TOWARDS A TOPOLOGICAL G_2 STRING

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Abstract We define new topological theories related to sigma models whose target space is a 7 dimensional manifold of G_2 holonomy. We show how to define the topological twist and identify the BRST operator and the physical states. Correlation functions at genus zero are computed and related to Hitchin's topological action for three-forms. We conjecture that one can extend this definition to all genus and construct a seven-dimensional topological string theory. In contrast to the four-dimensional case, it does not seem to compute terms in the low-energy effective action in three dimensions.

1. INTRODUCTION

Topological strings on Calabi-Yau manifolds describe certain solvable sectors of superstrings and as such provide simplified toy models of string theory. There are two inequivalent ways to twist the Calabi-Yau sigma model. This yields topological theories known as the A-model and the

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B-model, which at first sight depend on different degrees of freedom: the A-model apparently only involves the Kähler moduli and the B-model only the complex moduli. However, this changes once branes are included, and it has been conjectured that there is a version of S-duality which maps the A-model to the B-model [1]. Subsequently, several authors found evidence for the existence of seven and/or eight dimensional theories that unify and extend the A and B-model [2, 3, 4, 5, 6]. This was one of our original motivations to take a closer look at string theory on seven-dimensional manifolds of G_2 holonomy, and to see whether it allows for a topological twist, though we were motivated by other issues as well, such as applications to M-theory compactifications on G_2-manifolds, and as a possible tool to improve our understanding of the relation between supersymmetric gauge theories in three and four dimensions.

The outline of this note is as follows. We will first review sigma-models on target spaces of G_2 holonomy, and the structure of chiral the algebra of these theories. The latter is a non-linear extension of the $c = \frac{21}{2}$ $\mathcal{N} = 1$ superconformal algebra that contains an $\mathcal{N} = 1$ subalgebra with $c = \frac{7}{10}$. This describes a minimal model, the tricritical Ising model, which plays a crucial role in the twisting. We then go on to describe the twisting, the BRST operator, the physical states, and we end with a discussion of topological G_2 strings. Here we briefly summarize our findings. A more detailed discussion will appear elsewhere [7].

There is extensive literature about string theory and M-theory compactified on G_2 manifolds. The first detailed study of the world-sheet formulation of strings on G_2 manifolds appeared in [8]. The world-sheet chiral algebra was studied in some detail in [8, 9, 10, 11]. For more about type II strings on G_2 manifolds and their mirror symmetry, see e.g. [12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. A review of M-theory on G_2 manifolds with many references can be found in [22].

2. G_2 SIGMA MODELS

We start from an $\mathcal{N} = (1,1)$ sigma model describing d chiral superfields $X^\mu = \phi^\mu(z) + \theta\psi^\mu(z)$

$$S = \int d^2z \, d^2\theta \, (G_{\mu\nu} + B_{\mu\nu}) D_\theta \mathbf{X}^\mu D_{\bar\theta} \mathbf{X}^\nu. \tag{1.1}$$

The super stress-energy tensor is given by $T(z,\theta) = G(z) + \theta T(z) = -\frac{1}{2} G_{\mu\nu} D_\theta X^\mu \partial_z X^\nu$. This $\mathcal{N} = (1,1)$ sigma model can be formulated on an arbitrary target space. However, the target space theory will have some supersymmetry only when the manifold has special holonomy. This condition ensures the existence of covariantly constant spinors which are used to construct supercharges. The existence of a covariantly constant spinor on the manifold also implies the existence of covariantly constant *p-forms* given by

$$\phi_{(p)} = \epsilon^T \Gamma_{i_1 \ldots i_p} \epsilon \; dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{1.2}$$

This formal expression may be identically zero. The details of the target space manifold dictate which p-forms are actually present. If the manifold has special holonomy $H \subset SO(d)$, the non-vanishing forms (1.2) are precisely the forms that transform trivially under H.

