 for all Z while exact calculations for hydrogen yield -3.6 (Fano 1954).

For heavy incident particles these formulae for inelastic scattering cannot be used because of the limitation on the recoil energy Q that can be imparted to the recoiling electron. Equation (22) must be replaced by (Fano 1954)

$$Nt\sigma_{inel}(Q)\, dQ = Z\chi_c^2 S(v) \left(\frac{p^2}{2m_e} \right) \frac{dQ}{Q^2} \left(1 - \frac{Q\beta^2}{Q_{max}} \right). \quad (29)$$

This yields

$$\chi_c^2 \log \chi_\alpha = 4\pi \int_0^t \frac{dt'}{k^2(t')} \sum_i N_i(t') \eta^2(t') \left(\log \chi_{\alpha i}^{el} - \frac{1}{2Z_i} D_i \right) \quad (30)$$

where

$$D = \int_{Q_{min}}^{Q_{max}} \frac{dQ}{Q} \left(S(v) - \frac{Q}{Q_{max}} \right) \left(1 - \frac{Q\beta^2}{Q_{max}} \right) \quad (31)$$

$$= \log(1129\, Z^{-4/3} \beta^2 \gamma^2) - u_{inel} - \tfrac{1}{2}\beta^2 \quad (32)$$

where $\gamma = (1 - \beta^2)^{-1/2}$. For a homogeneous sample with no energy loss, χ_c^2 is again given by equation (17) and $\log \chi_\alpha$ is given by

$$\log \chi_\alpha \sum_i N_i Z_i^2 = \sum_i N_i Z_i^2 \left(\log \chi_{\alpha i}^{el} - \frac{1}{2Z_i} D_i \right). \quad (33)$$

3. Multiple scattering

Rossi and Greisen (1941) give the RMS scattering angle for multiple scattering when the energy loss can be neglected as

$$\theta_{RMS} = (E_s z / p\beta c)(\rho t / X_0)^{1/2} \quad (34)$$

where $E_s = (4\pi/\alpha)^{1/2} m_e c^2$ and X_0 is the radiation length (in units of mass per length^2). The definition of X_0 used by Rossi is

$$X_0 = (1/4\alpha)(A/N_A)[1/Z^2 r_e^2 \log(183\, Z^{-1/3})] \quad (35)$$

and hence

$$\theta_{RMS}^2 (\text{Rossi}) = 4\pi e^4 (zZ/p\beta c)^2 (N_A/A)\, \rho t\, 4 \log(183\, Z^{-1/3}). \quad (36)$$

A more accurate expression for the radiation length has been given by Tsai (1974) in the form

$$X_0 \propto A/[Z^2(L_{rad} - f) + ZL'_{rad}] \quad (37)$$

where L_{rad} is obtained from the atomic form factor, L'_{rad} is obtained from the incoherent scattering function and f is the Coulomb correction (Davies *et al* 1954)

$$f = (\alpha Z)^2\{[1+(\alpha Z)^2]^{-1} + 0.20206 - 0.0369(\alpha Z)^2 + 0.0083(\alpha Z)^4 \ldots\}. \tag{38}$$

From Molière's fit to the Thomas–Fermi potential, Tsai obtained

$$L_{rad} = \log(184.15\, Z^{-1/3}) \tag{39}$$

$$L'_{rad} = \log(1194\, Z^{-2/3}). \tag{40}$$

In Molière's model of multiple scattering the RMS angle of the Gaussian part of the distribution is given by

$$\theta_{RMS} = \chi_c B^{1/2} \tag{41}$$

where (Scott 1963)

$$B = 1.153 + 2.583 \lg (\chi_c/\chi_\alpha)^2. \tag{42}$$

Hence, using equation (17), we have

$$\theta^2_{RMS}(\text{Molière}) = 4\pi e^4 (zZ/\beta pc)^2 (N_A/A)\rho t B. \tag{43}$$

We now follow K M Hanson (1978 and 1979, private communication) and define a correction to the Rossi RMS angle of the form

$$\theta_{RMS}(C) = \theta_{RMS}(\text{Rossi})\ (1+\varepsilon) \tag{44}$$

and setting $\theta_{RMS}(C) \equiv \theta_{RMS}(M)$ we have

$$\varepsilon = \{0.25\, Z^2 B/[Z^2(L_{rad}-f) + Z\, L'_{rad}]\}^{1/2} - 1 \tag{45}$$

and

$$\theta_{RMS}(C) = \left(4\pi e^4 N_A z^2 \frac{\rho t}{A}\frac{4}{(p\beta c)^2}[Z^2(L_{rad}-f) + ZL'_{rad}\right)^{1/2}(1+\varepsilon). \tag{46}$$

