Physics/Astronomy 226, 2015, Final Exam

Rules: You've won a 2-day/3 night vacation in the fabulous land of General Relativity. Your trip began at 3:30 PM on Friday 3/13, and will end **Monday, 3/16, at 11:30 am**. Please give your completed exam to me by that time (I'll be in the office from 11 on; scanning and emailing is also acceptable.) I've made every effort to keep ambiguity out of the questions, but if some remains for you, I will readily remove it; please email (aguirre@scipp) or feel free to call (x92449 W, 325-6832 C) during reasonable hours. Feel free to consult whatever *non-computer* (and non-human!) sources you like[1], but your solutions should be your own.

Good luck!

1. **True/False section.** (30 Points; 2 each) For each of the following statements, say whether the statement is true or false. If false, provide a true statement that entails fairly minimal modification of the false one. (Note that most of these don't require any real calculation; a couple do, and are labeled by (*).)

   (a) Let V be a vector. Then under a Lorentz transform, V transforms into a new vector V'.

   (b) (*)Given that Kerr-Newman black holes are completely defined by mass M, charge Q and spin a, it's plausible that elementary particles, for example the electron, are secretly just teeny-tiny black holes with itsy-bitsy event horizons.

   (c) $\delta_{\mu\nu}$, which is one when $\mu = \nu$ and zero otherwise, is a good covariant tensor.

   (d) The commutator $[X, Y]$ of two vector fields X and Y has components $2X_{[\mu}Y_{\nu]}$.

   (e) Given an arbitrary metric $g_{\mu\nu}$, by a careful coordinate choice we can make $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu} = 0$ in a region R of finite size, but $\partial_\alpha\partial_\beta g_{\mu\nu}$ can be set to 0 only at a point.

   (f) For a closed achronal set S, $D^+(S) \cap D^-(S)$ is empty.

   (g) If a manifold contains a closed timelike curve, it cannot have a Cauchy surface.

   (h) Using Carroll's conventions, $\epsilon^{1023} = -1$ always.

   (i) In a Lorentzian spacetime, the path between points A and B is a geodesic either if the tangent vector is parallel transported by the Christoffel connection, or equivalently if the path minimizes the proper time between A and B.

   (j) (*) If the energy-momentum tensor for some scalar field is

   $$T_{\mu\nu} = \frac{1}{4\pi}\left[(\partial_\mu\phi)(\partial_\nu)\phi) - \frac{1}{2}g_{\mu\nu}(\partial_\alpha\phi)(\partial^\alpha\phi)\right]$$

   then the covariant conservation of $T_{\mu\nu}$ requires that its equation of motion be

   $$\Box\phi = 0.$$

[1] PDF documents already on your computer are OK

(k) If $R_{\mu\nu} = 0$ and $C_{\mu\nu\alpha\beta} = 0$ in some region (where the latter is the conformal tensor), then that region must be locally Minkowski space if the space has Lorentzian signature.

(l) Gravity will always cause two test particles to accelerate toward each other provided that the Null Energy Condition holds.

(m) In the system of units we are using, my mass is approximately (to order-of-magnitude) 10^{-13} Angstroms.

(n) If the geodesic worldline of a particle ends some finite proper time in the future or past, then the spacetime has a curvature singularity (i.e. $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \to \infty$.)

(o) For a massive particle with angular momentum $L > 0$ orbiting a point mass, the corrections dues to general relativity always make the radius of a stable circular orbit *smaller* than it would be in Newtonian gravity (or fail to exist).

2. (20 points) Consider a thin beam of light traveling in the $\hat{x}$-directioni n a nearly-flat background (i.e. in consider it in the weak-field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$).

   (a) Using just the 'physical' definition of the energy-momentum tensor $T_{\mu\nu}$ (fluxes of momenta in directions), argue that all components of $T_{\mu\nu}$ are zero except for T_{xx}, T_{00}, and $T_{0x} = T_{x0}$. Find a relation between these three quantities.

   (b) Consider a second thin beam of light initially running parallel to the first. Using the geodesic equation write down an expression for $\frac{d^2z}{d\lambda^2}$ in terms of the components of the metric perturbation $h_{\mu\nu}$. (λ is an affine parameter.)

   (c) In class (or Carroll), by a choice of variables and gauge we reduced the linearized Einstein's equations to $\Box\bar{h}_{\mu\nu} = 16\pi G T_{\mu\nu}$, and solved this in terms of Green's functions. Use this, along with parts (a) and (b), to show that $\frac{d^2z}{d\lambda^2}$ vanishes (and similarly $\frac{d^2y}{d\lambda^2}$).

   This shows that the second beam remains parallel to the first, thus two parallel beams of light do not attract each other in linear gravity. In fact it can be shown that this is true in general, outside of the weak field limit.

3. (25 points) The equation $z = x^2 + y^2$ defines a paraboloid. The surface of this paraboloid is a 2-dimensional curved space, describable by the coordinates $\rho = (x^2 + y^2)^{1/2}$ and ϕ (the angle about the z-axis). Let the metric $g_{\mu\nu}$ on the paraboloid be that inherited from $\mathcal{R}^3$ (which has metric $ds^2 = dx^2 + dy^2 + dz^2$).

   (a) What are the components of $g_{\mu\nu}$?

   (b) Find all non-vanishing Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ by differentiating the metric.

   (c) Compute the geodesic equations for ρ and ϕ by varying the action of $\int ds$ (or an equivalent one), or by using the Euler-Lagrange equations. Read off the Christoffel symbols and compare with the results of part (a).

   (d) Find all nonzero component of the Riemann tensor $R_{\alpha\beta\mu\nu}$.

4. (25 points) The Reissner-Nordstrom solution for a black hole of mass M and electric charge q is specified by metric

$$ds^2 = -\Delta(r)\, dt^2 + \Delta(r)^{-1} dr^2 + r^2 d\Omega^2,$$

with

$$\Delta(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},$$

and $A_v = (Q/r, 0, 0, 0)$. There is an outer horizon at

$$r_+ = M + \sqrt{M^2 - Q^2}.$$

(a) The motion of a test particle of mass μ and charge q is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} + \epsilon A_\alpha x^\alpha,$$

with $\epsilon = q/\mu$. Given that this does not depend on t or ϕ, derive two conserved quantities E and L corresponding to the particle's energy and angular momentum as measured at infinity.

(b) Confining trajectories to $\theta = \pi/2$, use these conserved quantities (plus $U \cdot U = -1$) to derive an equation of motion for the radius of the form

$$\left(\frac{dr}{d\tau} \right)^2 + V(r) = E^2/\mu^2,$$

and give an expression for $V(r; q, L, M, Q)$.

(c) Consider throwing an arbitrary charged particle of energy E and charge q into the RN black hole. Let $B(M, Q) = M^2 - Q^2$; if B goes negative then there is a naked singularity. Assuming that E and q correspond to changes in the black hole's mass and charge, write the change δB in terms of M, Q, E and q. This seems to suggest that we can just throw stuff in with lots of q and not much E and achieve a naked singularity. But...

(d) When our particle crosses r_+ it must have $dr/d\tau < 0$. What constraint does this place on E in term of q, M and Q?

(e) Use this to show that no sequence of throwing such objects into the BH can ever achieve $B = 0$, thus giving evidence for the 3rd law of black-hole thermodynamics (and cosmic censorship).