The existence of such covariantly constant p-forms on the target space manifold implies the existence of extra elements in the chiral algebra. For example, given a covariantly constant p form, $\phi_{(p)} = \phi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ satisfying $\nabla \phi_{i_1 \cdots i_p} = 0$, we can construct a holomorphic superfield current given by

$$J_{(p)}(z, \theta) = \phi_{i_1 \cdots i_p} D_\theta X^{i_1} \cdots D_\theta X^{i_p},$$

which satisfies $D_{\bar{\theta}} J_{(p)} = 0$ on shell. In components, this implies the existence of a dimension $\frac{p}{2}$ and a dimension $\frac{p+1}{2}$ current. For example, on a Kahler manifold, the existence of a covariantly constant Kahler two form $\omega = g_{i\bar{j}}(d\phi^i \wedge d\phi^{\bar{j}} - d\phi^{\bar{j}} \wedge d\phi^i)$ implies the existence of a dimension 1 current $J = g_{i\bar{j}} \psi^i \psi^{\bar{j}}$ and a dimension $\frac{3}{2}$ current $G'(z) = g_{i\bar{j}}(\psi^i \partial_z \phi^{\bar{j}} - \psi^{\bar{j}} \partial_z \phi^i)$, which add to the $(1,1)$ superconformal currents $G(z)$ and $T(z)$ to give a $(2,2)$ superconformal algebra.

A generic seven dimensional Riemannian manifold has $SO(7)$ holonomy. A G_2 manifold has holonomy which sits in a G_2 subgroup of $SO(7)$. Under this embedding, the eight dimensional spinor representation **8** of $SO(7)$ decomposes into a **7** and a singlet of G_2, and the latter corresponds to the covariantly constant spinor. The p-form (1.2) is non-trivial only when $p =$ 3, 4. In other words, there is a covariantly constant 3-form $\phi^{(3)} = \phi^{(3)}_{ijk} dx^i \wedge dx^j \wedge dx^k$. The hodge dual 4-form is then also automatically covariantly constant.

By the above discussion, the 3-form implies the existence of a superfield current $J_{(3)}(z, \theta) = \phi^{(3)}_{ijk} D_\theta X^i D_\theta X^j D_\theta X^k \equiv \Phi + \theta K$. Explicitly, Φ is a dimension $\frac{3}{2}$ current $\Phi = \phi^{(3)}_{ijk} \psi^i \psi^j \psi^k$ and K is its dimension 2 superpartner $K = \phi^{(3)}_{ijk} \psi^i \psi^k \partial \phi^k$. Similarly, the 4-from implies the existence of a dimension 2 current X and its dimension $\frac{5}{2}$ superpartner M. The chiral algebra of G_2 sigma models thus contains 4 extra currents on top of the two G, T that constitute the $\mathcal{N} = 1$ superconformal algebra. These six generators form a closed quantum algebra which appears explicitly e.g. in [9, 8, 10] (see also [11]).

An important fact, which will be crucial in almost all the remaining analysis, is that the generators Φ and X form a closed sub-algebra: if we define the supercurrent $G_I = \frac{i}{\sqrt{15}}\Phi$ and stress-energy tensor $T_I = -\frac{1}{5}X$ we recognize that this is the unique $\mathcal{N} = 1$ super-conformal algebra of the minimal model with central charge $c = \frac{7}{10}$ known as the *Tri-critical Ising Model*. This sub-algebra plays a role similar to the $U(1)$ R-symmetry of the $\mathcal{N} = 2$ algebra in compactifications on Calabi-Yau manifolds.

In fact, with respect to the conformal symmetry, the full Virasoro algebra decomposes in two commuting[1] Virasoro algebras: $T = T_I + T_r$ with $T_I(z)T_r(w) = 0$. This means we can classify conformal primaries by two quantum numbers, namely its tri-critical Ising model highest weight and its highest weight with respect to T_r: $|\text{primary}\rangle = |h_I, h_r\rangle$.

Perhaps it is worth emphasizing the logic here: classically, we find a conformal algebra with six generators in sigma-models on manifolds of G_2 holonomy. In the quantum theory we expect, in the absence of anomalies other than the conformal anomaly, to find a quantum version of this classical algebra. In [9] all quantum extensions were analyzed, and a two-parameter family of quantum algebras was found. Requiring that the quantum algebra has the right central charge (necessary to have a critical string theory) and that it contains the tricritical Ising model (necessary for space-time supersymmetry) fixes the two-parameters. This motivates why this is the appropriate definition for string theory on G_2 manifolds.

3. TRI-CRITICAL ISING MODEL

Unitary minimal models are labeled by a positive integer $p = 2, 3, \ldots$ and occur only on the "discrete series" at central charges $c = 1 - \frac{6}{p(p+1)}$. The Tri-Critical Ising Model is the second member ($p = 4$) which has central charge $c = \frac{7}{10}$. It is at the same time also a minimal model for the $\mathcal{N} = 1$ superconformal algebra.