Highland (1975) has investigated the angle $\theta_{1/e}$, which is the angle at which the measured distribution falls to $1/e$ of its value at $\theta = 0$, and is given by (Scott 1963)

$$\theta_{1/e} = \chi_c(1.007B - 1.33)^{1/2}. \tag{47}$$

This angle is preferred for examination of experimental data because it is least affected by the unmeasured tail of the distribution, and Highland (1975) suggests that it should be calculated from the formula

$$\theta_{1/e} = (17.5/p\beta c)(\rho t/X_0)^{1/2}(1+\varepsilon') \tag{48}$$

$$\varepsilon' = a' \lg(\rho t/b' X_0) \tag{49}$$

where a' and b' are constants, independent of p and z.

Hungerford *et al* (1972) have derived a correction to θ_{RMS} by applying a conversion factor from the laboratory to the centre-of-mass frame of reference. However, since all integrals in multiple-scattering theory are of the form

$$\int \sin\theta\, d\theta \int W(\theta, t)\, dt$$

and $W(\theta, t) \sin\theta\, d\theta$ is invariant with respect to the frame of reference it is not easy to see the justification for their procedure.

4. Calculations

Our ultimate aim is to calculate the lateral displacement of a beam of charged particles at various depths. This is given by

$$y_{RMS} = 6^{-1/2}\theta_{RMS}t. \tag{50}$$

We first carried out a complete calculation in the Molière (M) model, disregarding energy loss, i.e., calculating χ_c^2 from equation (17), $\chi_{\alpha i}^{el}$ from equation (19) with $R = 1.13$ for all Z, $\log \chi_\alpha$ from equations (32) and (33) with $u_{inel} = -5.8$ for all Z, B from equation (42) and finally $\theta_{RMS}(M)$ from equation (43). Then we calculated ε from equation (45) using L_{rad} and L'_{rad} given by equations (39) and (40). The value of X_0 was calculated from equation (37).

K M Hanson (1979, private communication) has recalculated χ_α^{el} from equation (18) using the atomic form factors tabulated by Hubbell *et al* (1975) and has deduced values of R_i for each Z. He has also recalculated L_{rad}, L'_{rad}, X_0 and u_{inel} using the atomic form factors and incoherent scattering factors tabulated by Hubbell *et al* (1975). These

Table 1. (*a*) Values of ε for incident protons calculated with the Molière model for copper ($Z = 29$, $A = 63.54$) with $R = 1.13$, $X_0 = 12.86$. (*b*) Values of ε calculated with Hanson's parameters $R = 1.287$, $X_0 = 13.02$, $u_{inel} = -5.068$. (The units for X and X_0 are g cm^{-2} ≡ 10 kg m^{-2}.)

(*a*)

	ε				
$\mathcal{X} = \rho t$	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
0.000	−0.4004	−0.4191	−0.4469	−0.4740	−0.4973
0.005	−0.3168	−0.3331	−0.3572	−0.3804	−0.4000
0.020	−0.2522	−0.2671	−0.2890	−0.3098	−0.3274
0.050	−0.2125	−0.2266	−0.2472	−0.2669	−0.2835
0.100	−0.1837	−0.1973	−0.2172	−0.2361	−0.2519
0.200	−0.1558	−0.1690	−0.1882	−0.2064	−0.2217
0.500	−0.1204	−0.1330	−0.1514	−0.1688	−0.1834
1.000	0.0945	−0.1068	−0.1246	−0.1415	−0.1556
2.000	−0.0694	−0.0813	−0.0986	−0.1150	−0.1286
5.000	−0.0371	−0.0486	−0.0653	−0.0811	−0.0943
10.000	−0.0134	−0.0246	−0.0409	−0.0563	−0.0691
20.000	0.0097	−0.0012	−0.0172	−0.0321	−0.0446
50.000	0.0395	0.0289	0.0134	−0.0011	−0.0132

(*b*)

