Physics/Astronomy 226, Problem set 4, Due 2/10

Reading: Carroll, Ch. 3

1. Derive the explicit expression for the components of the commutator (a.k.a. Lie bracket):

$$[X,Y]^u = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu.$$

2. Write down polar coordinates $x^{i'} = (r, \theta)$ and cartesian coordinates $x^i = x, y$ in terms of each other.

   (a) Write the cartesian ∂_i and $dx^i = (d\hat{x}, d\hat{y})$ basis vectors in terms of the polar basis vectors $\partial_{i'}$ and 1-forms $dx^{i'} = (d\hat{r}, d\hat{\theta})$.

   (b) Consider the tensor

$$T = y^2 d\hat{x} \otimes d\hat{x} + d\hat{y} \otimes d\hat{y}.$$

   Write this tensor in terms of the polar 1-form basis. Do this first by using the transformations of the 1-forms computed in part (a), then by explicitly transforming the components of the tensor, and check that your results agree.

3. Although there is much beauty in considering spacetime as a single entity, it is sometimes useful to break it into space and time separately, and in particular to break the metric $g_{\mu\nu}$ into (g_{00}, g_{0i} and g_{ij}). Two common ways to do this (1+3) splitting are as follows. (Assume below that γ^{ij} is the inverse of γ_{ij}, and similar for the 'hat' version.)

   (a) In the first, we can write the metric as

$$ds^2 = -M^2(dt - M_i dx^i)^2 + \gamma_{ij} dx^i dx^j.$$

   Show that in this case the metric components are given by

$$g_{00} = -M^2; \quad g_{0i} = M^2 M_i; \quad g_{ij} = \gamma_{ij} - M^2 M_i M_j$$

$$g^{00} = -(M^{-2} - M_i M^i); \quad g^{0i} = M^i; \quad g^{ij} = \gamma^{ij}.$$

   (b) In the second, we can write the metric as

$$ds^2 = -N^2 dt^2 + \hat{\gamma}_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$

   Show that in this case,

$$g_{00} = -(N^2 - N^i N_i); \quad g_{0i} = N_i; \quad g_{ij} = \hat{\gamma}_{ij}$$

$$g^{00} = -N^{-2}; \quad g^{0i} = N^{-2} N^i; \quad g^{ij} = \hat{\gamma}^{ij} - N^{-2} N^i N^j.$$

(c) The first procedure is sometimes called 'threading the spacetime' and the second is sometimes called 'slicing the spacetime'. Comment on the appropriateness of these terms.

4. Each point inside the forward lightcone of the origin (i.e. $-t^2 + r^2 < 0$ in spherical coordinates) in Minkowski space lies on some Lorentz hyperboloid of the form:

$$-t^2 + r^2 = -a^2$$

for some value of a. Such points can be labeled using a as a time coordinate and (χ, θ, ϕ) as spatial coordinates related to the Minkowski spherical coordiates by $t = a\cosh\chi$ and $r = a\sinh\chi$. Find the metric of flat spacetime in these new coordinates. Sketch a family of spacelike surfaces in a (t, r) spacetime diagram.

5. A guy walks up to you on the street and wants to sell you a 3-dimensional space with coordinates x, y, and z and metric

$$ds^2 = dx^2 + dy^2 + dz^2 - \left(\frac{3}{13}dx + \frac{4}{13}dy + \frac{12}{13}dz\right)^2.$$

Show that this guy is a hustler by demonstrating that this is really a 2-dimensional space, and find two new coordinates Z and W for which the metric takes the form:

$$ds^2 = dZ^2 + dW^2,$$

i.e. it's just a plain old plane. (Hint: think about the volume element.)