The conformal primaries of unitary minimal models are labeled by two integers $1 \leq n' \leq p$ and $1 \leq n < p$. The weights in this range are arranged into a "*Kac table*". The conformal weight of the primary $\Phi_{n'n}$ is $h_{n'n} = \frac{[pn' - (p+1)n]^2 - 1}{4p(p+1)}$. In the Tri-critical Ising model ($p = 4$) there are 6 primaries of weights $0, \frac{1}{10}, \frac{6}{10}, \frac{3}{2}, \frac{7}{16}, \frac{3}{80}$. Below we write the

[1] This decomposition only works for the Virasoro part of the corresponding $\mathcal{N} = 1$ algebras. The full $\mathcal{N} = 1$ structures do not commute. For example the superpartner of Φ with respect to the full $\mathcal{N} = 1$ algebra is K whereas its superpartner with respect to the $\mathcal{N} = 1$ of the tri-critical Ising model is X.

Kac table for the tricritical Ising model. Beside the Identity operator ($h = 0$) and the $\mathcal{N} = 1$ supercurrent ($h = \frac{3}{2}$) the NS sector (first and third columns) contains a primary of weight $h = \frac{1}{10}$ and its $\mathcal{N} = 1$ superpartner ($h = \frac{6}{10}$). The primaries of weight $\frac{7}{16}, \frac{3}{80}$ are in the Ramond sector (middle column).

$n' \setminus n$	1	2	3
1	$\mathbf{0}$	$\mathbf{\frac{7}{16}}$	$\frac{3}{2}$
2	$\mathbf{\frac{1}{10}}$	$\mathbf{\frac{3}{80}}$	$\frac{6}{10}$
3	$\mathbf{\frac{6}{10}}$	$\frac{3}{80}$	$\frac{1}{10}$
4	$\mathbf{\frac{3}{2}}$	$\frac{7}{16}$	0

The Hilbert space of the theory decomposes in a similar way, $\mathcal{H} = \oplus_{n,n'}\mathcal{H}_{n',n}$. A central theme in this work relies on the fact that since the primaries $\Phi_{n'n}$ form a closed algebra under the OPE they can be decomposed into *conformal blocks* which connect two Hilbert spaces. Conformal blocks are denoted by $\Phi^{l',l}_{n',n,m'm}$ which describes the restriction of $\Phi_{n',n}$ to a map that only acts from $\mathcal{H}_{m',m}$ to $\mathcal{H}_{l',l}$.

An illustrative example, which will prove crucial in what follows, is the block structure of the primary $\Phi_{2,1}$ of weight 1/10. General arguments show that the fusion rule of this field with any other primary $\Phi_{n'n}$ is $\phi_{(2,1)} \times \phi_{(n',n)} = \phi_{(n'-1,n)} + \phi_{(n'+1,n)}$. The only non-vanishing conformal blocks in the decomposition of $\Phi_{2,1}$ are those that connect a primary with the primary right above it and the primary right below in the Kac table, namely[2], $\phi^{n'-1,n}_{2,1,n',n}$ and $\phi^{n'+1,n}_{2,1,n',n}$. This can be summarized formally by defining the following decomposition[3]

$$\Phi_{2,1} = \Phi^{\downarrow}_{2,1} \oplus \Phi^{\uparrow}_{2,1}. \tag{1.3}$$

Similarly, the fusion rule of the Ramond field $\Phi_{1,2}$ with any primary is $\phi_{(1,2)} \times \phi_{(n',n)} = \phi_{(n,n-1)} + \phi_{(n',n+1)}$ showing that it is composed of two blocks, which we denote as follows $\Phi_{1,2} = \Phi^{-}_{1,2} \oplus \Phi^{+}_{1,2}$. Conformal blocks transforms under conformal transformations exactly like the primary field they reside in but are usually not single-valued functions of $z(\bar{z})$.

[2]Note the confusing notation where *down* the Kac table means *larger* n' and vice-versa.
[3]We stress that this decomposition is special to the field $\Phi_{2,1}$ and does not necessarily hold for other primaries which may contain other blocks.

