A Few Notes about Statistics

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So, you've taken some data with your counters, and now you want to draw conclusions. A typical situation is that you want to say, "I changed configuration XYZ", and I conclude that the rate changed in such-and-such a way.

Hypothesis Testing

Statistical interpretation of data is most clearly understood in terms of asking specific questions about the data. The simplest question you can ask is whether or not two rates are consistent. If you are actually searching to find a difference in two rates, one way to do this is to check that the "null hypothesis" that the two rates are the same is inconsistent with the data.

Gaussian Probability Distribution

A common probability distribution which is applicable to our case is the "normal" or Gaussian probability distribution.

$$P(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where μ is the "mean"value of the distribution and σ characterizes the width of the distribution. In a Gaussian distribution the RMS (root mean squared) of the distribution, another measure of its width, is equal to σ.

As shown in Figure 1, the fraction of probability between $\pm 2\sigma$ is approximately 95%. Therefore a 2σ discrepancy with a null hypothesis is often taken as evidence of a significant deviation. However, be aware that this standard that under this standard one out of every twenty tests will exhibit this "significant deviation", even if the null hypothesis is correct.
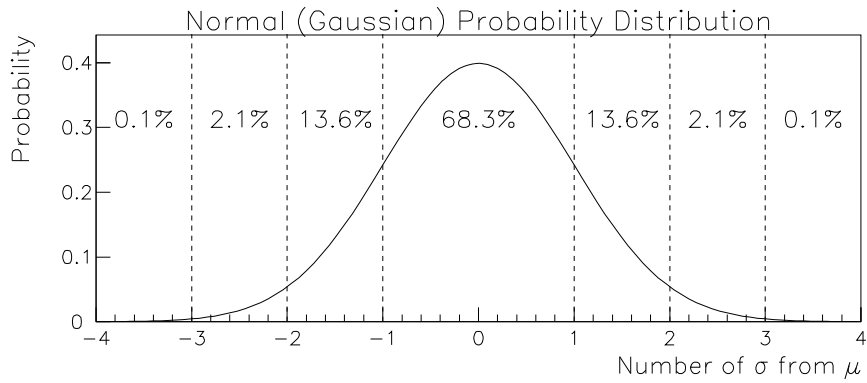
Figure 1: Gaussian (normal) probability distribution. The numbers on the plot show the integral fraction of the probability between the dashed lines.

Observed Events	-2σ	-1σ	$+1\sigma$	$+2\sigma$
1	0.23	0.71	3.3	5.7
2	0.60	1.4	4.6	7.3
3	1.1	2.1	5.9	8.9
5	2.2	3.6	8.4	11.8
7	3.4	5.2	10.8	14.6
10	5.4	7.7	14.3	18.6
15	9.0	12.0	20.0	25.0
25	16.8	20.9	31.1	37.2
50	37.7	43.9	58.1	66.3
7 (Gaussian)	1.7	4.4	9.6	12.3
50 (Gaussian)	35.9	42.9	57.1	64.1

Table 1: The equivalent one and two standard deviation σ points for a given number of observed events. The results from the Gaussian approximation are compared, note that the approximation is very good at $N = 50$ but has 50% errors at two σ for $N = 7$.

Gaussian Uncertainties in Counting Experiments

The types of experiments we do, measuring a rate of discrete events in a time interval, are sometimes referred to as "counting experiments". For a large number of counts, the probability distributions for results of counting experiments are very close to Gaussian, with a mean of N, the measured number of counts, and a σ of $\sqrt{N}$. So, for example, if your paddles measure a rate of 100 counts, the σ would be 10 counts. If you had reason to expect 80 counts, that would be a significant deviation by our 2σ standard above.

When the number of counts is small, the actual probability distribution is not as close to a Gaussian distribution. Table 1 lists the equivalent one and two "sigma" values for a small observed number of events. What is shown is actually the result of integrating the true probability distribution to find the same set integral probabilities for the ranges in Figure 1.

A common desire is to compare the results of two experiments and their uncertainties to determine consistency. The best way to do this is to calculate rates, using errors, and then form a difference of the two rates. The difference in rates and its error are given by

$$
\begin{aligned}
\Delta &= R_1 - R_2, \\
\sigma_\Delta &= \sqrt{\sigma_1^2 + \sigma_2^2}.
\end{aligned}
$$

Examples of Hypothesis Tests in Counting Experiments

1. Two experiments, each run for 10 seconds, are to be compared for consistency. The first run has 100 counts; the second has 121 counts. The rates of the two experiments are

$$\frac{100}{10 \text{ sec}} \pm \frac{\sqrt{100}}{10 \text{ sec}} = 10 \pm 1 \text{ sec}^{-1}$$

and

$$\frac{121}{10 \text{ sec}} \pm \frac{\sqrt{121}}{10 \text{ sec}} = 12.1 \pm 1.1 \text{ sec}^{-1}$$

The difference between these rates is then

$$(12.1 - 10.0) \pm \sqrt{(1.1)^2 + (1.0)^2} \approx 2.1 \pm 1.5 \text{ sec}^{-1}$$

This number exhibits a 1.4σ discrepancy, which is non-zero, but not significantly so according to our 2σ criteria. These results are therefore consistent.

2. The experimenters wish to repeat their experiments above with enough statistics so that if the central values for the two rate measurements were observed again, the discrepancy would be 3σ. How many counts should they accumulate in their measurements?
The errors are proportional to $\sqrt{N}$. In order to take the 1.4σ discrepancy above and make is a 3σ discrepancy, the errors would have to be reduced by $\frac{1.4}{3.0}$. This requires a factor of $\left(\frac{3.0}{1.4}\right)^2 = 4.6$ increase in events. Therefore, the experimenters should try to get about 500 counts per experiment.