Physics/Astronomy 226, Problem set 4, Due 2/10
Solutions

Reading: Carroll, Ch. 3

1. Derive the explicit expression for the components of the commutator (a.k.a. Lie bracket):

$$[X, Y]^u = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu.$$

Solution:

Our vectors consist of components and basis vectors:

$$X = X^\lambda \partial_\lambda.$$

We begin by breaking the commutator $[X, Y]$ into components and acting it upon a test function f.

$$\begin{aligned}
([X, Y]f)^\mu &= (X[Y(f)])^\mu - (Y[X(f)])^\mu \\
&= X^\lambda \partial_\lambda Y^\mu \partial_\mu f - Y^\lambda \partial_\lambda X^\mu \partial_\mu f \\
&= (X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu)\partial_\mu f.
\end{aligned}$$

The in the third step, the $\partial_\lambda \partial_\mu f$ terms cancel. Removing the test function, we are left with

$$([X, Y]f)^\mu = (X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu)$$

2. Write down polar coordinates $x^{i'} = (r, \theta)$ and cartesian coordinates $x^i = x, y$ in terms of each other.

   (a) Write the cartesian ∂_i and $dx^i = (d\hat{x}, d\hat{y})$ basis vectors in terms of the polar basis vectors $\partial_{i'}$ and 1-forms $dx^{i'} = (d\hat{r}, d\hat{\theta})$.

   (b) Consider the tensor

$$T = y^2 d\hat{x} \otimes d\hat{x} + d\hat{y} \otimes d\hat{y}.$$

   Write this tensor in terms of the polar 1-form basis. Do this first by using the transformations of the 1-forms computed in part (a), then by explicitly transforming the components of the tensor, and check that your results agree.

Solution:

(a) Our transformations are

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\frac{y}{x} \end{aligned}$$

The transformation law for a basis one-form is

$$\begin{aligned} dr &= \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy \\ &= \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy \end{aligned}$$

Similarly, for θ we find

$$\begin{aligned} d\theta &= \frac{dx}{1+(\frac{y}{x})^2}\frac{-y}{x^2} + \frac{dy}{1+(\frac{y}{x})^2}\frac{1}{x} \\ &= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \end{aligned}$$

Now for the basis vectors, we have the general transformation equation

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial r} &= \cos(\arctan\frac{y}{x})\frac{\partial}{\partial x} + \sin(\arctan\frac{y}{x})\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= -\sqrt{x^2+y^2}\sin(\arctan\frac{y}{x})\frac{\partial}{\partial x} + \sqrt{x^2+y^2}\cos(\arctan\frac{y}{x})\frac{\partial}{\partial y} \end{aligned}$$

(b)

$$T = y^2 d\hat{x}\otimes d\hat{x} + d\hat{y}\otimes d\hat{y}$$

The components are

$$T_{\mu\nu} = \begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix},$$

where as usual ν labels the columns and μ labels the rows. The transformation can be done in two ways, first by direct substitution of the basis 1-forms, and second by calculating and applying the transformation matrices. For the first method we need the basis one-forms $d\hat{x}$ and $d\hat{y}$, which are simple enough to compute.

$$\begin{aligned} d\hat{x} &= \cos\theta dr - r\sin\theta d\theta \\ d\hat{y} &= \sin\theta dr + r\cos\theta d\theta \end{aligned}$$

Now we plug it all in

$$\begin{aligned} T &= r^2\sin^2\theta(cos^2\theta(d\hat{r}\otimes d\hat{r}) - r\cos\theta\sin\theta(d\hat{r}\otimes d\hat{\theta} + d\hat{\theta}d\hat{r}) + r^2\sin^2\theta(d\hat{\theta}\otimes d\hat{\theta})) \\ &+ (\sin^2\theta(d\hat{r}\otimes d\hat{r}) - r\cos\theta\sin\theta(d\hat{r}\otimes d\hat{\theta} + d\hat{\theta}d\hat{r}) + r^2\cos^2\theta(d\hat{\theta}\otimes d\hat{\theta})) \end{aligned}$$

The components are no longer so concise:

$$T_{\mu'\nu'} = \begin{pmatrix} r^2 \sin^2\theta \cos^2\theta + \sin^2\theta & -r^3 \sin^3\theta \cos\theta + r \sin\theta \cos\theta \\ -r^3 \sin^3\theta \cos\theta + r \sin\theta \cos\theta & r^4 \sin^4\theta + r^2 \cos^2\theta \end{pmatrix}.$$

For the second method we want to use the transformation matrices

$$T_{\mu'\nu'} = \partial^{\mu}_{\mu'} \partial^{\nu}_{\nu'} T_{\mu\nu}$$

Here $\partial^{\mu}_{\mu'} \equiv \frac{\partial x^{\mu}}{\partial x^{\mu'}}$ Expanded as a matrix, this has the form

$$\partial^{\mu}_{\mu'} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Note that to sum over the tensors indices using using standard matrix multiplication, we want to multiply matrices of the form.

$$T' = \partial^T \begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix} \partial$$

This will yield the same result as the previous method.