3.1 CHIRAL PRIMARY STATES

The chiral-algebra associated with manifolds of G_2 holonomy[4] allows us to draw several conclusions about the possible spectrum of such theories. It is useful to decompose the generators of the chiral algebra in terms of primaries of the tri-critical Ising model and primaries of the remainder. The commutation relations of the G_2 algebra imply that the some of the generators of the chiral algebra decompose as [8]: $G(z) = \Phi_{2,1} \otimes \psi_{\frac{14}{10}}$, $K(z) = \Phi_{3,1} \otimes \psi_{\frac{14}{10}}$ and $M(z) = a\Phi_{2,1} \otimes \chi_{\frac{24}{10}} + b[X_{-1}, \Phi_{2,1}] \otimes \psi_{\frac{14}{10}}$, with ψ, χ primaries of the indicated weights in the T_r CFT and a, b constants.

Ramond ground states of the full $c = \frac{21}{2}$ SCFT are of the form $|\frac{7}{16}, 0\rangle$ and $|\frac{3}{80}, \frac{2}{5}\rangle$. The existence of the $|\frac{7}{16}, 0\rangle$ state living just inside the tricritical Ising model plays a crucial role in the topological twist. Coupling left and right movers, the only possible RR ground states compatible with the G_2 chiral algebra[5] are a single $|\frac{7}{16}, 0\rangle_L \otimes |\frac{7}{16}, 0\rangle_R$ ground state and a certain number of states of the form $|\frac{3}{80}, \frac{2}{5}\rangle_L \otimes |\frac{3}{80}, \frac{2}{5}\rangle_R$. By studying operator product expansions of the RR ground states we get the following "special" NSNS states $|0, 0\rangle_L \otimes |0, 0\rangle_R$, $|\frac{1}{10}, \frac{2}{5}\rangle_L \otimes |\frac{1}{10}, \frac{2}{5}\rangle_R$, $|\frac{6}{10}, \frac{2}{5}\rangle_L \otimes |\frac{6}{10}, \frac{2}{5}\rangle_R$ and $|\frac{3}{2}, 0\rangle_L \otimes |\frac{3}{2}, 0\rangle_R$ corresponding to the 4 NS primaries $\Phi_{n',1}$ with $n' = 1, 2, 3, 4$ in the tri-critical Ising model. Note that for these four states there is a linear relation between the Kac label n' of the tri-critical Ising model part and the total conformal weight $h_{total} = \frac{n'-1}{2}$. In fact, it can be shown that, similar to the BPS bound in the $\mathcal{N} = 2$ case, primaries of the G_2 chiral algebra satisfy a (non-linear) bound of the form

$$h_I + h_r \geq \frac{1 + \sqrt{1 + 80h_I}}{8}. \tag{1.4}$$

which is precisely saturated for the four NS states listed above. We will therefore refer to those states as "chiral primary" states. Just like in the case of Calabi-Yau, the $\frac{7}{16}$ field maps Ramond ground states to NS chiral primaries and is thus an analogue of the "spectral flow" operators in Calabi-Yau.

4. TOPOLOGICAL TWIST

To construct a topologically twisted CFT we usually proceed in two steps. First we define a new stress-energy tensor, which changes the

[4]We loosely refer to it as "the G_2 algebra" but it should not be confused with the Lie algebra of the group G_2.
[5]Otherwise the spectrum will contain a 1-form which will enhance the chiral algebra. Geometrically this is equivalent to demanding that $b_1 = 0$.

quantum numbers of the fields and operators of the theory under Lorentz transformations. Secondly, we identify a nilpotent scalar operator, usually constructed out of the supersymmetry generators of the original theory, which we declare to be the BRST operator. Often this BRST operator can be obtained in the usual way by gauge fixing a suitable symmetry. If the new stress tensor is exact with respect to the BRST operator, observables (which are elements of the BRST cohomology) are metric independent and the theory is called topological. In particular, the twisted stress tensor should have a vanishing central charge.

In practice [23, 24], for the $\mathcal{N} = 2$ theories, an n-point correlator on the sphere in the twisted theory can conveniently be defined[6] as a correlator in the *untwisted* theory of the same n operators plus two insertions of a spin-field, related to the space-time supersymmetry charge, that serves to trivialize the spin bundle. For a Calabi-Yau 3-fold target space there are two $SU(3)$ invariant spin-fields which are the two spectral flow operators $\mathcal{U}_{\pm\frac{1}{2}}$. This discrete choice in the left and the right moving sectors is the choice between the $+(-)$ twists [25] which results in the difference between the topological A/B models.