0.001	−0.3995	−0.4209	−0.4511	−0.4785	−0.5034
0.005	−0.3150	−0.3337	−0.3598	−0.3831	−0.4040
0.020	−0.2498	−0.2669	−0.2905	−0.3115	−0.3302
0.050	−0.2097	−0.2259	−0.2482	−0.2679	−0.2855
0.100	−0.1807	−0.1962	−0.2177	−0.2367	−0.2535
0.200	−0.1526	−0.1677	−0.1884	−0.2066	−0.2228
0.500	−0.1169	−0.1313	−0.1511	−0.1686	−0.1840
1.000	−0.0908	−0.1048	−0.1240	−0.1409	−0.1558
2.000	−0.0654	−0.0791	−0.0977	−0.1141	−0.1286
5.000	−0.0329	−0.0461	−0.0641	−0.0799	−0.0938
10.000	−0.0090	−0.0219	−0.0394	−0.0548	−0.0683
20.000	0.0143	0.0017	−0.0154	−0.0304	−0.0436
50.000	0.0443	0.0321	0.0155	0.0010	−0.0118

tabulations are based on the best available atomic wavefunctions and the parameters derived from them should represent an improvement on the values derived by Molière and others from the Thomas–Fermi model, particularly for light elements. We have used Hanson's values of R_i, L_{rad}, L'_{rad} and u_{inel} in a second set of calculations using the same equations as listed above.

The results obtained for ε for incident protons are given in tables 1–3 for copper, calcium and water. It can be seen that the magnitude and variation of ε is such that θ_{RMS} should always be calculated from a formula that takes this correction into account. The discrepancies between the values of ε calculated from Molière's model and those calculated with Hanson's parameters are small and arise mainly from differences in X_0. We have investigated the effect of replacing equation (19) by the approximate expression

$$\chi_a^2 = \chi_0^2 R(1+3.33\eta^2) \tag{51}$$

in the calculation with Hanson's parameters. The effect is negligible for $\beta \geq 0.6$, while

Table 2. (*a*) Values of ε for incident protons calculated with the Molière model for calcium ($Z = 20$, $A = 40.08$) with $R = 1.13$, $X_0 = 16.14$. (*b*) Values of ε calculated with Hanson's parameters $R = 1.223$, $X_0 = 16.24$, $u_{inel} = -5.25$. (The units for X and X_0 are g cm^{-2} ≡ 10 kg m^{-2}.)

(*a*)

	ε				
$X = \rho t$	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
0.001	−0.3789	−0.4088	−0.4437	−0.4734	−0.4970
0.005	−0.3019	−0.3284	−0.3589	−0.3846	−0.4046
0.020	−0.2419	−0.2662	−0.2940	−0.3172	−0.3352
0.050	−0.2047	−0.2279	−0.2542	−0.2761	−0.2931
0.100	−0.1777	−0.2001	−0.2254	−0.2466	−0.2628
0.200	−0.1515	−0.1732	−0.1977	−0.2181	−0.2338
0.500	−0.1181	−0.1390	−0.1625	−0.1820	−0.1969
1.000	−0.0937	−0.1139	−0.1368	−0.1557	−0.1702
2.000	−0.0699	−0.0896	−0.1118	−0.1302	−0.1442
5.000	−0.0393	−0.0584	−0.0799	−0.0976	−0.1111
10.000	−0.0168	−0.0355	−0.0564	−0.0737	−0.0869
20.000	0.0052	−0.0131	−0.0335	−0.0504	−0.0632
50.000	0.0335	0.0158	−0.0041	−0.0204	−0.0329

(*b*)

0.001	−0.3806	−0.4128	−0.4990	−0.4778	−0.5025
0.005	−0.3030	−0.3315	−0.3630	−0.3878	−0.4087
0.020	−0.2426	−0.2687	−0.2974	−0.3197	−0.3386
0.050	−0.2051	−0.2300	−0.2572	−0.2783	−0.2960
0.100	−0.1779	−0.2020	−0.2282	−0.2485	−0.2654
0.200	−0.1516	−0.1749	−0.2002	−0.2197	−0.2361
0.500	−0.1180	−0.1404	−0.1647	−0.1834	−0.1990
1.000	−0.0934	−0.1152	−0.1387	−0.1569	−0.1720
2.000	−0.0695	−0.0907	−0.1136	−0.1312	−0.1458
5.000	−0.0388	−0.0593	−0.0814	−0.0984	−0.1125
10.000	−0.0162	−0.0362	−0.0578	−0.0743	−0.0880
20.000	0.0059	−0.0136	−0.0347	−0.0509	−0.0642
50.000	0.0344	0.0154	−0.0051	−0.0207	−0.0337

Table 3. (*a*) Values of ε for incident protons calculated with the Molière model for water (mean $Z = 7.30$, $A = 18.02$) with $R = 1.13$, $X_0 = 36.70$. (*b*) Values of ε calculated with Hanson's parameters $R = 1.221$, $X_0 = 36.70$, $u_{\text{inel}} = -4.64$. (The units for X and X_0 are g cm$^{-2} \equiv 10$ kg m^{-2}.)