3. Although there is much beauty in considering spacetime as a single entity, it is sometimes useful to break it into space and time separately, and in particular to break the metric $g_{\mu\nu}$ into (g_{00}, g_{0i} and g_{ij}). Two common ways to do this (1+3) splitting are as follows. (Assume below that γ^{ij} is the inverse of γ_{ij}, and similar for the 'hat' version.)

   (a) In the first, we can write the metric as

   $$ds^2 = -M^2(dt - M_i dx^i)^2 + \gamma_{ij} dx^i dx^j.$$

   Show that in this case the metric components are given by

   $$g_{00} = -M^2; \quad g_{0i} = M^2 M_i; \quad g_{ij} = \gamma_{ij} - M^2 M_i M_j$$

   $$g^{00} = -(M^{-2} - M_i M^i); \quad g^{0i} = M^i; \quad g^{ij} = \gamma^{ij}.$$

   (b) In the second, we can write the metric as

   $$ds^2 = -N^2 dt^2 + \hat{\gamma}_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$

   Show that in this case,

   $$g_{00} = -(N^2 - N^i N_i); \quad g_{0i} = N_i; \quad g_{ij} = \hat{\gamma}_{ij}$$

   $$g^{00} = -N^{-2}; \quad g^{0i} = N^{-2} N^i; \quad g^{ij} = \hat{\gamma}^{ij} - N^{-2} N^i N^j.$$

(c) The first procedure is sometimes called 'threading the spacetime' and the second is sometimes called 'slicing the spacetime'. Comment on the appropriateness of these terms.

Solution:

(a) The metric expands out to

$$ds^2 = -M^2[(dt)^2 - M_i(dtdx^i + dx^i dt) + M_i M_j dx^i dx^j] + \gamma_{ij}dx^i dx^j.$$

We can read off the matrix elements of g from this expression:

$$\begin{aligned} g_{00} &= -M^2 \\ g_{0i} &= M^2 M_i \\ g_{ij} &= \gamma_{ij} - M^2 M_i M_j. \end{aligned}$$

We can check that the components of the inverse matrix are correct:

$$\begin{aligned} g^{0\rho}g_{\rho 0} &= g^{00}g_{00} + g^{0i}g_{i0} = M^2(M^{-2} - M_i M^i) + M^2 M_i M^i = 1 \\ g^{0\rho}g_{\rho i} &= g^{00}g_{0i} + g^{0j}g_{ji} = -M^2 M_i(M^{-2} - M_k M^k) + M^j(\gamma_{ij} - M^2 M_i M_j) \\ &= -M_i + M_i M^2 M_k M^k + M_i - M_i M^2 M_j M^j = 0 \\ g^{i\rho}g_{\rho 0} &= g^{i0}g_{00} + g^{ij}g_{j0} = -M^i M^2 + M^2\gamma^{ij}M_j = -M^i M^2 + M^i M^2 = 0 \\ g^{i\rho}g_{\rho j} &= g^{i0}g_{0j} + g^{ik}g_{kj} = M^2 M^i M_j + \gamma^{ik}(\gamma_{kj} - M^2 M_j M_k) \\ &= M^2 M^i M_j - M^2 M^i M_j + \gamma^{ik}\gamma_{kj} = \delta^i{}_j, \end{aligned}$$

where we used the fact that γ is the metric on the 3D leaves of this foliation of spacetime (ie, $\gamma^{ik}\gamma_{kj} = \delta^i{}_j$ and $\gamma^{ij}M_j = M^i$). This proves that $g^{\mu\rho}g_{\rho\nu} = \delta^\mu{}_\nu$, so the given inverse matrix elements are correct.

(b) Multiplying the line element out gives

$$\begin{aligned} ds^2 &= -N^2 dt^2 + \hat{\gamma}_{ij}dx^i dx^j + \hat{\gamma}_{ij}N^i dt dx^j + \hat{\gamma}_{ij}N^j dx^i dt + \hat{\gamma}_{ij}N^i N^j dt^2 \\ &= -(N^2 - N_i N^i)dt^2 + 2N_i dt dx^i + \hat{\gamma}_{ij}dx^i dx^j. \end{aligned}$$

This implies that

$$\begin{aligned} g_{00} &= -(N^2 - N_i N^i) \\ g_{0i} &= N_i \\ g_{ij} &= \hat{\gamma}_{ij}. \end{aligned}$$