In [8] a similar expression was written down for sigma models on G_2 manifolds, this time involving the single G_2 invariant spin field which is the unique primary $\Phi_{1,2}$ of weight $\frac{7}{16}$. It was proposed that this expression could be a suitable definition of the correlation functions of a putative topologically twisted G_2 theory. In other words the twisted amplitudes are defined as[7]

$$\langle V_1(z_1)\dots V_n(z_n)\rangle_{\texttt{twist}} \equiv \langle \Sigma(\infty)V_1(z_1)\dots V_n(z_n)\Sigma(0)\rangle_{\texttt{untwist}}. \quad (1.5)$$

In [8] further arguments were given, using the Coulomb gas representation of the minimal model, that there exists a twisted stress tensor with vanishing central charge. This argument is however problematic, since the twisted stress tensor proposed there does not commute with Felder's BRST operators [26] and therefore it does define a bona fide operator in the minimal model. In addition, a precise definition of a BRST operator was lacking.

We will proceed somewhat differently. We will first propose a BRST operator, study its cohomology, and then use a version of (1.5) to compute correlation functions of BRST invariant observables. We will then comment on the extension to higher genus and on the existence of a topologically twisted G_2 string.

[6] Up to proper normalization.
[7] Up to a coordinate dependent factor that we omit here for brevity and can be found in [7].

5. THE BRST OPERATOR

Our basic idea is that the topological theory for G_2 sigma models should be formulated not in terms of local operators of the untwisted theory but in terms of its (non-local)[8] conformal blocks.

By using the decomposition (1.3) into conformal blocks, we can split any field whose tri-critical Ising model part contains just the conformal family $\Phi_{2,1}$ into its *up* and *down* parts. In particular, the $\mathcal{N}=1$ supercurrent $G(z)$ can be split as

$$G(z) = G^{\downarrow}(z) + G^{\uparrow}(z). \tag{1.6}$$

We claim that $G^{\downarrow}$ is the BRST current and $G^{\uparrow}$ carries many features of an anti-ghost.

The basic $\mathcal{N}=1$ relation

$$G(z)G(0) = \Big(G^{\downarrow}(z) + G^{\uparrow}(z)\Big)\Big(G^{\downarrow}(0) + G^{\uparrow}(0)\Big) \sim \frac{2c/3}{z^3} + \frac{2T(0)}{z} \tag{1.7}$$

proves the nilpotency of this BRST current (and of the candidate anti-ghost) because the RHS contains descendants of the identity operator only and has trivial fusion rules with the primary fields of the tri-critical Ising model and so $(G^{\downarrow})^2 = (G^{\uparrow})^2 = 0$.

More formally, denote by $P_{n'}$ the projection operator on the subspace $\mathcal{H}_{n'}$ of states whose tri-critical Ising model part lies within the conformal family of one of the four NS primaries $\Phi_{n',1}$. The 4 projectors add to the identity

$$P_1 + P_2 + P_3 + P_4 = 1, \tag{1.8}$$

because this exhaust the list of possible highest weights in the NS sector of the tri-critical Ising model[9]. We can now define our BRST operator in the NS sector more rigorously

$$Q = G^{\downarrow}_{-\frac{1}{2}} \equiv \sum_{n'} P_{n'+1} G_{-\frac{1}{2}} P_{n'}. \tag{1.9}$$

The nilpotency $Q^2 = 0$ is easily proved

$$Q^2 = \sum_{n'} P_{n'+2} G^2_{-\frac{1}{2}} P_{n'} = \sum_{n'} P_{n'+2} L_{-1} P_{n'} = 0, \tag{1.10}$$

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[8] It should be stressed that this splitting into conformal blocks is non-local in the simple sense that conformal blocks may be multi-valued functions of $z(\bar{z})$.

[9] For simplicity, we will set $P_{n'} = 0$ for $n' \leq 0$ and $n' \geq 5$, so that we can simply write $\sum_{n'} P_{n'} = 1$ instead of (1.8).

where we could replace the intermediate $P_{n'+1}$ by the identity because of property 1.6 and the last equality follows since L_{-1} maps each $\mathcal{H}_{n'}$ to itself.