(*a*)

	ε				
$X = \rho t$	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
0.001	−0.3703	−0.4282	−0.4664	−0.4929	−0.5094
0.005	−0.3064	−0.3585	−0.3923	−0.4155	−0.4297
0.020	−0.2557	−0.3041	−0.3351	−0.3562	−0.3691
0.050	−0.2241	−0.2703	−0.2998	−0.3199	−0.3321
0.100	−0.2010	−0.2458	−0.2743	−0.2936	−0.3053
0.200	−0.1785	−0.2220	−0.2496	−0.2683	−0.2796
0.500	−0.1497	−0.1917	−0.2182	−0.2361	−0.2469
1.000	−0.1286	−0.1695	−0.1953	−0.2126	−0.2231
2.000	−0.1079	−0.1478	−0.1730	−0.1899	−0.2001
5.000	−0.0814	−0.1200	−0.1444	−0.1607	−0.1705
10.000	−0.0617	−0.0996	−0.1233	−0.1393	−0.1489
20.000	−0.0425	−0.0796	−0.1028	−0.1184	−0.1277
50.000	−0.0177	−0.0538	−0.0764	−0.0915	−0.1006
(*b*)					
0.001	−0.3739	−0.4349	−0.4741	−0.4970	−0.5153
0.005	−0.3087	−0.3635	−0.3980	−0.4179	−0.4336
0.020	−0.2572	−0.3078	−0.3395	−0.3575	−0.3717
0.050	−0.2250	−0.2734	−0.3035	−0.3205	−0.3340
0.100	−0.2015	−0.2484	−0.2774	−0.2939	−0.3068
0.200	−0.1787	−0.2242	−0.2523	−0.2682	−0.2806
0.500	−0.1494	−0.1933	−0.2203	−0.2355	−0.2474
1.000	−0.1280	−0.1707	−0.1969	−0.2117	−0.2233
2.000	−0.1070	−0.1487	−0.1742	−0.1886	−0.1998
5.000	−0.0801	−0.1205	−0.1452	−0.1590	−0.1698
10.000	−0.0602	−0.0997	−0.1238	−0.1373	−0.1479
20.000	−0.0407	−0.0794	−0.1030	−0.1162	−0.1262
50.000	−0.0156	−0.0533	−0.0761	−0.0889	−0.0989

for small β and large values of $X = \rho t$ the maximum change is ~9%. Since $B \sim 15$ the difference between θ_{RMS} and $\theta_{1/e}$ is small in all cases.

Typical behaviour of ε as a function of $\rho t/X_0$ is shown in figure 1. It is clear that the simple formula (49) for ε could not reproduce this behaviour over the whole range of β and for different absorbing materials.

Figure 2 shows the behaviour of ε for different ions calculated at $\beta = 0.65$ as a function of $X = \rho t$ while figure 3 shows the behaviour of ε as a function of β for a fixed depth $t = 200$ mm in water. It is interesting to note that the correction to the Rossi formula (34) for θ_{RMS} is negligible for heavy ions with mass numbers in the range 12–40. For the same target material ε varies with β/z.

The RMS lateral displacement y_{RMS} of a proton beam as it passes through water has been calculated using equation (50) with θ_{RMS} given by equation (46). Results are given in table 4 and show that inclusion of the correction ε gives an apparent improvement in the lateral resolution of ~10%.

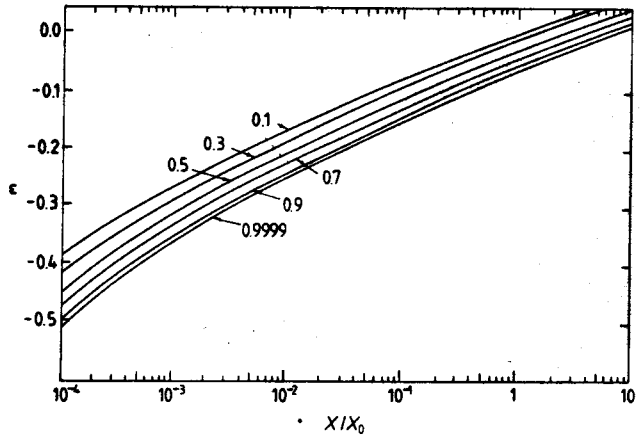

Figure 1. Behaviour of ε for protons passing through copper as a function of X/X_0 where $X = \rho t$ and X_0 is the radiation length. The values of ε correspond to those tabulated in table 1(b). The values of β are given on the curves.