Again, we will verify that the given inverse metric is correct, using the fact that $\hat{\gamma}$ is the metric on the leaves of the foliation:

$$\begin{aligned} g^{0\rho}g_{\rho 0} &= g^{00}g_{00} + g^{0i}g_{i0} = N^{-2}(N^2 - N_i N^i) + N_i N^{-2}N^i = 1 \\ g^{0\rho}g_{\rho i} &= g^{00}g_{0i} + g^{0j}g_{ji} = -N^{-2}N_i + N^{-2}N^j\hat{\gamma}_{ji} = 0 \\ g^{i\rho}g_{\rho 0} &= g^{i0}g_{00} + g^{ij}g_{j0} = N^{-2}N^i(N_k N^k - N^2) + N_j(\hat{\gamma}^{ij} - N^{-2}N^i N^j) \\ &= N^i(N^{-2}N_k N^k - 1) + N^i - N^{-2}N^i N_j N^j = 0 \\ g^{i\rho}g_{\rho j} &= g^{i0}g_{0j} + g^{ik}g_{kj} = N^{-2}N^i N_j + (\hat{\gamma}^{ik} - N^{-2}N^i N^k)\hat{\gamma}_{jk} \\ &= N^{-2}N^i N_j - N^{-2}N^i N_j + \hat{\gamma}^{ik}\hat{\gamma}_{kj} = \delta^i{}_j. \end{aligned}$$

This proves that the given inverse matrix elements are correct.

(c) In the second procedure, when t is held constant the line element reduces to $\hat{\gamma}_{ij}dx^i dx^j$. Spacetime is "sliced" into 3D spacelike submanifolds in this $(3+1)$ splitting.

In the first case, the line element reduces to $\gamma_{ij}dx^i dx^j$ when $dt = M_i dx^i$. This expression defines a curve in spacetime at each point that "threads" through the full 4D manifold, splitting it into $(1+3)$ dimensions.

4. Each point inside the forward lightcone of the origin (i.e. $-t^2 + r^2 < 0$ in spherical coordinates) in Minkowski space lies on some Lorentz hyperboloid of the form:

$$-t^2 + r^2 = -a^2$$

for some value of a. Such points can be labeled using a as a time coordinate and (χ, θ, ϕ) as spatial coordinates related to the Minkowski spherical coordinates by $t = a\cosh\chi$ and $r = a\sinh\chi$. Find the metric of flat spacetime in these new coordinates. Sketch a family of spacelike surfaces in a (t, r) spacetime diagram.

Solution:

There's not actually that much to this problem: taking differentials of the given coordinate transform gives:

$$dt = a\sinh\chi d\chi + (\cosh\chi)da, \quad dr = a\cosh\chi d\chi + (\sinh\chi da).$$

Plugging this into the metric for flat space in spherical coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

gives

$$ds^2 = -da^2 + a^2(d\chi^2 + \sinh^2\chi d\Omega^2).$$

What's fun about this problem, though is, that as we will see later in the course, a spatial geometry

$$d\sigma^2 = d\chi^2 + \sinh^2\chi d\Omega^2$$

constitutes an infinite homogeneous space of constant (negative) curvature. Scaled by a, the metric we have actually then corresponds to an infinite expanding universe. On the other hand, if we look at a $t = const.$ slice, the region we've described looks like a finite-size sphere, which expands as t increases. As it turns out, our universe might actually have just this structure, being infinite while sitting inside of a sort of bubble expanding into the ambient spacetime.

5. A guy walks up to you on the street and wants to sell you a 3-dimensional space with coordinates x, y, and z and metric

$$ds^2 = dx^2 + dy^2 + dz^2 - \left(\frac{3}{13}dx + \frac{4}{13}dy + \frac{12}{13}dz\right)^2.$$

Show that this guy is a hustler by demonstrating that this is really a 2-dimensional space, and find two new coordinates Z and W for which the metric takes the form:

$$ds^2 = dZ^2 + dW^2,$$

i.e. it's just a plain old plane. (Hint: think about the volume element.)

Solution:

One way to see that this is 2-dimensional is to compute the 3-dimensional volume element:

$$dV = \sqrt{g} dx\, dy\, dz.$$

working out the determinant we find $\sqrt{g} = 0$, meaning this is either 1 or 2-dimensional in reality. Now, since the metric does not depend on, say, z, we can just throw this coordinate away: we can experience the whole space while staying at constant z. The remaining metric is then:

$$ds^2 = dx^2 + dy^2 - \left(\frac{3}{13} dx + \frac{4}{13} dy\right)^2.$$

the determinant of this metric is nonzero, so the space really is 2D. To find the coordinate transformation we need to diagonalize the metric, then rescale. we find:

$$\xi = \frac{12}{5} \left(\frac{3}{13} x + \frac{4}{13} y\right),$$

$$\eta = \frac{13}{5} \left(\frac{-4}{13} x + \frac{3}{13} y\right),$$

which gives

$$ds^2 = d\xi^2 + d\eta^2.$$

Note that this metric has nonzero determinant, showing that the original space was two dimensional, not one dimensional.