Q does not commute with the local operator $\mathcal{O}_{\Delta(1),0}$, $\mathcal{O}_{\Delta(2),\frac{2}{5}}$, $\mathcal{O}_{\Delta(3),\frac{2}{5}}$ and $\mathcal{O}_{\Delta(4),0}$ corresponding to the chiral states $|0,0\rangle, |\frac{1}{10},\frac{2}{5}\rangle, |\frac{6}{10},\frac{2}{5}\rangle$ and $|\frac{3}{2},0\rangle$ (for brevity we will denote those 4 local operators juts by their minimal model Kac label $\mathcal{O}_i,\ i=1,2,3,4$). However, it can be checked that the following blocks,

$$\mathcal{A}_{n'} = \sum_m P_{n'+m-1}\mathcal{O}_{n'}P_m, \tag{1.11}$$

which pick out the maximal "down component" of the corresponding local operator, do commute with Q and are thus in its operator cohomology. Thus the chiral *operators* of the twisted model are represented in terms of the blocks 1.11 of the local operators corresponding to the chiral *states*. Furthermore, it can be shown easily that those chiral operator form a ring under the OPE.

By doing a calculation at large volume, we see that the BRST cohomology has one operator of type $\mathcal{O}_1\otimes\mathcal{O}_1$, b_2+b_3 of type $\mathcal{O}_2\otimes\mathcal{O}_2$, b_4+b_5 of type $\mathcal{O}_3\otimes\mathcal{O}_3$, and one of type $\mathcal{O}_4\otimes\mathcal{O}_4$. The total BRST cohomology is thus precisely given by $H^*(M)$. Also, one finds that the b_3 operators of type $\mathcal{O}_2\otimes\mathcal{O}_2$ are precisely the geometric moduli of the G_2 target space[10].

In the topological G_2 theory, genus zero correlation functions of chiral primaries between BRST closed states are position independent. Indeed, the generator of translations on the plane, namely L_{-1}, is BRST exact

$$\{G^{\downarrow}_{-\frac{1}{2}}, G^{\uparrow}_{-\frac{1}{2}}\} = \sum_{n'} P_{n'}G_{-\frac{1}{2}}\left(P_{n'-1}+P_{n'+1}\right)G_{-\frac{1}{2}}P_{n'} = \sum_{n'} P_{n'}L_{-1}P_{n'} = L_{-1}. \tag{1.12}$$

This is a crucial ingredient of topological theories.

Moreover, it can be shown [8] that the upper components $\tilde{G}_{-\frac{1}{2}}|\frac{1}{10},\frac{2}{5}\rangle_L\otimes G_{-\frac{1}{2}}|\frac{1}{10},\frac{2}{5}\rangle_R$ correspond to exactly marginal deformations of the CFT preserving the G_2 chiral algebra, completely in agreement with the identification of them as the geometric moduli of the theory. Focusing momentarily on the left movers, we can show that $[Q,\{G_{-\frac{1}{2}},\mathcal{O}_2\}] = \partial\mathcal{A}_2$ [11] so that the very same deformation is physical, namely Q exact, also in

[10] In this work we ignore the b_2 moduli corresponding to the B-field.
[11] Recall that $\mathcal{A}_2$ was defined in (1.11).

the topological theory. Note that the deformation is given by a conventional operator that does not involve any projectors. Combining with the right-movers, we find that the deformations in the action of the topological string are exactly the same as the deformations of the non-topological string as is expected because both should exist on an arbitrary manifold of G_2 holonomy.

Most correlation functions at genus zero vanish. The most interesting one is the three-point function of three operators $Y = \mathcal{O}_2 \otimes \mathcal{O}_2$. These correspond to geometric moduli. If we introduce coordinates t_i on the moduli space of G_2 metrics, then we obtain from a large volume calculation

$$\langle Y_i Y_j Y_k \rangle = \int_M d^7 x \sqrt{g} \phi_{abc} \frac{\partial g^{aa'}}{\partial t_i} \frac{\partial g^{bb'}}{\partial t_j} \frac{\partial g^{cc'}}{\partial t_k} \phi_{a'b'c'}. \tag{1.13}$$

One might expect, based on general arguments, that this is the third derivative of some prepotential if suitable 'flat' coordinates are used. We do not know the precise definition of flat coordinates for the moduli space of G_2 metrics, but if we take for example $M = T^7$ and take coordinates such that ϕ is linear in them, then we can verify

$$\langle Y_i Y_j Y_k \rangle = -\frac{1}{21} \frac{\partial^3}{\partial t_i \partial t_j \partial t_k} \int \phi \wedge *\phi. \tag{1.14}$$

The prepotential appearing on the right hand side is exactly the same as the action functional introduced by Hitchin in [27, 28]. A similar action was also used as a starting point for topological M-theory in [3] (see also [2]). This strongly suggests that our topological G_2 field theory is somehow related to topological M-theory.