From these results, we conclude that it is necessary to include the correction ε, i.e. to correct the Rossi formula for θ_{RMS}, for protons and α-particles but not for heavy ions. The Molière model is sufficiently accurate for the calculation of ε and θ_{RMS}.

For thick targets energy loss should be taken into account by using equations (16) and (30) instead of equations (17) and (33). Equations (16) and (30) can be rewritten as

$$\chi_c^2 = \int_0^t \mathrm{d}t' \chi_c^2(t') \tag{52}$$

$$\chi_c^2 \log \chi_\alpha^2 = \int_0^t \mathrm{d}t' \chi_c^2(t') \log \chi_\alpha^2(t'). \tag{53}$$

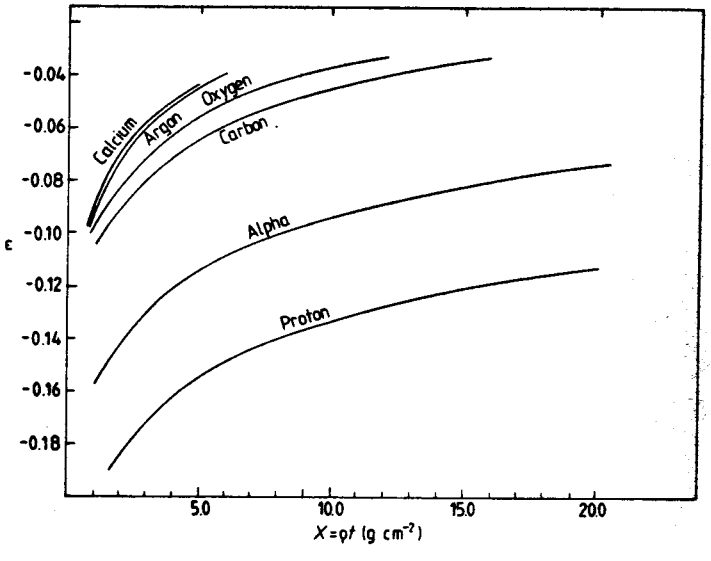

Figure 2. Behaviour of ε as a function of $X = \rho t$ for different ions with $\beta = 0.65$. (The units of X are g cm$^{-2} \equiv 10$ kg m^{-2}.)

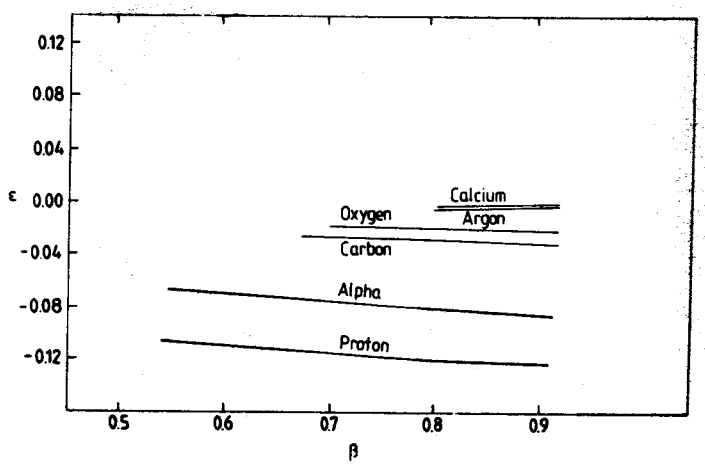

Figure 3. Behaviour of ε as a function of β for different ions at a depth of 200 mm in water.

These equations can be converted to integrals over the kinetic energy T of the form (Berger and Seltzer 1964)

$$\chi_c^2 = \int_T^{T_0} dT' \chi_c^2(T') \left(-\frac{1}{\rho} \frac{dE(T')}{dx} \right)^{-1} \tag{54}$$

$$\chi_c^2 \log \chi_\alpha^2 = \int_T^{T_0} dT' \chi_c^2(T') \log \chi_\alpha^2(T') \left(-\frac{1}{\rho} \frac{dE(T')}{dx} \right)^{-1} \tag{55}$$

where dE/dx is the stopping power, T_0 is the initial kinetic energy and T is the kinetic energy at thickness t. These formulae are reliable when the final energy is not near to zero. This condition is satisfied in all our examples.

Results for a proton beam at various energies passing through various thicknesses in water are given in table 5. Both χ_c and χ_α increase quite substantially compared with the results for no energy loss, leading to an increase in B of only a few per cent and to a decrease in ε of 10–30% depending on the energy and thickness. The large increase in χ_c is reflected in the large increases in the values of θ_{RMS} and y given in table 5 compared

Table 4. The RMS scattering angle and lateral displacement for a proton beam passing through water. The last two columns give the values for $\varepsilon = 0$.

Proton energy (MeV)	β	Depth, t (mm)	ε	θ_{RMS} (rad)	y_{RMS} (mm)	Uncorrected θ_{RMS} (rad)	Uncorrected y_{RMS} (mm)
190	0.555	100	−0.127	0.0274	1.12	0.0314	1.28
		200	−0.101	0.0399	3.26	0.0445	3.63
210	0.568	250	−0.100	0.0404	4.11	0.0449	3.76
240	0.604	300	−0.099	0.0407	4.98	0.0452	5.51
250	0.613	100	−0.132	0.0212	0.86	0.0245	0.99
		200	−0.111	0.0307	2.51	0.0346	2.82
		300	−0.099	0.0382	4.67	0.0424	5.19
350	0.685	100	−0.136	0.0156	0.64	0.0181	0.73
		200	−0.115	0.0220	1.79	0.0256	2.09
		300	−0.103	0.0282	3.44	0.0314	3.84

Table 5. Results for a proton beam passing through water when energy loss is taken into account and using the Molière model.

Proton energy (MeV)	Depth, t (mm)	ε	θ_{RMS} (rad)	y_{RMS} (mm)
190	200	−0.093	0.0684	5.78
210	250	−0.078	0.0754	7.69
240	300	−0.086	0.0661	8.80
250	200	−0.101	0.0398	3.25
	300	−0.087	0.0604	7.40
350	200	−0.107	0.0267	2.17
	300	−0.095	0.0348	4.26

with those given in table 4. Thus for protons passing through thick targets it is essential to take energy loss into account rather accurately. It is also evident from tables 4 and 5 that spatial resolution in proton tomography will be poor for thick targets, as already noted by Hanson (1978).

The uncertainties in the values given in tables 4 and 5 are small. We estimate that, in the energy region of interest, uncertainties due to errors in measured values of stopping power of a single element and to departures from the Bragg rule should be less than 1%. This will have a negligible effect on our calculations. Uncertainties of ~10% in the mean residual energy T lead to a change in y_{RMS} of ~0.5%. Hence, provided that T does not fall below about 10 MeV, the results given by such calculations should be very reliable.

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Resumé

Diffusions multiples selon des petits angles et résolution spatiale dans la tomographie avec des particules chargées.

Les formules, données par la théorie de la diffusion multiple selon des petits angles, pour l'écart type de l'angle de diffusion sont discutées et il en est déduit une correction de la formule standard de Rossi. Les valeurs du facteur correctif sont calculées pour des protons, des particules alpha et des ions lourds: ces résultats montrent que cette correction est importante pour les protons et les particules alpha, mais négligeable pour les ions lourds de nombre de masse compris entre 12 et 40. Nous avons calculé les valeurs de l'écart-type de l'angle de diffusion et de l'écart-type du déplacement latéral pour un faisceau de protons traversant de l'eau. Pour les protons, la correction de la perte d'énergie à travers des cibles épaisses est encore plus importante.

Zusammenfassung

Vielfachstreuung bei kleinen Winkeln und räumliche Auflösung bei der Tomographie mit geladenen Teilchen.

Die Formeln für den quadratischen Mittelwert des Streuwinkels, der durch die Vielfachstreutheorie bei kleinen Winkeln gegeben ist, werden diskutiert und Korrekturen zur Standard-Rossi-Formel werden abgeleitet. Korrekturfaktoren wurden berechnet für Protonen und α-Teilchen und schwere Ionen. Die

Ergebnisse zeigen, daß die Korrekturen für Protonen und α-Teilchen wichtig sind, für schwere Ionen mit Massenzahlen von 12–40 jedoch vernachlässigbar sind. Die Werte für den quadratischen Mittelwert des Streuwinkels und die mittlere seitliche Verschiebung werden berechnet für einen Protonenstrahl im Wasser. Für Protonen ist die Korrektur des Energieverlustes in dicken Targets noch wichtiger.

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