6. TOPOLOGICAL G_2 STRINGS

In the case of $N = 2$ theories, the computation of correlation functions at genus zero outlined above can be generalized to higher genera [23, 24]. An n-point correlator on a genus-g Riemann surface in the twisted theory can be defined as a correlator in the untwisted theory of the same n operators plus $(2-2g)$ insertions of the spin-field that is related to the space-time supersymmetry charge. For a Calabi-Yau 3-fold target space on a Riemann surface with $g > 1$, the meaning of the above prescription is to insert $2g-2$ of the conjugate spectral flow operator.

To generalize this to the G_2 situation, we would like to have something similar. However, there is only one G_2 invariant spin-field. This is where the decomposition in conformal blocks given in section 3. is useful: the spin-field $\Phi_{2,1}$ could be decomposed[12] in a block $\Phi^+_{2,1}$ and in a block $\Phi^-_{2,1}$. At genus zero we needed two insertions of $\Phi^+_{2,1}$, so the natural guess is that at genus g we need $2g-2$ insertions of $\Phi^-_{2,1}$. However, this is not the full story. We also need to insert $3g-3$ copies of the anti-ghost and integrate over the moduli space of Riemann surfaces to properly define a topological string theory. The anti-ghost is very close to $G^\uparrow$, and the fusion rules of the tri-critical Ising model tell us that there is indeed a non-vanishing contribution to correlation functions of $2g-2$ $\Phi^-_{2,1}$'s and $3g-3$ $G^\uparrow$.

This prescription would therefore work very nicely if we would have found the right anti-ghost. The candidates we tried so far all seem to fail in one way or another. One possible conclusion might be that a twisted stress tensor does not exist and that there is only a sensible notion of topological G_2 sigma models but not of topological G_2 strings. The fact that so far so many properties of the $\mathcal{N}=2$ topological theories appeared to hold also in our G_2 model leads us to believe that a sensible extension to higher genera indeed exists. Identifying the correct twisted stress tensor remains an open problem. Barring this important omission the coupling to topological gravity goes pretty much along the same lines as for the $\mathcal{N}=2$ topological string (details can be found in [7]).

7. CONCLUSIONS

An important application of topological strings stems from the realization [23, 24, 25] that its amplitudes agree with certain amplitudes of the physical superstring. Just like $\mathcal{N}=2$ topological strings compute certain F-terms in four dimensional $\mathcal{N}=2$ gauge theories, one might wonder whether the G_2 topological string similarly computes F-terms in three dimensional $\mathcal{N}=2$ gauge theories.

Since G_2 manifolds are Ricci-flat we can consider compactifying the type II superstring on $R^{1,2}\times\mathcal{N}_7$ where $\mathcal{N}_7$ is a 7 dimensional manifold of G_2 holonomy. This reduces the supersymmetry down to two real supercharges in 3 dimension from each worldsheet chirality so we end up with a low energy field theory in 3 dimensions with $\mathcal{N}=2$ supergravity. By studying amplitudes in some detail we observe that, except perhaps at genus zero, the amplitudes do involve a sum over conformal blocks, and the topological G_2 string therefore seems to compute only one of many contributions to an amplitude. This is in contrast to the four

[12] In terms of the Coulomb gas representation, one of these can be represented as an ordinary vertex operator, the other one involves a screening charge.

dimensional case where the possibility to look only at the (anti)-self dual gravitons allowed to isolate the topological contributions.

Nevertheless, we believe that topological G_2 strings are worthwhile to study. They may possibly provide a good definition of topological M-theory, and a further study may teach us many things about non-topological G_2 compactifications as well. We leave this, as well as the generalization to $spin(7)$ compactifications [29] and the study of branes in these theories to future work.

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